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On Extensions of Singular Fourth Order Dynamic Operators on Time Scales

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Abstract. In this work, we consider a singular fourth order dynamic operator on time scales. We construct a space of boundary values. Later, we give a description of all maximal dissipative, self-adjoint and other extensions of singular fourth order differential operators on unbounded time scales.

1. Introduction

By the time scale calculus, we mean a unification of continuous and discrete analysis. In the early 1990s, the time scale calculus was introduced in [30] and since then, it has received a lot of attention. The theory of dynamic equations on time scales unifies the theories of differential equations and difference equations. The dynamic equations on time scales have several important applications, e.g. in the study of heat transfer, insect population models, epidemic models, stock market and neural networks, etc. (see [30]-[34]). On the other hand, we encounter the extensions of symmetric operators in many areas of mathematical physics, e.g. in solvable models of quantum mechanics and quantization problems. The extension theory was developed originally by J. von Neumann [19]. In [2], Calkin gave the description of all self-adjoint extensions of a symmetric operator, in terms of abstract boundary conditions. Later, Rofe- Beketov [20] described self-adjoint extensions of a symmetric operator, in terms of abstract boundary conditions with the aid of linear relations. The notion of a space of boundary values were introduced by Bruk [1] and Kochubei [7]. Using this notion, they described all maximal dissipative, accretive, self-adjoint extensions of symmetric operators. This problem has been investigated by many mathematicians (see [5],[6],[8]-[17]). For a more comprehensive discussion of extension theory of symmetric operators, the reader is referred to [3]. In this paper, a space of boundary values is constructed for singular fourth-order dynamic operators. We describe all maximal dissipative, accretive, self-adjoint and other extensions, in terms of boundary conditions. The organization of this paper is as follows. In Section 2, certain essential properties of time scales are included for the convenience of the reader. In Section 3, we construct a space of boundary values for singular fourth order dynamic operators in Lim-2 Case. We describe all maximal dissipative, accretive, self-adjoint and other extensions, in terms of boundary conditions. Finally, in Section 4, we give a description of all extensions of singular fourth order dynamic operators in Lim-4 Case.

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2. Preliminaries

First, we recall some necessary fundamental concepts related to time scales, and we refer to [23]-[30] for more details.

Definition 2.1. *Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad t \in \mathbb{T}.$$

It is convenient to have graininess operators $\mu_\sigma : \mathbb{T} \to [0,\infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. A point $t \in \mathbb{T}$ is left-scattered if $\mu_\rho(t) \neq 0$ and left-dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right-scattered if $\mu_\sigma(t) \neq 0$ and right-dense if $\mu_\sigma(t) = 0$. We introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 2.2. *A function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}^k$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

Analogously, one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show that

$$f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions (see [23]).

Example 2.3. *If $\mathbb{T} = \mathbb{R}$, then we have*

$$\sigma(t) = t, \; f^\Delta(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then we have

$$\sigma(t) = t + 1, \; f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

If $\mathbb{T} = q^{\mathbb{N}_0} = \left\{q^k : q > 1, \; k \in \mathbb{N}_0\right\}$, then we have

$$\sigma(t) = qt, \; f^\Delta(t) = \frac{f(qt) - f(t)}{qt - t}.$$

Definition 2.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$, such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula*

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Analogously, one may define the notion of ∇-antiderivative of some function.

Let $L^2_\Delta(\mathbb{T}^*)$ be the space of all functions defined on $\mathbb{T}^*$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2 \,\Delta t\right)^{1/2} < \infty.$$

Let $\mathbb{T}$ be a time scale which is bounded from below and unbounded from above such that $\inf \mathbb{T} = a > -\infty$ and $\sup \mathbb{T} = \infty$. We will denote $\mathbb{T}$ also as $[a,\infty)_{\mathbb{T}}$. The space $L^2_\Delta[a,\infty)_{\mathbb{T}}$ is a Hilbert space with the inner product

$$\langle f, g\rangle := \int_a^\infty f(t)\,\overline{g(t)}\Delta t,\ \ f,g \in L^2_\Delta[a,\infty)_{\mathbb{T}}\ .$$

