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On an Application of Herzberger's Matrix Method to Multipoint Families of Root-solvers

Jovana Džunić[a], Ljiljana D. Petković[b], Miodrag S. Petković[a]

[a]*Faculty of Electronic Engineering, University of Niš, Serbia*
[b]*Faculty of Mechanical Engineering, University of Niš, Serbia*

Abstract. An application of Herzberger's matrix method, very rarely used in the topic of multipoint methods for solving nonlinear equations, is presented. It is shown that the area of application of Herzberger's matrix method is wider than it is presented in [J. Herzberger, Über Matrixdarstellungen für Iterationverfahren bei nichtlinearen Gleichungen, Computing, 12 (1974) 215–222]. This method is applied for the determination of the order of convergence of multipoint families of methods, Steffensen's type and Newton's type, with and without memory. The advantage and the elegance of this method arise from ease in handling matrices.

In Memory of Professor Jürgen Herzberger.

1. Introduction

In this paper we demonstrate the application of Herzberger's matrix method [1] (HMM for short) for calculation of the order of convergence of multipoint methods for solving nonlinear equations. Herzberger has presented a matrix procedure which reduces the problem of determination of the order of convergence of Hermite's class of one-point methods, single-step and total-step methods to the problem of finding the spectral radius of a certain matrix. We prove that Herzberger's approach can be used to an arbitrary class of multipoint methods with or without memory. The requirement is the knowledge of an error relation of the method considered.

The main advantage of multipoint methods is their high computational efficiency since they can attain convergence order 2^n consuming only $n+1$ function evaluations. Methods with this property are usually called *optimal methods* and support the Kung-Traub hypothesis [2] on the upper bound of convergence order of multipoint methods. For example, Kung-Traub's families [2], Zheng-Li-Huang's family [3], a general class based on Hermite's interpolation polynomials [4] and on Hermite's interpolation by rational functions [5], belong to the class of $n-$point optimal methods of general form. In our study we shall consider both Newton's type and Steffensen's type methods with and without memory. It will be shown that under the

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Research supported by Serbiam Ministry of Education and Science
Corresponding author: Jovana Džunić
Email addresses: jovana.dzunic@elfak.ni.ac.rs (Jovana Džunić), ljiljana.petkovic@masfak.ni.ac.rs (Ljiljana D. Petković), miodragpetkovi@gmail.com (Miodrag S. Petković)

same amount of Hermite's sampled data, Steffensen's type of methods produce higher convergence order for methods with memory.

The paper is organized as follows. In Section 2 we recall some basic definitions and theorems of matrix and Perron-Frobenius theory. Section 3 is devoted to Herzberger's work [1] and provides evidence of applicability of HMM in a broader sense than that defined in [1]. In the next section application of HMM is explored. A multipoint derivative free class of iterative methods without memory, which starts with an arbitrary two-point method of fourth order, has been proposed in a condensed form as a five-line note in [6]. In Section 4 we present this class of methods in an expanded form and its generalization, suitable for the application of HMM. With HMM we give an alternative proof that the order of these two multipoint families is 2^n. In Section 4 we also use HMM to compare acceleration effects using information from the current and one previous iteration to Newton's and Steffensen's type of multipoint methods.

2. Non-negative Matrices

We recall some basic notions of matrix theory regarding positive and non-negative matrices.

Definition 2.1. *A real $n \times m$ matrix $A = [a_{ij}]$ is non-negative if*

$$a_{ij} \geq 0, \ \forall i, j, \ i \in \{1, 2, \dots, n\}, \ j \in \{1, 2, \dots, m\}.$$

If $a_{ij} > 0, \ \forall i, j, \ i \in \{1, 2, \dots, n\}, \ j \in \{1, 2, \dots, m\}$, then the matrix A is positive.

Definition 2.2. *A non-negative $n \times n$ matrix A is primitive if there exists $k \in \mathbb{N}$ such that A^k is positive.*

Definition 2.3. *A permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column, and zeros elsewhere.*

Definition 2.4. *A matrix is irreducible if it is not similar via a permutation to a block upper triangular matrix, that has more than one block of positive size.*

For a reducible matrix A there is a permutation matrix P such that

$$PAP^T = \left[\begin{array}{c|c} X & Y \\ \hline O & Z \end{array}\right],$$

where X and Z are square matrices of order at least 1. O denotes a rectangular zero-matrix. This definition is equivalent to the existence of a permutation matrix Q such that

$$QAQ^T = \left[\begin{array}{c|c} Z & O \\ \hline Y & X \end{array}\right].$$

A list of theorems used in the rest of the paper and appropriate references are given as follows. These results originate from [7] and [8] and they can be found, e.g., in [9] or [10].

Theorem 2.5. *Every primitive matrix is irreducible.*

Theorem 2.6 (Perron-Frobenius). *Let A be a non-negative irreducible $n \times n$ matrix. Then,*

1. *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.*
2. *To $\rho(A)$ there corresponds a positive eigenvector $\boldsymbol{x}$.*
3. *$\rho(A)$ increases when any entry of A increases.*
4. *$\rho(A)$ is a simple eigenvalue of A.*

Theorem 2.7. *Let A be a non-negative irreducible $n \times n$ matrix. There exists an $n \times n$ matrix B such that*

$$\lim_{j\to\infty}\left(\frac{A}{\rho(A)}\right)^j = B$$

if and only if A is primitive.

Theorem 2.8. *If $A = [a_{ij}]$ is a non-negative irreducible $n \times n$ matrix, then either*

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \text{for all} \quad 1 \le i \le n,$$

or

$$\min_{1\le i\le n}\sum_{j=1}^{n} a_{ij} < \rho(A) < \max_{1\le i\le n}\sum_{j=1}^{n} a_{ij}.$$

For matrix manipulations we use Sylvester's theorem stated in 1851 (see [11] for details and variants of Sylvester's theorem).

