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Multiplicity of Positive Solutions for Critical Fractional Kirchhoff Type Problem with Concave-convex Nonlinearity

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Abstract. This paper is devoted to study a class of Kirchhoff type problem with critical fractional exponent and concave nonlinearity. By means of variational methods, the multiplicity of the positive solutions to this problem is obtained.

1. Introduction and Main Results

This paper is concerned with the multiplicity of positive solutions for the following problem:

$$\begin{cases} M\left(\int_{R^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)(-\Delta)^s u = \lambda u^{q-1} + u^{2_s^*-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } R^N \setminus \Omega, \end{cases} \tag{1}$$

where $0 \in \Omega$ is a regular bounded domain in R^N, $M(t) = a + bt^k$ and the parameters $a,\ b,\ \lambda > 0$, $0 < s < 1 < q < 2$, $N > 2s$, $0 \le k < \frac{2s}{N-2s}$, $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev exponent. Here $(-\Delta)^s$ is the fractional Laplace operator(see [11]) defined, up to a normalization factor, by

$$-(-\Delta)^s u(x) = \int_{R^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in R^N.$$

In recent years, more and more attention have been paid to nonlocal diffusion problems, in particular to the ones driven by the fractional Laplace operator. This type of operator seems to have a dislocations in mechanical systems or in crystals. In addition, these operators arise in modelling diffusion and transport in a prevalent role in physical situations such as combustion and highly heterogeneous medium. As to the concave-convex nonlinearity, this type of problems has been studied by many authors [2, 3, 4, 10, 12, 16] and the references therein. In the case $k = 0$, the authors in [3] have investigated the following equation:

$$\begin{cases} (-\Delta)^s u = \lambda u^{q-1} + u^{2_s^*-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } R^N \setminus \Omega, \end{cases} \tag{2}$$

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where $\lambda > 0$, $0 < s < 1$, $N > 2s$ and $1 < q < 2$. They found that there exists $\Lambda > 0$ such that the equation (2) admits at least two positive solutions for $0 < \lambda < \Lambda$, has a positive solution for $\lambda = \Lambda$ and no positive solution exists for $\lambda > \Lambda$. However, we may not find Λ such that problem (1) have the same result. In fact, this problem is still unsolved for the semilinear elliptic equation(the case $s = 1$), see the Remark 6.4 in [9].

The main purpose of this paper is to generalize the partial results of [3]. Using the variational method, we prove that the equation (1) has at least two positive solutions for λ sufficiently small when the weight functions satisfy some conditions. The main results of this paper are as follows.

Theorem 1.1 Let $0 < s < 1$, $N > 2s$ and $1 < q < 2$. Suppose that $M(t) = a + bt^k$, $a,\ b > 0$, $0 \le k < \dfrac{2s}{N-2s}$, then there exists $\lambda_0 > 0$ such that problem (1) for all $\lambda \in (0, \lambda_0)$ has at least one positive solution.

Theorem 1.2 Let $0 < s < 1$, $N > 2s$ and $\frac{N}{N-2s} < q < 2$. Suppose that $M(t) = a + bt^k$, $a > 0$, $0 \le k < \dfrac{2s}{N-2s}$, $b > 0$ is small enough, then there exists $\lambda^* > 0$ such that problem (1) for all $\lambda \in (0, \lambda^*)$ has at least two positive solutions.

This paper is organized as follows. In section 2 we give the functional framework necessary to work with the fractional Laplacian operator and we will give some auxiliary results. The proof of Theorem 1.1 and Theorem 1.2 is provided in section 3.