(see [21]). We consider the fourth order dynamic equation

$$\Upsilon x(t) := (p_0 x^{\Delta\nabla})^{\nabla\Delta}(t) - (p_1 x^{\nabla})^{\Delta} + p_2(t)\,x(t) = \lambda x(t),\, t \in [a,\infty)_{\mathbb{T}}, \tag{1}$$

and assume that p_0, p_1 and p_2 are real-valued, p_0^{-1}, p_1 and p_2 are locally $\Delta-$integrable functions on $[a,\infty)_{\mathbb{T}}$, and $p_0 > 0$ on $[a,\infty)_{\mathbb{T}}$. For simplicity of notations, we write

$$\begin{aligned} x^{[0]} &= x \\ x^{[1]} &= x^{\Delta} \\ x^{[2]} &= p_0 x^{\Delta\nabla} \\ x^{[3]} &= p_1 x^{\nabla} - \left(x^{[2]}\right)^{\nabla} \\ x^{[4]} &= p_2 x - \left(x^{[3]}\right)^{\Delta} \end{aligned}$$

Now, we will convert (1) to the Hamiltonian system form. Thus we put

$$X = \begin{bmatrix} x \\ x^{\Delta} \\ -(p_0 x^{\Delta\nabla})^{\Delta} + p_1 x^{\Delta} \\ p_0 x^{\Delta\Delta} \end{bmatrix},\ \widehat{X} = \begin{bmatrix} x \\ x^{\Delta} \\ -(p_0 x^{\Delta\nabla})^{\nabla} + p_1 x^{\nabla} \\ p_0 x^{\Delta\nabla} \end{bmatrix}$$

which brings (1) to the form

$$J\,\widehat{X}^{\Delta} = (\lambda A + B)\,X, \tag{2}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} -p_2 & 0 & 0 & 0 \\ 0 & -p_1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/p_0 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that A and B are real and symmetric. Green's formula for solutions $x(t,\lambda)$ and $z(t,\lambda)$ is

$$\int_a^\infty (\Upsilon x)(t)\,\overline{z(t)}\Delta t - \int_a^\infty x(t)\,\overline{(\Upsilon z)(t)}\Delta t = [x,z]_\infty - [x,z]_a, \tag{3}$$

where $[x,z]_t := x^{[0]}(t)\overline{z}^{[3]}(t) - x^{[3]}(t)\overline{z}^{[0]}(t) + x^{[1]}(t)\overline{z}^{[2]}(t) - x^{[2]}(t)\overline{z}^{[1]}(t)$ and $[x,z]_\infty := \lim_{t\to\infty}[x,z]_t$ (see [22]). It is clear that $[x,z]_\infty$ exists and is finite. Letting $X(t,\lambda)$ and $Z(t,\lambda)$ be the corresponding vectors from (2), we get

$$X^T J Z(t) = [x,z]_t.$$

Let $y_i, 1 \leq i \leq 4$, be the solutions of Eq. (1) subject to the following normalization conditions:

$$p_0^2(t)\, W(y_1, y_2, y_3, y_4) = 1$$

and

$$\begin{bmatrix} [y_1, y_1] & [y_2, y_1] & [y_3, y_1] & [y_4, y_1] \\ [y_1, y_2] & [y_2, y_2] & [y_3, y_2] & [y_4, y_2] \\ [y_1, y_3] & [y_2, y_3] & [y_3, y_3] & [y_4, y_3] \\ [y_1, y_4] & [y_2, y_4] & [y_3, y_4] & [y_4, y_4] \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where the Wronskian of y_1, y_2, y_3 and y_4 is defined by (see [22])

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1^{[1]} & y_2^{[1]} & y_3^{[1]} & y_4^{[1]} \\ y_1^{[2]} & y_2^{[2]} & y_3^{[2]} & y_4^{[2]} \\ y_1^{[3]} & y_2^{[3]} & y_3^{[3]} & y_4^{[3]} \end{vmatrix}.$$

Then, we have a

Lemma 2.5. *For every $f, g \in L_\Delta^2[a, \infty)_\mathbb{T}$, we have the Plücker identity*

$$[f, g]_t = \begin{vmatrix} [y_2, g]_t & [g, y_4]_t \\ [y_2, f]_t & [f, y_4]_t \end{vmatrix} + \begin{vmatrix} [y_1, g]_t & [g, y_3]_t \\ [y_1, f]_t & [f, y_3]_t \end{vmatrix}. \quad (4)$$