Theorem 2.9 (Sylvester). *If A and B are matrices of size $m \times k$ and $k \times m$ ($m \ge k$), respectively, then*

$$\det(\lambda I_m - AB) = \lambda^{m-k}\det(\lambda I_k - BA),$$

where I_j is the identity matrix of order j.

3. Herzberger's Matrix Method

Iterative root-finding methods are one of the oldest and most persistent subjects of investigation. Basic problem is to approximate a zero α of a function f by iterative means. This has evolved to finding an iteration function that would solve such a problem in the least of time with the prescribed amount of data. This is in close relation to order of convergence and optimality of an iterative function. In [1] a connection was established between order of convergence of an iterative method and eigenvalues of a matrix formed in a special manner. The connection was determined through an error relation of the analysed iteration. In [1] Herzberger organized presentation around one-point interpolatory iteration functions φ with memory of the form

$$x_{k+1} = \varphi(x_k, x_{k-1}, \dots, x_{k-n}),$$

due to the beforehand knowledge of their error relation. The specific estimate reads

$$|x_{k+1} - \alpha| = C_k \prod_{j=0}^{n} |x_{k-j} - \alpha|^{m_j}, \tag{1}$$

where $m_j \in \mathbb{N}$ is the number of function evaluations calculated at the point x_{k-j} : $f(x_{k-j}), f'(x_{k-j}), \dots, f^{(m_j-1)}(x_{k-j})$ and for some expression

$$C_k = C(x_k, x_{k-1}, \dots, x_{k-n}) \to C = \text{const. as } x_k \to \alpha.$$

This class of one-point interpolatory root-finding algorithms was coined Hermite interpolatory iteration functions (HIF for short) because it operates on Hermite interpolation (both direct and indirect).

We wish to prove that HMM has a wider range of use to that defined in [1]. Specifically, HMM is applicable for stationary iterative methods whenever the error relation of an iterative method is known and takes form (1) for $m_j \ge 0$, $j \in \{0, 1, \dots, n\}$. Therefore, we are broadening the domain of coefficients m_j from $\mathbb{N}$ to $\mathbb{N}_0$ and the application of HMM to a wider class of methods. This will particulary become of importance when analysing multipoint iterations.

A wider application range of Herzberger's method (HMM) is obtained by adapting the original proof from [1] to validate conclusions for a broader class of single root finding methods.

Theorem 3.1. *Let the error relation* (1) *be valid for some iterative root finding method* $\varphi = \varphi(x_k, x_{k-1}, \ldots, x_{k-n})$, *where all* $m_j \geq 0$. *Then the order of convergence of the stationary iteration*

$$x_{k+1} = \varphi(x_k, x_{k-1}, \ldots, x_{k-n}),\ k \geq n,\ k \in \mathbb{N},$$

equals the spectral radius of the associated matrix

$$M(\varphi) = \begin{bmatrix} m_0 & m_1 & m_2 & \ldots & m_{n-1} & m_n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{2}$$

Proof. Having in mind that we are analyzing error relation of an iteration function, we will not consider the trivial case when all $m_j = 0$.

First, it is assumed that all $m_j > 0$. Introduce

$$\epsilon_k = -\log|x_k - \alpha|, \qquad c_k = -\log C_k,$$

and apply logarithm to (1). The following is obtained

$$\epsilon_{k+1} = c_k + \sum_{j=0}^{n} m_j \epsilon_{k-j}. \tag{3}$$

Convergence of $\{x_k\}$ to α is assumed and that the order of convergence exists and equals $r = \lim_{j\to\infty} \frac{\epsilon_{j+1}}{\epsilon_j}$. Divide (3) by ϵ_{k-n} and allow $k \to \infty$. Since $\epsilon_k \to \infty$ and

$$\lim_{j\to\infty} \frac{\epsilon_{j+i}}{\epsilon_j} = \lim_{j\to\infty} \frac{\epsilon_{j+i}}{\epsilon_{j+i-1}} \frac{\epsilon_{j+i-1}}{\epsilon_{j+i-2}} \cdots \frac{\epsilon_{j+1}}{\epsilon_j} = r^i,$$

we obtain that the order of convergence r satisfies the equation

$$P_\varphi(x) := x^{n+1} - \sum_{j=0}^{n} m_j x^{n-j} = 0. \tag{4}$$

By Descartes' rule of signs, P_φ defined in (4) has a unique positive root - the order of convergence r. Note that Descartes' rule of signs provides the same conclusion in the case when some $m_j = 0$. The Frobenius companion matrix [12] of the polynomial (4) reads

$$M(\varphi) = \begin{bmatrix} m_0 & m_1 & \ldots & m_{n-1} & m_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{5}$$

Characteristic polynomial and the minimal polynomial of $M(\varphi)$ coincide with $P_\varphi(\lambda) = \det\left(\lambda I_{n+1} - M(\varphi)\right)$, and $M(\varphi)$ is a primitive matrix (see e.g. [9], [10], [12]), thus $r = \rho(M(\varphi))$.

Now we shall consider the case when some m_j vanish. There exists an array of matrices $\{M_k\}_{k\in\mathbb{N}}$ of the form (5) with all positive entries in the first raw that converge to the matrix $M(\varphi)$. Basically, if $m_j = 0$ in $M(\varphi)$ we substitute it with a positive value δ_k to form the matrix M_k, and then allow δ_k to be arbitrarily close to zero as k approaches infinity. Each M_k has a unique positive eigenvalue $r_k = \rho(M_k)$. Polynomial zeros are continuous functions of its coefficients, so by the argument of continuity we can conclude that the spectral radius of $M(\varphi)$ is its eigenvalue. Therefore, the order of convergence r of the iteration function φ equals spectral radius of the matrix $M(\varphi)$. It is also the unique positive eigenvalue of multiplicity one, by Descartes' rule of signs. □

Remark 3.1. The error relation (3) can as well be presented in a matrix form

$$\begin{bmatrix} \epsilon_{k+1} \\ \epsilon_k \\ \epsilon_{k-1} \\ \vdots \\ \epsilon_{k-n+1} \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & \dots & m_{n-1} & m_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \epsilon_k \\ \epsilon_{k-1} \\ \epsilon_{k-2} \\ \vdots \\ \epsilon_{k-n} \end{bmatrix} + \begin{bmatrix} c_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{6}$$

which establishes an even more profound and natural connection between iteration formula φ and its associated matrix $M(\varphi)$.