2. Some Auxiliary Results

We will denote by $H^s(R^N)$ the usual fractional Sobolev space endowed with the Gagliardo norm

$$||u||_{H^s(R^N)} = ||u||_{L^2(R^N)} + \left(\int_{R^N\times R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}.$$

We consider the function space

$$X_0 = \left\{u \in H^s(R^N) : u = 0 \ a.e. \ in \ R^N \setminus \Omega\right\},$$

with the norm

$$||u||_{X_0} = \left(\int_{R^N\times R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}.$$

We also recall that $(X_0, ||\cdot||_{X_0})$ is a Hilbert space(see [3] or [13]), with scalar product

$$\langle u, v\rangle = \int_{R^N\times R^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy.$$

Let $u^\pm = \max\{\pm u, 0\}$, the corresponding functional of problem (1) is

$$I(u) = \frac{a}{2}||u||^2_{X_0} + \frac{b}{2(k+1)}||u||^{2(k+1)}_{X_0} - \frac{\lambda}{q}\int_\Omega (u^+)^q dx - \frac{1}{2^*_s}\int_\Omega (u^+)^{2^*_s}dx, \quad u \in X_0.$$

It is well known that the critical points of the functional I in X_0 are positive solutions of problem (1). By the definition of weak solution u of problem (1), it means that $u \in X_0$ satisfies

$$\langle I'(u), v\rangle \ = \ \left(a + b||u||^{2k}_{X_0}\right)\int_{R^N\times R^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy - \lambda\int_\Omega (u^+)^{q-1}v dx - \int_\Omega (u^+)^{2^*_s-1}v dx$$

for any $v \in X_0$.

Define the best Sobolev constant

$$S = \inf_{u\in H^s(R^N)\setminus\{0\}} \frac{\|u\|_{X_0}^2}{\left(\int_{R^N} |u|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}. \tag{3}$$

From [7], we know that S is attained by functions

$$v_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}.$$

Now we give some definitions, and show that the corresponding functional of problem (1) satisfies the $(PS)_c$ condition.

Definition 2.1 A sequence $\{u_n\} \subset X_0$ is called a $(PS)_c$ sequence of I if $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_n\} \subset X_0$ of I has a convergent subsequence.

Lemma 2.2 Let $0 < s < 1$, $N > 2s$ and $1 < q < 2$. Suppose that $M(t) = a + bt^k$, $a,\ b > 0$, $0 \le k < \dfrac{2s}{N-2s}$. If $\{u_n\} \subset X_0$ is a $(PS)_c$ sequence of I, then $\{u_n\}$ is bounded in X_0.

Proof. By the Hölder inequality and the Young inequality, it follows from (3) that

$$\left|\frac{\lambda}{q}\int_\Omega (u^+)^q dx\right| \le \frac{\lambda}{q} S^{-\frac{q}{2}} |\Omega|^{\frac{2_s^*-q}{2_s^*}} \|u\|_{X_0}^q \le \eta\|u\|_{X_0}^2 + C(\eta)\lambda^{\frac{2}{2-q}} \tag{4}$$

for any $u \in X_0$, where $C(\eta) = \frac{2-q}{q} 2^{-\frac{2}{2-q}} |\Omega|^{\frac{(2-q)N+2qs}{(2-q)N}} (\eta S)^{\frac{q}{q-2}}$. Let $\{u_n\}$ be a $(PS)_c$ sequence of I. Note that $k < \frac{2s}{N-2s}$, it follows from (4) that

$$\begin{aligned} 2_s^* I(u_n) - \langle I'(u_n), u_n\rangle &= \frac{2as}{N-2s}\|u_n\|_{X_0}^2 + \frac{2s-(N-2s)k}{(k+1)(N-2s)} b\|u_n\|_{X_0}^{2(k+1)} - \frac{(2-q)N+2qs}{q(N-2s)}\lambda\int_\Omega (u_n^+)^q dx \\ &\ge \left(\frac{2as}{N-2s} - \frac{(2-q)N+2qs}{N-2s}\eta\right)\|u_n\|_{X_0}^2 - \frac{(2-q)N+2qs}{N-2s} C(\eta)\lambda^{\frac{2}{2-q}}, \end{aligned}$$

which implies

$$\left(\frac{2as}{N-2s} - \frac{(2-q)N+2qs}{N-2s}\eta\right)\|u_n\|_{X_0}^2 \le 2_s^* c + \frac{(2-q)N+2qs}{N-2s} C(\eta)\lambda^{\frac{2}{2-q}} + o(\|u_n\|_{X_0}).$$

Set $\eta < \frac{2as}{(2-q)N+2qs}$, we obtain $\{u_n\}$ is bounded in X_0. □

Lemma 2.3 Let $0 < s < 1$, $N > 2s$ and $1 < q < 2$. Suppose that $M(t) = a + bt^k$, $a,\ b > 0$, $0 \le k < \dfrac{2s}{N-2s}$, then there exists a positive constant A depending on N, q, s, S and a such that I satisfies the $(PS)_c$ condition with $c < c^* = \frac{s}{N}(aS)^{\frac{N}{2s}} - A\lambda^{\frac{2}{2-q}}$.