Proof. The proof is similar to that of Lemma 1 in [35], thus we skip it. □

We will denote by $D_{\max}$ the set of all functions x in $L_\Delta^2\,[a, \infty)_\mathbb{T}$ such that the first three Δ derivatives are locally $\Delta-$ absolutely continuous in $[a, \infty)_\mathbb{T}$, and $\Upsilon(x) \in L_\Delta^2[a, \infty)_\mathbb{T}$. We define the maximal operator $\Gamma_{\max}$ on $D_{\max}$ by the equality $\Gamma_{\max} x = \Upsilon x$. Let $D_{\min}$ denote the linear set of all vectors $x \in D_{\max}$ satisfying the conditions

$$x^{[0]}(a) = x^{[1]}(a) = x^{[2]}(a) = x^{[3]}(a) = [x, z]_\infty = 0, \ \forall z \in D_{\max}. \quad (5)$$

If we restrict the operator $\Gamma_{\max}$ to the set $D_{\min}$, then we obtain the minimal operator $\Gamma_{\min}$. It is clear that $\Gamma_{\min}^* = \Gamma_{\max}$, and $\Gamma_{\min}$ is a closed symmetric operator with deficiency indices (2,2), (3,3), (4,4) (see [18], [35]). Now we recall the following.

Definition 2.6. *A linear operator M (with dense domain $D(M)$) acting on some Hilbert space H is called dissipative (accumulative) if* $\operatorname{Im}(\mathrm{Mf}, \mathrm{f}) \geq 0$ *(*$\operatorname{Im}(\mathrm{Mf}, \mathrm{f}) \leq 0$*) for all $f \in D(M)$ and maximal dissipative (maximal accumulative) if it does not have a proper dissipative (accumulative) extension (see [9]-[13]).*

Definition 2.7. *A triplet (H, Φ_1, Φ_2) is called a space of boundary values of a closed symmetric operator M on a Hilbert space H if Φ_1 and Φ_2 are linear maps from $D(M^*)$ to H, with equal deficiency numbers and such that:*

i) For every $f, g \in D(M^*)$ we have

$$\langle M^* f, g\rangle_H - \langle f, M^* g\rangle_H = \langle \Phi_1 f, \Phi_2 g\rangle_\mathbb{H} - \langle \Phi_2 f, \Phi_1 g\rangle_\mathbb{H};$$

ii) For any $F_1, F_2 \in H$ there is a vector $f \in D(M^*)$ such that $\Phi_1 f = F_1$ and $\Phi_2 f = F_2$ (see [4]).

3. Lim-2 Case

In this section, we will consider singular fourth order dynamic operators in the Lim-2 Case. Using the concept of boundary value space, we will describe all maximal dissipative, accretive, self-adjoint and other extensions, in terms of boundary conditions. Let the symmetric operator $\Gamma_{\min}$ has deficiency indices (2,2), i.e., the Lim-2 Case. Then $[y,z]_\infty = 0$ for all $y,z \in D_{\min}$ (see [18]). The domain $D_{\min}$ of the symmetric operator $\Gamma_{\min}$ consists of precisely those vectors $x \in D_{\min}$ satisfying the conditions: $x^{[0]}(a) = x^{[1]}(a) = x^{[2]}(a) = x^{[3]}(a) = 0$. Let us define the linear maps S_1 and S_2 from $D_{\max}$ to $\mathbb{C}^2$ by the formulae

$$S_1 f = \begin{pmatrix} -x^{[0]}(a) \\ x^{[1]}(a) \end{pmatrix},\ S_2 f = \begin{pmatrix} x^{[3]}(a) \\ x^{[2]}(a) \end{pmatrix}. \tag{6}$$

Now, we will state and prove some lemmas.

Lemma 3.1. *For arbitrary $y,z \in \mathrm{D}_{\max}$ we have*

$$\langle \Gamma_{\max} x, z\rangle_{L^2_\Delta} - \langle x, \Gamma_{\max} z\rangle_{L^2_\Delta} = \langle S_1 x, S_2 z\rangle_{\mathbb{C}^2} - \langle S_2 x, S_1 z\rangle_{\mathbb{C}^2}. \tag{7}$$

Proof. For every $x,z \in D_{\max}$, we have the Green's formula

$$\langle \Gamma_{\max} x, z\rangle_{L^2_\Delta} - \langle x, \Gamma_{\max} z\rangle_{L^2_\Delta} = -[x,z]_a\ . \tag{8}$$

Then,

$$\begin{aligned}\langle S_1 x, S_2 z\rangle_{\mathbb{C}^2} - \langle S_2 x, S_1 z\rangle_{\mathbb{C}^2} &= -x^{[0]}(a)\,\bar{z}^{[3]}(a) - x^{[1]}(a)\,\bar{z}^{[2]}(a) + x^{[3]}(a)\,\bar{z}^{[0]}(a) + x^{[2]}(a)\,\bar{z}^{[1]}(a) \\ &= -[x,z]_a\ .\end{aligned}$$