Remark 3.2. Iterative root finding methods are applied in real life problems only when their order of convergence $r \geq 1$. This is in accordance with a demand that there is no sign change of $P_\varphi(x)$ between 0 and 1, or equivalently

$$P_\varphi(1) \leq 0 \iff m_0 + \dots + m_n \geq 1.$$

For $m_j \in \mathbb{N}_0$ where not all m_j vanish, this inequality holds.

Consideration of the case when some m_j take zero value is important for analysis of iterations without memory and multipoint iterations. For example, the matrix $M(N)$, associated to Newton's method (N), takes one of the forms

$$M(N) = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \qquad M(N) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \dots$$

This is a consequence of the fact that any iteration can be considered as a function of more variables where some of these variables do not participate in the error relation. Not all matrices associated to Newton's method are primitive or irreducible, however HMM formula works as much expected.

With powerful numerical tools for estimating dominant eigenvalues and modern computers, HMM method allows ease in computing order of convergence of one point iterations with memory. Newton's method can be successfully used to determine the dominant eigenvalue in this case. Results such as those presented in [13] and [14] provide good initial value estimates.

The application of HMM expands to compositions of iteration functions. Composition of functions is naturally related to matrix multiplication. Using Perron-Frobenius theory of irreducible matrices it was proved in [1] that the order of convergence of a composition of HIF iterations $\varphi = \varphi_m \circ \varphi_{m-1} \circ \dots \circ \varphi_1$ with associated matrices $M_m, M_{m-1}, \dots, M_1$, where $M_k = M(\varphi_k)$, preserves feature:

$$r(\varphi) = \rho(M_m \cdot M_{m-1} \cdots M_1), \quad \text{and} \quad r(\varphi) \in Sp(M_m \cdot M_{m-1} \cdots M_1),$$

where $Sp(M)$ is the spectrum of the matrix M. We shall prove that this feature is not restricted to interpolatory type of iteration functions. Also, matrices M_j need not be primitive while of the specific form.

Composition of one-point iteration functions with memory φ_1 and φ_2 is defined through extensions $\phi_j : D^{n+1} \to D^{n+1},\ j = 1, 2$

$$\phi_j(x_k, x_{k-1}, \dots, x_{k-n}) = \Big(\varphi_j(x_k, x_{k-1}, \dots, x_{k-n}), x_k, \dots, x_{k-n+1}\Big)$$

where $D \subset \mathbb{C}$ is some open set containing α. Let $y_{k+1} = \varphi_1(x_k, x_{k-1}, \dots, x_{k-n})$. Composition $\varphi = \varphi_1 \circ \varphi_2$ is defined by an implicit equation

$$\Big(\varphi(x_k, x_{k-1}, \dots, x_{k-n}), y_{k+1}, x_k, \dots, x_{k-n+2}\Big) = (\phi_2 \circ \phi_1)(x_k, x_{k-1}, \dots, x_{k-n}).$$

In other words,

$$\varphi(x_k, x_{k-1}, \dots, x_{k-n}) = \varphi_2\Big(\varphi_1(x_k, x_{k-1}, \dots, x_{k-n}), x_k, \dots, x_{k-n+1}\Big).$$

Theorem 3.2. *Let iterations $\varphi_j = \varphi_j(x_k, x_{k-1}, \dots, x_{k-n})$ all have error relations*

$$|x_{k+1} - \alpha| = C_k^{(j)} \prod_{i=0}^{n} |x_{k-i} - \alpha|^{m_i^{(j)}}, \tag{7}$$

where all $m_i^{(j)} \in \mathbb{N}_0$. Then the order of convergence of the stationary iteration

$$x_{k+1} = \varphi(x_k, x_{k-1}, \dots, x_{k-n}) = (\varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1)(x_k, x_{k-1}, \dots, x_{k-n})$$

equals the spectral radius of the matrix

$$M = M_m \cdot M_{m-1} \cdots M_1, \quad M_j = M(\varphi_j). \tag{8}$$

Proof. We first consider the case when all $m_i^{(j)} > 0$. It can be verified that any product of associated matrices $M_j = M(\varphi_j)$ with positive first rows is still a primitive matrix. Also, for the entries a_{ij} of the product of M_s it is easily verified that they are all nonnegative integers and $\min_{1\le i\le n} \sum_{j=1}^{n} a_{ij} \ge 1$ and $\max_{1\le i\le n} \sum_{j=1}^{n} a_{ij} > 1$. From Theorem 2.8 it follows $\rho\Big(\prod M_s\Big) > 1$.

Error relations (7) for iterations φ_j, $j = 1, 2 \dots, m$, in matrix form (6) read

$$\begin{bmatrix} \epsilon_{k+1} \\ \epsilon_k \\ \epsilon_{k-1} \\ \vdots \\ \epsilon_{k-n+1} \end{bmatrix} = \begin{bmatrix} m_0^{(j)} & m_1^{(j)} & \dots & m_{n-1}^{(j)} & m_n^{(j)} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \epsilon_k \\ \epsilon_{k-1} \\ \epsilon_{k-2} \\ \vdots \\ \epsilon_{k-n} \end{bmatrix} + \begin{bmatrix} c_k^{(j)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $c_k^{(j)} \to c^{(j)}$ when $k \to \infty$. For iteration φ we thus have

$$\boldsymbol{e}_{k+1} = M_m \cdot M_{m-1} \cdots M_1 \cdot \boldsymbol{e}_k + \boldsymbol{v}_k = M \cdot \boldsymbol{e}_k + \boldsymbol{v}_k, \tag{9}$$

where

$$\boldsymbol{e}_k = \begin{bmatrix} \epsilon_k & \epsilon_{k-1} & \dots & \epsilon_{k-n} \end{bmatrix}^T, \quad M = M_m \cdot M_{m-1} \cdots M_1,$$

$$\boldsymbol{v}_k = \sum_{j=1}^{m} \left(\Big(\prod_{s=j+1}^{m} M_s \Big) \cdot \boldsymbol{c}_k^{(j)} \right), \qquad \boldsymbol{c}_k^{(j)} = \begin{bmatrix} c_k^{(j)} & 0 & \dots & 0 \end{bmatrix}^T.$$