Proof. Let $\{u_n\} \subset X_0$ be a $(PS)_c$ sequence of I with $c < c^*$. By Lemma 2.2, we know that $\{u_n\}$ is bounded. Up to a subsequence, there exists $u \in X_0$ such that u_n converges u weakly in X_0, strongly in $L^r(\Omega)$ with $1 \le r < 2_s^*$ and a.e. in Ω. By the Dominated Convergence Theorem, we have

$$\lambda\int_\Omega (u_n^+)^{q-1}(u_n - u)dx + \int_\Omega (u_n^+)^{2_s^*-1}(u_n - u)dx \to 0.$$

Thus, by using also the fact that $\langle I'(u_n), u_n - u\rangle \to 0$, we get

$$\left(a + b\|u\|_{X_0}^{2k}\right)\int_{R^N\times R^N} \frac{(u_n(x) - u_n(y))((u_n(x) - u_n(y)) - (u(x) - u(y)))}{|x-y|^{N+2s}} dxdy \to 0,$$

from which it follows that $u_n \to u$ in X_0. Since I is C^1, we obtain $I'(u) = 0$. In particular, we have $\langle I'(u), u\rangle = 0$, which implies that

$$a\|u\|^2_{X_0} + b\|u\|^{2(k+1)}_{X_0} = \lambda \int_\Omega (u^+)^q dx + \int_\Omega (u^+)^{2^*_s} dx.$$

It follows from (4) that

$$I(u) = I(u) - \frac{1}{2^*_s}\langle I'(u), u\rangle \geq \frac{1}{2^*_s}\left(\frac{2s}{N-2s} - \frac{(2-q)N+2qs}{N-2s}\eta\right)\|u\|^2_{X_0} - \frac{(2-q)N+2qs}{N-2s}C(\eta)\lambda^{\frac{2}{2-q}}.$$

Set $\eta = \frac{2as}{(2-q)N+2qs}$, $A = \frac{2-q}{qN}\left(\frac{(2-q)N+2qs}{4}\right)^{\frac{2}{2-q}} |\Omega|^{\frac{(2-q)N+2qs}{(2-q)N}} (saS)^{\frac{q}{q-2}}$, we have

$$I(u) \geq -A\lambda^{\frac{2}{1-q}}. \tag{5}$$

By the Dominated Convergence Theorem, we obtain

$$\int_\Omega (u_n^+)^q dx = \int_\Omega (u^+)^q dx + o(1).$$

Let $w_n = u_n - u$, by the Brezis-Lieb lemma(see [5]), one has

$$\|u_n\|^2_{X_0} = \|w_n\|^2_{X_0} + \|u\|^2_{X_0} + o(1), \qquad \|u_n\|^{2(k+1)}_{X_0} = \left(\|w_n\|^2_{X_0} + \|u\|^2_{X_0} + o(1)\right)^{k+1},$$

and

$$\int_\Omega (u_n^+)^{2^*_s} dx = \int_\Omega (w_n^+)^{2^*_s} dx + \int_\Omega (u^+)^{2^*_s} dx + o(1).$$

Since $I(u_n) = c + o(1)$, we obtain

$$\begin{aligned}&\frac{a}{2}\|w_n\|^2_{X_0} + \frac{b}{2(k+1)}\left(\left(\|w_n\|^2_{X_0} + \|u\|^2_{X_0} + o(1)\right)^{k+1} - \|u\|^{2(k+1)}_{X_0}\right) - \frac{1}{2^*_s}\int_\Omega (w_n^+)^{2^*_s} dx \\ &= c - I(u) + o(1).\end{aligned} \tag{6}$$