Using the equality (8), we obtain the equality (7). □

Lemma 3.2. *For any complex numbers $\alpha_1, \alpha_2, \alpha_3$,and α_4 there is a function $x \in \mathrm{D}_{\max}$ satisfying*

$$x^{[0]}(a) = \alpha_1,\ x^{[1]}(a) = \alpha_2,\ x^{[2]}(a) = \alpha_3,\ x^{[3]}(a) = \alpha_4\ .$$

Proof. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$. Then the vector-valued function

$$x(t) = \alpha_1(t)\,u_1 + \alpha_2(t)\,v_1 + \alpha_3(t)u_2 + \alpha_4(t)v_2,$$

where $\alpha_i(t) \in L^2_\Delta[a,\infty)_{\mathbb{T}}$ $(i = 1, ..., 4)$, satisfying the conditions

$$\begin{array}{llll} \alpha_1^{[0]}(a) = 1 & \alpha_1^{[1]}(a) = 0 & \alpha_1^{[2]}(a) = 0 & \alpha_1^{[3]}(a) = 0 \\ \alpha_2^{[0]}(a) = 0 & \alpha_2^{[1]}(a) = 0 & \alpha_2^{[2]}(a) = 0 & \alpha_2^{[3]}(a) = 1 \\ \alpha_3^{[0]}(a) = 0 & \alpha_3^{[1]}(a) = -1 & \alpha_3^{[2]}(a) = 0 & \alpha_3^{[3]}(a) = 0 \\ \alpha_4^{[0]}(a) = 0 & \alpha_4^{[1]}(a) = 0 & \alpha_4^{[2]}(a) = 1 & \alpha_4^{[3]}(a) = 0 \end{array}$$

belongs to the set $D_{\max}$ and $S_1 x = u,\ S_2 x = v$. □

Thus, the following hold.

Theorem 3.3. *The triple $\langle \mathbb{C}^2, S_1, S_2\rangle$ defined by (6) is a boundary value space of the operator $\Gamma_{\min}$.*

Corollary 3.4. *For any contraction K in $\mathbb{C}^2$ the restriction of the operator $\Gamma_{\min}$ to the set of functions $x \in \mathrm{D}_{\max}$ satisfying either*

$$(K - I)\,S_1 x + i\,(K + I)\,S_2 x = 0 \tag{9}$$

or

$$(K-I)\,S_1x - i\,(K+I)\,S_2x = 0 \tag{10}$$

is respectively the maximal dissipative and accretive extension of the operator $\Gamma_{\min}$. *Conversely, every maximal dissipative (accretive) extension of the operator* $\Gamma_{\min}$ *is the restriction of* $\Gamma_{\max}$ *to the set of functions* $x \in \mathrm{D}_{\max}$ *satisfying (9) ((10)), and the extension uniquely determines the contraction K. Conditions (9) ((10)), in which K is an isometry describe the maximal symmetric extensions of* $\Gamma_{\min}$ *in* $L^2_\Delta[a,\infty)_{\mathbb{T}}$. *If K is unitary, these conditions define self-adjoint extensions.*

In particular, the boundary conditions

$$x^{[3]}(a) - h_1 x^{[0]}(a) = 0$$
$$x^{[1]}(a) - h_2 x^{[2]}(a) = 0$$

with $Imh_1 \geq 0$ or $h_1 = \infty$, $Imh_2 \geq 0$ or $h_2 = \infty$, ($Imh_1 = 0$ or $h_1 = \infty$, $Imh_2 = 0$ or $h_2 = \infty$, $Imh_3 = 0$) describe the maximal dissipative (self-adjoint) extensions of $\Gamma_{\min}$ with separated boundary conditions.

4. Lim-4 Case

In this section, we will consider singular fourth order dynamic operators in the Lim-4 Case. We will describe all maximal dissipative, accretive, self-adjoint and other extensions, in terms of boundary conditions. Now we assume that $\Gamma_{\min}$ has deficiency indices (4,4). Then $y_i \in L^2_\Delta[a,\infty)_{\mathbb{T}}, 1 \leq i \leq 4$, and $y_i \in D_{\max}, 1 \leq i \leq 4$.