Note that components of the vector $\boldsymbol{v}_k$ converge to some constant expression v as $k \to \infty$, that is $\boldsymbol{v}_k \to \boldsymbol{v} = [v\ v\ \dots\ v]^T$, when $k \to \infty$. Since every convergent sequence is bounded, there exists $\delta > 0$ such that

$$\|\boldsymbol{v}_k - \boldsymbol{v}\| < \delta, \ \text{ for all } k \in \mathbb{N}. \tag{10}$$

For arbitrary $j \in \mathbb{N}$ we have

$$\begin{aligned} \boldsymbol{e}_{k+j} &= M^j \boldsymbol{e}_k + \sum_{s=0}^{j-1} M^s \boldsymbol{v}_{k+j-s-1} \\ &= \rho(M)^j \Big(\frac{M}{\rho(M)} \Big)^j \boldsymbol{e}_k + \sum_{s=0}^{j-1} \rho(M)^s \Big(\frac{M}{\rho(M)} \Big)^s \boldsymbol{v}_{k+j-s-1}. \end{aligned} \tag{11}$$

Let B be a matrix from Theorem 2.7, such that $\lim\limits_{j\to\infty}\left(\dfrac{M}{\rho(M)}\right)^j = B$. Sequences

$$\left\{\Big\|\Big(\frac{M}{\rho(M)}\Big)^j\Big\|\right\}_{j\in\mathbb{N}} \quad\text{and}\quad \left\{\Big\|\Big(\frac{M}{\rho(M)}\Big)^j - B\Big\|\right\}_{j\in\mathbb{N}}$$

are convergent and are bounded. There exist $\delta_1, \delta_2 > 0$ such that for all $j \in \mathbb{N}$

$$\Big\|\Big(\frac{M}{\rho(M)}\Big)^j\Big\| < \delta_1, \qquad \Big\|\Big(\frac{M}{\rho(M)}\Big)^j - B\Big\| < \delta_2. \tag{12}$$

Error relation (11) may be written in the form,

$$\boldsymbol{e}_{k+j} = \rho(M)^j\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k + \sum_{s=0}^{j-1} \rho(M)^s\bigg(\Big(\frac{M}{\rho(M)}\Big)^s(\boldsymbol{v}_{k+j-s-1} - \boldsymbol{v}) + \bigg(\Big(\frac{M}{\rho(M)}\Big)^s - B\bigg)\boldsymbol{v} + B\boldsymbol{v}\bigg). \tag{13}$$

Using properties of the induced matrix norm, vector norm and modulus (triangle inequalities)

$$\big|\|a\| - \|b\|\big| \le \|a + b\| \le \|a\| + \|b\|,$$

from (10)–(13) it follows

$$\begin{aligned}
\|\boldsymbol{e}_{k+j}\| &\le \rho(M)^j\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| + \varepsilon\sum_{s=0}^{j-1}\rho(M)^s \\
&= \rho(M)^j\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| + \varepsilon\frac{\rho(M)^j - 1}{\rho(M) - 1} \\
&\le \rho(M)^j\left(\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| + \frac{\varepsilon}{\rho(M) - 1}\right),
\end{aligned}$$

and

$$\|\boldsymbol{e}_{k+j}\| \ge \rho(M)^j\left(\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| - \frac{\varepsilon}{\rho(M) - 1}\right),$$

for all $j \in \mathbb{N}$, and $\varepsilon = \delta_1\delta + \delta_2\|\boldsymbol{v}\| + \|B\boldsymbol{v}\|$. Having in mind $\Big\|\Big(\dfrac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| \to \infty,\ k \to \infty$, for $k \in \mathbb{N}$ large enough the inequality

$$\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| - \frac{\varepsilon}{\rho(M) - 1} > 0$$

is valid.

We conclude

$$0 < \rho(M)^j\left(\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| + \alpha_k\right) \le \|\boldsymbol{e}_{k+j}\| \le \rho(M)^j\left(\Big\|\Big(\frac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| + \beta_k\right), \tag{14}$$

where for fixed k, α_k and β_k are some constants.

Among several equivalent ways to obtain the order of convergence r, in [15] (see also [16]) it was proved that $r = \lim\limits_{j\to\infty}\|e_j\|^{1/j}$. Taking $(k + j)$−th root of inequalities (14) and allowing $j \to \infty$, relying on limits:

$\lim\limits_{j\to\infty}\Big\|\Big(\dfrac{M}{\rho(M)}\Big)^j \boldsymbol{e}_k\Big\| = \|B\boldsymbol{e}_k\|$ and $\lim\limits_{j\to\infty}\sqrt[j]{C} = 1$ we get $r = \rho(M)$.