According to $I'(u_n) = o(1)$ and $\langle I'(u), u\rangle = 0$, we get

$$a\|w_n\|^2_{X_0} + b\left(\left(\|w_n\|^2_{X_0} + \|u\|^2_{X_0} + o(1)\right)^{k+1} - \|u\|^{2(k+1)}_{X_0}\right) - \int_\Omega (w_n^+)^{2^*_s} dx = o(1). \tag{7}$$

Assume that $\|w_n\|_{X_0} \to l_1$, we have

$$\left(\|w_n\|^2_{X_0} + \|u\|^2_{X_0} + o(1)\right)^{k+1} - \|u\|^{2(k+1)}_{X_0} \to \left(l_1^2 + \|u\|^2_{X_0}\right)^{k+1} - \|u\|^{2(k+1)}_{X_0} \doteq l_2 \geq 0.$$

it follows from (7) that

$$\int_\Omega (w_n^+)^{2^*_s} dx \to al_1^2 + bl_2.$$

From (3), we have

$$\|w_n\|^{2^*_s}_{X_0} \geq S^{\frac{2^*_s}{2}} \int_{R^n} |w_n|^{2^*_s} dx \geq S^{\frac{2^*_s}{2}} \int_\Omega (w_n^+)^{2^*_s} dx.$$

As $n \to \infty$, we have $l_1^{2^*_s} \geq S^{\frac{2^*_s}{2}}(al_1^2 + bl_2) \geq S^{\frac{2^*_s}{2}} al_1^2$. Therefore, one has $l_1 \geq a^{\frac{N-2s}{4s}} S^{\frac{N}{4s}}$. Note that $2(k+1) < 2^*_s$, it follows from (5), (6) and (7) that

$$c = \frac{s}{N}al_1^2 + \left(\frac{1}{2(k+1)} - \frac{1}{2^*_s}\right)bl_2 + I(u) \geq \frac{s}{N}(aS)^{\frac{N}{2s}} + I(u) \geq c^*,$$

which contradicts the fact $c < c^*$. Therefore, we have $l_1 = 0$, which implies that $u_n \to u$ in X_0. Hence I satisfies the $(PS)_c$ condition with $c < c^*$. □

3. The Proof of Main Results

In this section, we complete the proof of our Theorems 1.1 and 1.2. Before the proof of Theorem 1.1, we first recall the following Lemma in [6].

Lemma 3.1 Let $r,\ p > 1$, $\psi(x) \in L^p(\Omega)$ and $\psi^+ = \max\{\psi, 0\} \neq 0$. Then there exists $w_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega \psi(x)(w_0^+)^r dx > 0$.

Proof of Theorem 1.1. It follows from (3) and (4) that

$$I(u) \geq \left(\frac{a}{2} - \eta\right)\|u\|^2_{X_0} - C(\eta)\lambda^{\frac{2}{2-q}} - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}_{X_0}.$$

Let $\eta = \frac{a}{4}$, we can find $\rho > 0$ and $\Lambda_1 > 0$ such that for all $\lambda \in (0, \Lambda_1)$

$$I(u) > 0 \ \text{ if } \ \|u\| = \rho \quad \text{and} \quad I(u) > -C_1 \quad \text{if} \quad \|u\| \leq \rho, \tag{8}$$

where $C_1 = \frac{2-q}{q}\left(\frac{\Lambda_1}{2}\right)^{\frac{2}{2-q}} |\Omega|^{\frac{(2-q)N+2qs}{(2-q)N}} \left(\frac{2}{aS}\right)^{\frac{q}{2-q}}$.