Theorem 4.1. *The domain* $\mathrm{D}_{\min}$ *of the operator* $\Gamma_{\min}$ *consists of precisely those functions* $x \in \mathrm{D}_{\max}$ *satisfying the following conditions:*

$$\begin{aligned} x^{[0]}(a) &= x^{[1]}(a) = x^{[2]}(a) = x^{[3]}(a) = 0, \\ [x,y_1]_\infty &= [x,y_2]_\infty = [x,y_3]_\infty = [x,y_4]_\infty = 0. \end{aligned} \tag{11}$$

Proof. From (5) and (4), we get the desired result. □

Let us denote by Ω_1 and Ω_2 the linear mappings of $D_{\max}$ to $\mathbb{C}^4$ defined by

$$\Omega_1 x = \begin{pmatrix} -x^{[0]}(a) \\ -x^{[1]}(a) \\ [x,y_2]_\infty \\ [x,y_1]_\infty \end{pmatrix}, \Omega_2 y = \begin{pmatrix} x^{[3]}(a) \\ x^{[2]}(a) \\ [x,y_4]_\infty \\ [x,y_3]_\infty \end{pmatrix}. \tag{12}$$

Then we have the following

Theorem 4.2. *The triple* $\langle \mathbb{C}^4, \Omega_1, \Omega_2 \rangle$ *defined by (12) is a boundary space of the operator* $\Gamma_{\min}$.

Proof. For every $x, z \in D_{\max}$, we have

$$\begin{aligned} \langle \Omega_1 x, \Omega_2 z\rangle - (\langle \Omega_2 x, \Omega_1 z\rangle &= -x^{[0]}(a)\,\bar{z}^{[3]}(a) - x^{[1]}(a)\,\bar{z}^{[2]}(a) + [x,y_2]_\infty\,[z,y_4]_\infty + [x,y_1]_\infty\,[z,y_3]_\infty \\ &\quad + x^{[3]}(a)\,\bar{z}^{[0]}(a) + x^{[2]}(a)\,\bar{z}^{[1]}(a) - [x,y_4]_\infty\,[z,y_2]_\infty - [x,y_3]_\infty\,[z,y_1]_\infty \\ &= -x^{[0]}(a)\,\bar{z}^{[3]}(a) - x^{[1]}(a)\,\bar{z}^{[2]}(a) + x^{[3]}(a)\,\bar{z}^{[0]}(a) + x^{[2]}(a)\,\bar{z}^{[1]}(a) \\ &\quad + [x,y_4]_\infty\,[y_2,z]_\infty - [x,y_2]_\infty\,[y_4,z]_\infty + [x,y_3]_\infty\,[y_1,z]_\infty - [y_1,x]_\infty\,[z,y_3]_\infty\,. \end{aligned}$$

From the Plucker identity (4), we have

$$\langle \Omega_1 x, \Omega_2 z\rangle - (\langle \Omega_2 x, \Omega_1 z\rangle = [x,z]_\infty - [x,z]_a\,.$$

By the Green formula (3), we get

$$\langle \Omega_1 x, \Omega_2 z\rangle - (\langle \Omega_2 x, \Omega_1 z\rangle = \langle \Gamma_{\max} x, z\rangle_{L^2_\Delta} - \langle x, \Gamma_{\max} z\rangle_{L^2_\Delta},$$

i.e., the first condition of the definition of a space of boundary values holds. □

The second condition will prove the following lemma.

Lemma 4.3. *For any complex numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$, and β_3 there is a function $x \in \mathrm{D}_{\max}$ satisfying*

$$\begin{aligned} x^{[0]}(a) &= \alpha_0,\ x^{[1]}(a) = \alpha_1,\ x^{[2]}(a) = \alpha_2,\ x^{[3]}(a) = \alpha_3 \\ [x, y_1]_\infty &= \beta_0,\ [x, y_2]_\infty = \beta_1,\ [x, y_3]_\infty = \beta_2,\ [x, y_4]_\infty = \beta_3 . \end{aligned}$$

Proof. *Let f be an arbitrary element of $L^2_\Delta[a,\infty)_{\mathbb{T}}$ satisfying*

$$\begin{aligned} \langle f, y_1\rangle_{L^2_\Delta} &= \beta_0 + \alpha_3,\ \langle f, y_2\rangle_{L^2_\Delta} = \beta_1 - \alpha_1 \\ \langle f, y_3\rangle_{L^2_\Delta} &= \beta_2 - \alpha_0,\ \langle f, y_4\rangle_{L^2_\Delta} = \beta_3 + \alpha_2 . \end{aligned} \tag{13}$$