As in the conclusion of the proof of Theorem 3.1, using the argument of continuity we verify the statement in the case when some $m_i^{(j)}$ are equal to zero. □

When considering multipoint methods without memory, the following is observed: Multipoint methods are a composition $\varphi = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1$ of one-point iteration functions with/without memory $\varphi_2, \varphi_3, \dots, \varphi_m$ and φ_1 a one-point iteration function without memory. For example, Traub-Steffensen's method (S)

$$x_{k+1} = x_k - \frac{\gamma f(x_k)^2}{f\big(x_k + \gamma f(x_k)\big) - f(x_k)}, \ \gamma \neq 0,$$

can be viewed as a composition of Secant method (φ_2) and Traub-Steffensen's correction (φ_1) :

$$\varphi_2(x_k) = x_k - \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad \text{and} \quad \varphi_1(x_k) = x_k + \gamma f(x_k).$$

The associated matrices read

$$M(\varphi_2) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M(\varphi_1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Their product, according to (8), gives

$$M(S) = M_2 \cdot M_1 = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \rho(M(S)) = 2.$$

Matrix $M(\varphi_1)$ is neither a primitive matrix, nor it is irreducible. However, HMM can still be used to calculate the order of convergence.

Additional examples of the application of HMM methods can be found in the early works [18], [19] of B. Neta.

Multipoint methods with memory are a composition of one-point iteration functions with/without memory $\varphi = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1$. With such interpretation it is obvious that HMM can be used to calculate the order of convergence of multipoint methods as long as the error relations are known. This application is the subject of the following section. Although in this communication we restrict our attention to the scalar case, it is important to note that HMM is applicable to the non-scalar case as well.

4. Application of Herzberger's Matrix Method to Multipoint Methods

4.1. *Family of multipoint derivative free methods without memory*

Let α be a simple zero of a given function f, and x an approximation to α. The multipoint family of derivative free methods for solving nonlinear equations proposed in [6]

$$\begin{cases} y_{k,0} = x_k, \ y_{k,1} = y_{k,0} + \gamma f(y_{k,0}), \ \gamma \neq 0, \\ y_{k,2} = y_{k,0} - \dfrac{f(y_{k,0})}{f[y_{k,0}, y_{k,1}]}, \\ y_{k,3} = \theta_4(y_{k,0}, y_{k,1}, y_{k,2}), \\ y_{k,j+1} = y_{k,j} - \dfrac{f(y_{k,j})}{P_j'(y_{k,j})}, \ (j = 3, \dots, n), \\ x_{k+1} = y_{k,n+1}, \ k \in \mathbb{N}_0, \end{cases} \tag{15}$$

relies on the method from [3]. It is a generalization due to the fact that pre-conditioner (steps $y_{k,0}$ to $y_{k,3}$) is an arbitrary two-point optimal family of methods starting with Steffensen-Traub's correction. Steps that define $y_{k,j+1}, \ j = 3, \dots, n$, use Newton's interpolating polynomials $P_j(t) = P_j(t; y_{k,0}, \dots, y_{k,j})$.

As noted in [6], this approach of increase in convergence by Newton's polynomial is of general nature. In this manner we can define optimal families of methods

$$\begin{cases} y_{k,0} = x_k, \ y_{k,1} = y_{k,0} + \gamma f(y_{k,0}), \ \gamma \neq 0, \\ y_{k,2} = y_{k,0} - \dfrac{f(y_{k,0})}{f[y_{k,0}, y_{k,1}]}, \\ y_{k,j+1} = \theta_{2^j}(y_{k,0}, \dots, y_{k,j}), \ (j = 2, \dots, p) \\ y_{k,j+1} = y_{k,j} - \dfrac{f(y_{k,j})}{P_j'(y_{k,j})}, \ (j = p+1, \dots, n), \\ x_{k+1} = y_{k,n+1}, \ k \in \mathbb{N}_0, \end{cases} \tag{16}$$

where $\theta_{2^{j-1}}(y_{k,0}, \dots, y_{k,j-1})$ is an arbitrary optimal derivative-free method of order 2^{j-1}. We note that dealing with $j > 4$ is only of academic interest, which has been mentioned several times in the existing literature. First we analyze methods (15).

Theorem 4.1. *The order of convergence of the family of derivative free methods* (15) *is* 2^n.

Proof. According to error relations

$$\varepsilon_{k,0} = y_{k,0} - \alpha, \qquad \varepsilon_{k,1} = y_{k,1} - \alpha \sim \varepsilon_{k,0},$$
$$\varepsilon_{k,2} = y_{k,2} - \alpha \sim \varepsilon_{k,0}\varepsilon_{k,1}, \quad \varepsilon_{k,3} = y_{k,3} - \alpha \sim \varepsilon_{k,0}^4,$$
$$\varepsilon_{k,j+1} = y_{k,j+1} - \alpha \sim \prod_{i=0}^{j} \varepsilon_{k,i}, \ j = 3, \dots, n+1, \ k \in \mathbb{N}_0,$$

we form companion matrices of dimensions $(n+1) \times (n+1)$. Indices of companion matrices M_j match the second index of the approximation $y_{k,j}$. For brevity, block matrices notation is used, with

$$a_j = [\underbrace{1 \ \dots \ 1}_{j} \ \underbrace{0 \ \dots \ 0}_{n+1-j}] \quad \text{and} \quad b = [0\,0\,4\ \underbrace{0\,0 \dots \ 0}_{n-2}]$$

row matrices, $O_{j\times i} = [0]_{j\times i}$ a zero rectangular matrix and I_j is the identity matrix of dimension j :

$$M_j = \left[\begin{array}{c|c} \multicolumn{2}{c}{a_j} \\ \hline I_n & O_{n\times 1} \end{array}\right], \ j \neq 3, \quad M_3 = \left[\begin{array}{c|c} \multicolumn{2}{c}{b} \\ \hline I_n & O_{n\times 1} \end{array}\right].$$

It is easily verified that when M_3 is not a factor, the following equalities hold

$$M_{j+1}M_j = \left[\begin{array}{c|c} \multicolumn{2}{c}{2a_j} \\ \hline \multicolumn{2}{c}{a_j} \\ \hline I_{n-1} & O_{(n-1)\times 2} \end{array}\right], \ M_{j+2}M_{j+1}M_j = \left[\begin{array}{c|c} \multicolumn{2}{c}{2^2 a_j} \\ \hline \multicolumn{2}{c}{2a_j} \\ \hline \multicolumn{2}{c}{a_j} \\ \hline I_{n-2} & O_{(n-2)\times 3} \end{array}\right].$$