From Lemma 3.1, we obtain that there exists $\varphi_0 \in C_0^\infty(\Omega) \subset X_0$ such that

$$\int_\Omega (\varphi_0^+)^q dx > 0. \tag{9}$$

Therefore, one has

$$I(K\varphi_0) \leq \frac{a}{2}K^2\|\varphi_0\|^2_{X_0} - \frac{\lambda}{q}K^q \int_\Omega (\varphi_0^+)^q dx - \frac{1}{2_s^*}K^{2_s^*}\int_\Omega (\varphi_0^+)^{2_s^*} dx.$$

Fix $\lambda \in (0, \Lambda_1)$, noticing that $1 < q < 2$, it implies from (9) that there exists $K_0 = K(\lambda) > 0$ small enough such that $I(K_0\varphi_0) < 0$. Thus we deduce that

$$c_\lambda = \inf_{u \in B_\rho(0)} I(u) < 0 < \inf_{u \in \partial B_\rho(0)} I(u).$$

By applying the Ekeland's variational principle in $\overline{B_\rho(0)}$(see [8]), we obtain that there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset \overline{B_\rho(0)}$ of I.

By the expression of c^*, we can choose $0 < \lambda_0 < \Lambda_1$ such that $c^* > 0$ for all $\lambda \in (0, \lambda_0)$. It follows from $c_\lambda < 0$ and Lemma 2.3 that I satisfies the $(PS)_{c_\lambda}$ condition. Therefore, one has a subsequence still denoted by $\{u_n\}$ and $u_\lambda \in X_0$ such that $u_n \to u_\lambda$ in X_0 and

$$I(u_\lambda) = c_\lambda, \quad I'(u_\lambda) = 0,$$

which implies that u_λ is a solution of problem (1). After a direct calculation, we derive that $\|u_\lambda^-\|_{X_0} = \langle I'(u_\lambda), -u_\lambda^-\rangle = 0$, which implies $u_\lambda \geq 0$. Since $I(u_\lambda) = c_\lambda < 0 = I(0)$, we have $u_\lambda \neq 0$. Applying the Strong Maximum Principle(see [14]), we obtain u_λ is a positive solution of problem (1). The proof of Theorem 1.1 is completed. □

Lemma 3.2 Let $0 < s < 1, N > 2s$ and $\frac{N}{N-2s} < q < 2$. Suppose that $M(t) = a + bt^k, a > 0, 0 \leq k < \dfrac{2s}{N-2s}, b > 0$ is small enough, then there exists $\Lambda^* > 0$, for any $\lambda \in (0, \Lambda^*)$, we can find $\bar{u}_\lambda \in X_0$ such that $\sup\limits_{t \geq 0} I(t\bar{u}_\lambda) < c^*$.

Proof. For convenience, we consider the functional $J_b : X_0 \to R$ defined by

$$J_b(u) = \frac{a}{2}\|u\|^2_{X_0} + \frac{b}{2(k+1)}\|u\|^{2(k+1)}_{X_0} - \frac{1}{2_s^*}\int_\Omega (u^+)^{2_s^*} dx, \quad u \in X_0.$$

For the constant $\delta_0 > 0$, we can choose such a cut-off function $\phi(x) \in C_0^\infty(\Omega)$ that $\phi(x) = 1$ for $x \in B(0, \delta_0)$, $\phi(x) = 0$ for $x \in R^n \setminus B(0, 2\delta_0)$ and $0 \le \phi(x) \le 1$. Define $u_\varepsilon(x) = \frac{\phi(x)\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2s}{2}}}$. Similar to the calculation of [15], we have the following estimate(as $\varepsilon \to 0$)

$$\begin{aligned} \|u_\varepsilon\|_{X_0}^2 &= \int_{R^N\times R^N} \frac{|U(x)-U(y)|^2}{|x-y|^{N+2s}}dxdy + O(\varepsilon^{N-2s}),\\ \int_\Omega |u_\varepsilon|^{2_s^*}dx &= \|U\|_{L^{2_s^*}(R^N)}^{2_s^*} + O(\varepsilon^N),\\ \frac{\|u_\varepsilon\|_{X_0}^2}{\left(\int_\Omega |u_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}} &= S + O\left(\varepsilon^{N-2s}\right), \end{aligned} \tag{10}$$

where $U(x) = \left(1+|x|^2\right)^{-\frac{N-2s}{2}}$ satisfies

$$\frac{\int_{R^N\times R^N} \frac{|U(x)-U(y)|^2}{|x-y|^{N+2s}}dxdy}{\|U\|_{L^{2_s^*}(R^N)}^2} = S = \inf_{u\in H^s(R^N)\setminus\{0\}} \frac{\|u\|_{X_0}^2}{\left(\int_{R^N}|u|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}.$$