There is such an f, even among the linear combinations of y_1, y_2, y_3 and y_4. Indeed, if we set

$$f = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4,$$

then the conditions in (13) constitute a system of equations in the constants c_1, c_2, c_3, c_4, whose determinant is the Gram determinant of the linearly independent functions $y_1, y_2, y_3\ y_4$, and is therefore nonzero. Let $x(t)$ denote the solution of the equation $\Upsilon(x) = f$ satisfying the initial conditions

$$x^{[0]}(a) = \alpha_0,\ x^{[1]}(a) = \alpha_1,\ x^{[2]}(a) = \alpha_2,\ x^{[3]}(a) = \alpha_3.$$

Assume that $x(t)$ is the desired element. If we apply Green's formula (3) to $x(t)$ and y_j, we obtain

$$\langle f, y_j\rangle_{L^2_\Delta} = \langle \Upsilon(x), y_j\rangle_{L^2_\Delta} = [x, y_j]_\infty - [x, y_j]_0,\ j = 1,2,3,4.$$

But $\Upsilon\left(y_j\right) = 0\ (j = 1,2,3,4)$. Since $x^{[0]}(a) = \alpha_0,\ x^{[1]}(a) = \alpha_1,\ x^{[2]}(a) = \alpha_2,\ x^{[3]}(a) = \alpha_3$, we have

$$\left[x, y_j\right]_a = \begin{cases} -\alpha_3, & \text{for } j = 1 \\ \alpha_1, & \text{for } j = 2 \\ \alpha_0, & \text{for } j = 3 \\ -\alpha_2, & \text{for } j = 4 \end{cases} .$$

Hence we get

$$\begin{aligned} \langle f, y_1\rangle_{L^2_\Delta} &= [x, y_1]_\infty - [x, y_1]_a \\ &= [x, y_1]_\infty + \alpha_3, \\[1em] \langle f, y_2\rangle_{L^2_\Delta} &= [x, y_2]_\infty - [x, y_2]_a \\ &= [x, y_2]_\infty - \alpha_1, \\[1em] \langle f, y_3\rangle_{L^2_\Delta} &= [x, y_3]_\infty - [x, y_3]_a \\ &= [x, y_3]_\infty - \alpha_0, \\[1em] \langle f, y_4\rangle_{L^2_\Delta} &= [x, y_4]_\infty - [x, y_4]_a \\ &= [x, y_4]_\infty + \alpha_2. \end{aligned}$$

According to (13), we have

$$[x, y_1]_\infty = \beta_0,\ [x, y_2]_\infty = \beta_1,\ [x, y_3]_\infty = \beta_2,\ [x, y_4]_\infty = \beta_3.$$

□

Corollary 4.4. *For any contraction K in $\mathbb{C}^4$ the restriction of the operator $\Gamma_{\min}$ to the set of functions $x \in D_{\max}$ satisfying either*

$$(K-I)\,\Omega_1 x + i\,(K+I)\,\Omega_2 x = 0 \tag{14}$$

or

$$(K-I)\,\Omega_1 x - i\,(K+I)\,\Omega_2 x = 0 \tag{15}$$

is respectively the maximal dissipative and accretive extension of the operator $\Gamma_{\min}$. Conversely, every maximal dissipative (accretive) extension of the operator $\Gamma_{\min}$ is the restriction of $\Gamma_{\max}$ to the set of functions $x \in D_{\max}$ satisfying (14) ((15)), and the extension uniquely determines the contraction K. Conditions (14) ((15)), in which K is an isometry describe the maximal symmetric extensions of $\Gamma_{\min}$ in $L^2_\Delta[a,\infty)_{\mathbb{T}}$. If K is unitary, these conditions define self-adjoint extensions.

In particular, the boundary conditions

$$x^{[3]}(a) - h_1 x^{[0]}(a) = 0$$

$$x^{[2]}(a) - h_2\, x^{[1]}(a) = 0$$

$$[x,y_4]_\infty - h_3[x,y_2]_\infty = 0$$

$$[x,y_3]_\infty - h_4[x,y_1]_\infty = 0$$

with $Imh_1 \geq 0$ or $h_1 = \infty$, $Imh_2 \geq 0$ or $h_2 = \infty$, $Imh_3 \geq 0$ or $h_3 = \infty$ and $Imh_4 \geq 0$ or $h_4 = \infty$ ($Imh_1 = 0$ or $h_1 = \infty$, $Imh_2 = 0$ or $h_2 = \infty$, $Imh_3 = 0$ or $h_3 = \infty$ and $Imh_4 = 0$ or $h_4 = \infty$) describe the maximal dissipative (self-adjoint) extensions of $\Gamma_{\min}$ with separated boundary conditions.

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