Using induction it is easily demonstrated that when M_3 is not a factor, the matrix product reads

$$M_{j+s} \cdots M_j = \left[\begin{array}{c|c} \multicolumn{2}{c}{A_{s+1}(j)} \\ \hline I_{n-s} & O_{(n-s)\times(s+1)} \end{array}\right], \ 1 \leq s \leq n+1-j, \tag{17}$$

where $A_{s+1}(j)$ is the $(s+1)\times(n+1)$ matrix with i–th row equal to $2^{s+1-i}a_j$, that is

$$A_{s+1}(j) = \left[\begin{array}{c} 2^s a_j \\ \hline 2^{s-1}a_j \\ \hline \vdots \\ \hline 2^0 a_j \end{array}\right] = \begin{array}{c} \overbrace{\qquad\qquad\quad}^{j} \quad \overbrace{\qquad\quad}^{n+1-j} \\ \begin{bmatrix} 2^s & \dots & 2^s & 0 & \dots & 0 \\ 2^{s-1} & \dots & 2^{s-1} & 0 & \dots & 0 \\ & & \vdots & & & \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix} \end{array}. \tag{18}$$

Thus, regard to Theorem 3.2, for the method (15) we obtain

$$\begin{aligned} M &= (M_{n+1}\cdots M_4)M_3(M_2\cdot M_1) \\ &= \left[\begin{array}{c|c} \multicolumn{2}{c}{A_{n-2}(4)} \\ \hline I_3 & O_{3\times(n-2)} \end{array}\right]\cdot M_3\cdot \left[\begin{array}{c|c} \multicolumn{2}{c}{A_2(1)} \\ \hline I_{n-1} & O_{(n-1)\times 2} \end{array}\right] \\ &= \left[\begin{array}{c|c} \multicolumn{2}{c}{A_{n-2}(4)} \\ \hline I_3 & O_{3\times(n-2)} \end{array}\right]\cdot \left[\begin{array}{c|c} \multicolumn{2}{c}{A_3(1)} \\ \hline I_{n-2} & O_{(n-2)\times 3} \end{array}\right] \\ &= \begin{bmatrix} 2^{n-3} & 2^{n-3} & 2^{n-3} & 2^{n-3} & 0 & \dots & 0 \\ 2^{n-4} & 2^{n-4} & 2^{n-4} & 2^{n-4} & 0 & \dots & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \end{bmatrix}\cdot \begin{bmatrix} 4 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \dots & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^n & 0 & \dots & 0 \\ 2^{n-1} & 0 & \dots & 0 \\ \vdots & & & \\ 1 & 0 & \dots & 0 \end{bmatrix}. \end{aligned}$$

We now easily conclude $r = \rho(M) = 2^n$. □

Theorem 4.2. *The order of convergence of the family of derivative free methods* (16) *is* 2^n.

Proof. Error relations of (16) for $k \in \mathbb{N}_0$ read

$$\begin{aligned} \varepsilon_{k,0} &= y_{k,0}-\alpha, \\ \varepsilon_{k,1} &= y_{k,1}-\alpha \sim \varepsilon_{k,0}, \\ \varepsilon_{k,2} &= y_{k,2}-\alpha \sim \varepsilon_{k,0}\varepsilon_{k,1} \sim \varepsilon_{k,0}^2, \end{aligned}$$

and

$$\varepsilon_{k,j} = y_{k,j}-\alpha \sim \begin{cases} \varepsilon_{k,0}^{2^{j-1}}, & j = 3,\dots,p+1, \\ \prod_{i=0}^{j-1} \varepsilon_{k,i}, & j = p+2,\dots,n+1. \end{cases}$$

Using earlier notation $a_j = [\underbrace{1 \dots 1}_{j} \ \underbrace{0 \dots 0}_{n+1-j}]$ and $b_j = 2^{j-1}(a_j - a_{j-1})$, $(b_1 = a_1)$, $O_{j\times i} = [0]_{j\times i}$ a zero rectangular matrix and I_j the identity matrix of dimension j, associated matrices are

$$M_j = \begin{cases} \left[\begin{array}{c|c} \multicolumn{2}{c}{b_j} \\ \hline I_n & O_{n\times 1} \end{array}\right], & j = 1,2,\dots,p+1, \\ \left[\begin{array}{c|c} \multicolumn{2}{c}{a_j} \\ \hline I_n & O_{n\times 1} \end{array}\right], & j = p+2,\dots,n+1. \end{cases}$$

By induction we easily verify

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{b_{p+1}} \\ \hline I_n & O_{n\times 1} \end{array}\right] \cdots \left[\begin{array}{c|c} \multicolumn{2}{c}{b_1} \\ \hline I_n & O_{n\times 1} \end{array}\right] = \left[\begin{array}{c|c} \multicolumn{2}{c}{B_{p+1}} \\ \hline I_{n-p} & O_{(n-p)\times(p+1)} \end{array}\right],$$

where $B_{p+1} = \begin{bmatrix} 2^p & 0 & \dots & 0 \\ 2^{p-1} & 0 & \dots & 0 \\ \vdots & & & \\ 1 & 0 & \dots & 0 \end{bmatrix}$ is a matrix of dimension $(p+1)\times(n+1)$. In view of (17) we have

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{a_{n+1}} \\ \hline I_n & O_{n\times 1} \end{array}\right] \cdots \left[\begin{array}{c|c} \multicolumn{2}{c}{a_{p+2}} \\ \hline I_n & O_{n\times 1} \end{array}\right] = \left[\begin{array}{c|c} \multicolumn{2}{c}{A_{n-p}(p+2)} \\ \hline I_{p+1} & O_{(p+1)\times(n-p)} \end{array}\right],$$

where $A_{s+1}(j)$ is defined in (18). Therefore

$$M = M_{n+1}\cdots M_1 = \begin{bmatrix} 2^n & 0 & \dots & 0 \\ 2^{n-1} & 0 & \dots & 0 \\ \vdots & & & \\ 1 & 0 & \dots & 0 \end{bmatrix}, \ r = \rho(M) = 2^n. \ \square$$

We conclude with a remark that similar families of methods can be defined for Newton's type methods using Hermite's interpolating polynomials. HMM can again be used for the convergence analysis of these methods.