Define

$$h_b(t) = J_b(tu_\varepsilon) = \frac{a}{2}t^2\|u_\varepsilon\|_{X_0}^2 + \frac{bt^{2(k+1)}}{2(k+1)}\|u_\varepsilon\|_{X_0}^{2(k+1)} - \frac{1}{2_s^*}t^{2_s^*}\int_\Omega |u_\varepsilon|^{2_s^*}dx$$

for all $t \ge 0$. According to $2(k+1) < 2_s^*$ and (10), we have $\lim_{t\to+\infty} h(t) = -\infty$. Note that $h_b(0) = 0$ and $h_b(t) > 0$ for $t \to 0^+$, so $\sup_{t\ge0} h_b(t)$ attains for some $t_{b,\varepsilon} > 0$. By

$$0 = h_0'(t_{0,\varepsilon}) = t_{0,\varepsilon}\left(a\|u_\varepsilon\|_{X_0}^2 - t_{0,\varepsilon}^{2_s^*-2}\int_\Omega |u_\varepsilon|^{2_s^*}dx\right),$$

one has

$$t_{0,\varepsilon} = \left(\frac{a\|u_\varepsilon\|_{X_0}^2}{\int_\Omega |u_\varepsilon|^{2_s^*}dx}\right)^{\frac{1}{2_s^*-2}}.$$

Therefore, we deduce from (10) that

$$\sup_{t\ge0} J_b(tu_\varepsilon) = h_b(t_{b,\varepsilon}u_\varepsilon) \le h_0(t_{b,\varepsilon}u_\varepsilon) \le h_0(t_{0,\varepsilon}u_\varepsilon) = \frac{S}{N}(aS)^{\frac{N}{2s}} + O\left(\varepsilon^{N-2s}\right) = c^* + A\lambda^{\frac{2}{2-q}} + O\left(\varepsilon^{N-2s}\right). \tag{11}$$

By the expression of c^*, we can choose $\Lambda_2 > 0$ such that $c^* > 0$ for all $\lambda \in (0, \Lambda_2)$. Using the definitions of I and u_ε, we have

$$I(tu_\varepsilon) \le \frac{a}{2}t^2\|u_\varepsilon\|_{X_0}^2 + \frac{bt^{2(k+1)}}{2(k+1)}\|u_\varepsilon\|_{X_0}^{2(k+1)}$$

for all $t \ge 0$ and $\lambda > 0$. It follows from (10) that there exist $T \in (0,1)$, $b_1 > 0$ and $\varepsilon_1 > 0$ such that

$$\sup_{0\le t\le T} I(tu_\varepsilon) < c^* \tag{12}$$

for all $0 < \lambda < \Lambda_2$, $0 < b < b_1$ and $0 < \varepsilon < \varepsilon_1$. Moreover, it implies from (11) that

$$\begin{aligned} \sup_{t\ge T} I(tu_\varepsilon) &= \sup_{t\ge T}\left(J_b(tu_\varepsilon) - \frac{\lambda}{q}t^q\int_\Omega |u_\varepsilon|^q dx\right)\\ &\le c^* + A\lambda^{\frac{2}{2-q}} + O\left(\varepsilon^{N-2s}\right) - \frac{\lambda}{q}T^q\int_{B(0,\varepsilon)} |u_\varepsilon|^q dx. \end{aligned}$$

Let $\varepsilon = \lambda^{\frac{2}{(2-q)(N-2s)}} \in (0, \delta_0)$, It follows that

$$\int_{B(0,\varepsilon)} |u_\varepsilon|^q dx = \int_{B(0,\varepsilon)} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{(N-2s)q}{2}} dx \geq \int_{B(0,\varepsilon)} \frac{1}{(2\varepsilon)^{\frac{(N-2s)q}{2}}} dx = C_3 \varepsilon^{\frac{2N-(N-2s)q}{2}}.$$