4.2. *Families of methods with memory*

In this section we will use HMM to compare order of convergence of multipoint methods with memory. We are exploring root-finders for a nonlinear equation $f(x) = 0$ used to approximate an isolated simple real zero α. We restrict our attention to methods that use Hermitian information: if the method uses $f^{(s)}(x)$ at some point x, function values $f(x), f'(x), \dots, f^{(s-1)}(x)$ are used as well. These type of methods were in detail discussed in [17] by Woźniakowski. He proved that for this type of methods those relayed on Hermite's interpolation attain the highest order of convergence. Kung and Traub in [2] have proved that two types of sampling in interpolatory iteration functions (HIF for short) provide optimal order of convergence. These are

1^o Newton's type: $f(y_1), f'(y_1), f(y_2), \dots, f(y_n)$;
2^o Steffensen's type: $f(y_0), f(y_1), f(y_2), \dots, f(y_n)$,

with iteration index k omitted. For these reasons we explore convergence acceleration of optimal Newton's type and Steffensen's type methods and compare outcomes. Using HMM we will show that convergence acceleration in HIF using information from one previous iteration gives better results for Steffensen's type of methods.

Let φ_N and φ_S denote two optimal multipoint iterations of order 2^n. φ_N will denote Newton's type of methods, using sampling 1^o, and φ_S Steffensen's type of methods , using sampling 2^o.

$$\varphi_N : \begin{cases} y_{k,1} = x_k, \\ y_{k,j} = N_j(y_{k,j-1}, \dots, y_{k,1}), \ j = 2, \dots, n+1, \\ x_{k+1} = y_{k,n+1}, \ k \in \mathbb{N}_0, \end{cases}$$

$$\varphi_S : \begin{cases} y_{k,0} = x_k, \quad y_{k,1} = S_1(y_{k,0}) = y_{k,0} + \gamma f(y_{k,0}), \\ y_{k,j} = S_j(y_{k,j-1}, \dots, y_{k,1}, y_{k,0}), \ j = 2, \dots, n+1, \\ x_{k+1} = y_{k,n+1} \ k \in \mathbb{N}_0. \end{cases}$$

Therefore, $\varphi_N = N_{n+1} \circ \cdots \circ N_2$, and $\varphi_S = S_{n+1} \circ \cdots \circ S_1$ in the context of composition of iteration functions.

We wish to calculate the highest order of convergence obtainable for methods φ_N and φ_S when they are accelerated using information from one previous iteration. Modifications with memory of methods φ_N and φ_S will be denoted by $\widetilde{\varphi}_N$ and $\widetilde{\varphi}_S$, respectively. We are examining the case where all information from the current and one previous iteration are used in each step. Let the modifications with memory read

$$\widetilde{\varphi}_N : \begin{cases} y_{k,1} = x_k, \\ y_{k,j} = \widetilde{N}_j(y_{k,j-1}, \dots, y_{k,1}, y_{k-1,n}, \dots, y_{k-1,1}), \ j = 2, \dots, n+1, \\ x_{k+1} = y_{k,n+1}, \ k \in \mathbb{N}_0, \end{cases} \tag{19}$$

$$\widetilde{\varphi}_S : \begin{cases} y_{k,0} = x_k, \\ y_{k,j} = \widetilde{S}_j(y_{k,j-1}, \dots, y_{k,0}, y_{k-1,n}, \dots, y_{k-1,0}), \ j = 1, \dots, n+1, \\ x_{k+1} = y_{k,n+1} \ k \in \mathbb{N}_0. \end{cases} \tag{20}$$

It is assumed that corresponding error relations of each step in $\widetilde{\varphi}_N$ and $\widetilde{\varphi}_S$ are of HIF form

$$\begin{aligned} \widetilde{\varphi}_N :\ y_{k,j} - \alpha &= \widetilde{N}_j(y_{k,j-1}, \dots, y_{k-1,1}) - \alpha \\ &\sim (y_{k,1} - \alpha) \prod_{i=1}^{j-1} (y_{k,i} - \alpha) \cdot (y_{k-1,1} - \alpha) \prod_{i=1}^{n} (y_{k-1,i} - \alpha), \\ j &= 2, \dots, n+1; \\ \widetilde{\varphi}_S :\ y_{k,j} - \alpha &= \widetilde{S}_j(y_{k,j-1}, \dots, y_{k-1,0}) - \alpha \\ &\sim \prod_{i=0}^{j-1} (y_{k,i} - \alpha) \cdot \prod_{i=0}^{n} (y_{k-1,i} - \alpha), \qquad j = 1, \dots, n+1. \end{aligned}$$

Theorem 4.3. $r(\widetilde{\varphi}_S) > r(\widetilde{\varphi}_N)$.

Proof. Associated matrices for Newton's type methods with memory $\widetilde{\varphi}_N$ are

$$M(\widetilde{N}_j) = \left[\begin{array}{c|c} \multicolumn{2}{c}{c_j} \\ \hline I_{2n-1} & O_{(2n-1)\times 1} \end{array} \right],$$

where $c_j = [\overbrace{1 \ \dots \ 1}^{j-2} \ 2 \ \overbrace{1 \ \dots \ 1}^{n-1} \ 2 \ \overbrace{0 \ \dots \ 0}^{n+1-j}]$, for $j = 2, \dots, n+1$.

It is easily proved that for $1 \le j \le n-1$ the product of associated matrices reads

$$M(\widetilde{N}_{2+j})\cdots M(\widetilde{N}_2) = \left[\begin{array}{c|c} \multicolumn{2}{c}{C_j} \\ \hline I_{2n-j-1} & O_{(2n-j-1)\times 1} \end{array}\right], \quad C_j = \begin{bmatrix} 2^j c_2 \\ \hline 2^{j-1}c_2 \\ \hline \vdots \\ \hline 2^0 c_2 \end{bmatrix},$$

C_j is the matrix of dimension $(j+1)\times 2n$. Then

$$M(\widetilde{\varphi}_N) = \left[\begin{array}{c|c} \multicolumn{2}{c}{C_{n-1}} \\ \hline I_n & O_{n\times n} \end{array}\right].$$

Since $(n+1)-$dimension leading principal submatrix of $M(\widetilde{\varphi}_N)$ is

$$\begin{bmatrix} 2^n & \overbrace{2^{n-1} & \dots & 2^{n-1}}^{n-1} & 2^n \\ 2^{n-1} & 2^{n-2} & \dots & 2^{n-2} & 2^{n-1} \\ & & \dots & & \\ 2 & 1 & \dots & 1 & 2 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2^{n-1} & 0 \\ 2^{n-2} & 0 \\ \vdots & \\ 2^0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \overbrace{1 & \dots & 1}^{n-1} & 2 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix},$$

introduce $A = \begin{bmatrix} 2^{n-1} & 2^{n-2} & \dots & 2^0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T$ and $B = \begin{bmatrix} 2 & \overbrace{1 & \dots & 1}^{n-1} & 2 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$. The characteristic polynomial of $M(\widetilde{\varphi}_N)$ is obtained using Laplace's expansion and Theorem 2.9:

$$\begin{aligned} P_{\widetilde{\varphi}_N}(\lambda) &= \det(\lambda I_{2n} - M(\widetilde{\varphi}_N)) = \lambda^{n-1}\det(\lambda I_{n+1} - AB) \\ &= \lambda^{2n-2}\det(\lambda I_2 - BA) = \lambda^{2n-2}\begin{vmatrix} \lambda - 3\cdot 2^{n-1} + 1 & -2 \\ -2^{n-1} & \lambda \end{vmatrix}. \end{aligned}$$

This gives

$$\begin{aligned} r(\widetilde{\varphi}_N) &= \rho(M(\widetilde{\varphi}_N)) = \tfrac{1}{2}\left(3\cdot 2^{n-1} - 1 + \sqrt{1 + 2^n + 9\cdot 2^{2(n-1)}}\right) \\ &= \tfrac{1}{2}\left(3\cdot 2^{n-2} - 1 + \sqrt{2^{2n+1} + (2^{n-1}+1)^2}\right). \end{aligned}$$

With notation similar to previous section, with $a_j = [\,\underbrace{1\ \dots\ 1}_{j}\ \underbrace{0\ \dots\ 0}_{2n+2-j}\,]$, associated matrices for Steffensen's type of methods read

$$M(\widetilde{S}_j) = \left[\begin{array}{c|c} \multicolumn{2}{c}{a_{n+j+1}} \\ \hline I_{2n+1} & O_{(2n+1)\times 1} \end{array}\right] \in \mathcal{M}_{(2n+2)\times(2n+2)},\ 1 \le j \le n+1.$$

Proceeding in the same manner as earlier, we obtain

$$M(\widetilde{\varphi}_S) = \left[\begin{array}{c|c} \multicolumn{2}{c}{A_{n+1}(n+2)} \\ \hline I_{n+1} & O_{(n+1)\times(n+1)} \end{array}\right],$$

where $A_{n+1}(n+2)$ is the $(n+1)\times(2n+2)$ matrix with i−th row equal to $2^{n+1-i}a_{n+2}$, that is

$$A_{n+1}(n+2) = \begin{bmatrix} 2^n a_{n+2} \\ \hline 2^{n-1}a_{n+2} \\ \hline \vdots \\ \hline 2^0 a_{n+2} \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} 2^n & \dots & 2^n \end{matrix}}^{n+2} & \overbrace{\begin{matrix} 0 & \dots & 0 \end{matrix}}^{n} \\ \begin{matrix} 2^{n-1} & \dots & 2^{n-1} \end{matrix} & \begin{matrix} 0 & \dots & 0 \end{matrix} \\ \vdots & \\ \begin{matrix} 1 & \dots & 1 \end{matrix} & \begin{matrix} 0 & \dots & 0 \end{matrix} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 2^n & 2^n & \dots & 2^n \\ 2^{n-1} & 2^{n-1} & \dots & 2^{n-1} \\ & & \vdots & \\ 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 2^{n-1} & 0 \\ \vdots & \\ 2^0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix},$$

we get $P_{\widetilde{\varphi}_S}(\lambda) = \lambda^{2n} \begin{vmatrix} \lambda - 2^n + 1 & -1 \\ -2^n & \lambda \end{vmatrix}$. Therefore

$$r(\widetilde{\varphi}_S) = \rho(M(\widetilde{\varphi}_S)) = \frac{1}{2}\left(2^{n+1} - 1 + \sqrt{2^{2(n+1)} + 1}\,\right).$$

Having in mind

$$r(\widetilde{\varphi}_N) = \tfrac{1}{2}\left(3\cdot 2^{n-1} - 1 + \sqrt{(3\cdot 2^{n-1}-1)^2 + 2^{n+2}}\,\right),$$
$$r(\widetilde{\varphi}_S) = \tfrac{1}{2}\left(4\cdot 2^{n-1} - 1 + \sqrt{(4\cdot 2^{n-1}-1)^2 + 2^{n+2}}\,\right),$$

we conclude that Steffensen's type methods produce better order of convergence with the same information volume. □

Note that similar analysis can be performed for different type of memory usage and different error relations.

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