By the above two inequalities, for any $0 < \lambda < \delta_0^{\frac{(2-q)(N-2s)}{2}}$, we have

$$\sup_{t \geq T} I(tu_\varepsilon) \leq c^* + O\left(\lambda^{\frac{2}{2-q}}\right) - \frac{C_3 T^q}{q} \lambda^{\frac{4(N-s)-2(N-2s)q}{(2-q)(N-2s)}}. \tag{13}$$

Note that $\frac{N}{N-2s} < q < 2$, we have $\frac{4(N-s)-2(N-2s)q}{(2-q)(N-2s)} < \frac{2}{2-q}$. Hence, we can choose $\Lambda_3 > 0$ such that

$$O\left(\lambda^{\frac{2}{2-q}}\right) - \frac{C_3 T^q}{q} \lambda^{\frac{4(N-s)-2(N-2s)q}{(2-q)(N-2s)}} < 0$$

for all $0 < \lambda < \Lambda_3$. Therefore, we have

$$\sup_{t \geq T} I(tu_\varepsilon) < c^*. \tag{14}$$

Set $\Lambda^* = \min\left\{\Lambda_2, \Lambda_3, \varepsilon_1^{\frac{(2-q)(N-2s)}{2}}\right\}$. Let $\lambda \in (0, \Lambda^*)$, $\varepsilon = \lambda^{\frac{2}{(2-q)(N-2s)}}$ and $\bar{u}_\lambda = u_\varepsilon$, we deduce from (12) and (14) that $\sup_{t \geq 0} I(t\bar{u}_\lambda) < c^*$ for all $0 < \lambda < \Lambda^*$ and $0 < b < b_1$. $\square$

Proof of Theorem 1.2. Choose $\lambda^* = \min\{\lambda_0, \Lambda^*\}$, from the proof of Theorem 1.1, we have already obtained that problem (1) for any $\lambda \in (0, \lambda^*)$ has a positive solution u_λ with $I(u_\lambda) < 0$. Now we only need to find that the second positive solution of problem (1). It follows from Lemma 3.1 that there exists $\phi_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega (\phi_0^+)^{2_s^*} dx > 0$. According to (4), we have

$$I(t\phi_0) \leq \left(\frac{a}{2} + \eta\right) t^2 \|\phi_0\|_{X_0}^2 - \frac{1}{2_s^*} t^{2_s^*} \int_\Omega (\phi_0^+)^{2_s^*} dx + C(\eta) \lambda^{\frac{2}{2-q}},$$

which implies that $I(t\phi_0) \to -\infty$ as $t \to +\infty$. Hence, there exists a positive number t_0 such that $\|t_0\phi_0\| > \rho$ and $I(t_0\phi_0) < 0$ for any $\lambda \in (0, \lambda^*)$. It implies from (8) that the functional I has the mountain pass geometry. Define

$$\Gamma = \{\gamma \in C([0,1], X_0) |\, \gamma(0) = 0,\ \gamma(1) = t_0\phi_0\}, \ \ \tilde{c}_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)).$$

From Lemma 3.2, we have $\tilde{c}_\lambda < c^*$. Applying Lemma 2.3, we know that I satisfies the $(PS)_{\tilde{c}_\lambda}$ condition. By the mountain pass theorem(see [1]), we obtain that problem (1) has the second solution $\tilde{u}_\lambda$ with $I(\tilde{u}_\lambda) > 0$. After a direct calculation, we derive that $\|\tilde{u}_\lambda^-\|_{X_0}^2 = \langle I'(\tilde{u}_\lambda), -\tilde{u}_\lambda^- \rangle = 0$, which implies that $\tilde{u}_\lambda^- = 0$. Hence, $\tilde{u}_\lambda \geq 0$. Since $I(\tilde{u}_\lambda) > 0 = I(0)$, we have $\tilde{u}_\lambda \neq 0$. By the Strong Maximum Principle, we obtain $\tilde{u}_\lambda$ is the second positive solution of problem (1). $\square$

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