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THE SEMILATTICE OF COMPACTIFICATIONS OF A TYCHONOFF FUNCTION

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Abstract. In this paper we continue the study of the $\mathcal{P}$–funtions started in [CN], [CFP] and [N_1], in particular, for the property $\mathcal{P}$ = Tychonoff. We describe a simple tecnique to obtain a Tychonoff compactification of any Tychonoff function and we study the set of all the Tychonoff compactifications of a Tychonoff function, showing that it is a complete upper semilattice.

1. Introduction and preliminaries

It is well-known that the set $K(X)$ of all the compactifications of a Tychonoff space X is a complete upper semilattice and that it is a complete lattice if and only if X is locally compact (see for example [Cn]). In this paper we continue the study of the $\mathcal{P}$–functions started in [CN], [CFP] and [N_1], obtaining a corresponding result for the set of the compactifications of a function with the property $\mathcal{P}$ =Tychonoff.

For a topological property, we define a property $\mathcal{P}$ for a function such that every continuous function on a space with the corresponding property is always a $\mathcal{P}$–function.

We will describe a simple tecnique to obtain a compactification of any Tychonoff function and we will study the set $K(X, f)$ of all the Tychonoff compactifications of a given Tychonoff function $f \in C(X, Y)$ showing that it is a complete upper semilattice.

In the following, all the functions are assumed to be continuous unless it is stated otherwise.

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If X is a topological space, $\tau(X)$ will denote the set of the open sets of X, $\sigma(X)$ will denote the set of the closed sets of X, $C^*(X)$ will be the set of all the real–valued bounded continuous functions on X and, if $x \in X$, we will use $\mathcal{N}_x$ to denote the filter of the neighbourhoods of x.

For notations, definitions or basic properties not explicitly mentioned here we refer to [PW] and [E].

2. Tychonoff functions and compactifications

Let $f \in C(X,Y)$ be a function. We will say that f is *Hausdorff* [U, P_3] if for every $x, x' \in X$ such that $f(x) = f(x')$ there are some $U, V \in \tau(X)$ such that $x \in U$, $x' \in V$ and $U \cap V = \emptyset$; that f is *completely regular* [P_3] if for every $F \in \sigma(X)$ and $x \in X \backslash F$ there exist $O \in \mathcal{N}_{f(x)}$ and $\varphi \in C^*(X)$ such that $\varphi(x) = 1$ and $\varphi(F \cap f^{-1}(O)) \subseteq \{0\}$ and that f is *Tychonoff* [P_3] if it is completely regular and Hausdorff.

Remark 2.1. The notions above defined are valid definitions of $\mathcal{P}$–functions in the sense of [CN]. In fact, every continuous function defined on a Hausdorff (respectively, completely regular, Tychonoff) space is Hausdorff (respectively, completely regular, Tychonoff).

We will say that a function is compact if it is perfect, i.e. closed and fibre-wise compact.

Let us note that, recently in [N_2], the author – generalizing some results contained in [D], [W_1] and [W_2] – has obtained a filter based method which allow us to build a perfect extension of any function (not necessariely continuous) between two arbitrary topological spaces.

Remark 2.2 We observe that, in general, a compact Hausdorff function is not Tychonoff. In fact, it was proved in [Cb] (see also [HI]) that the property $T_{3\frac{1}{2}}$ is not an inverse invariant of the class of perfect functions and it is known by Proposition 1.5 [P_3] that if a space Y and a function $f : X \to Y$ are both Tychonoff then the domain X is Tychonoff too.

In despite of this limitation, we will prove that it is possible to build a dense compact Tychonoff extension of any Tychonoff function.

Definition 2.3 Let $f \in C(X,Y)$, we will say that $F \in C(Z,Y)$ is a *Tychonoff compactification* of f if F is compact Tychonoff, X is dense in Z and $F|_X = f$.

Let $f \in C(X,Y)$ be a Tychonoff function.

The tecnique we use to obtain a compactification of f requires a particular extension of the domain X obtained in the following way.

Let $\Phi_X : X \to \left(\prod_{g\in C^*(X)} I_g\right) \times Y$ be the function defined by $\Phi_X(x) = (e_X(x), f(x))$ for each $x \in X$, where $I_g = [0,1]$ and $e_X : X \to \prod_{g\in C^*(X)} I_g$ is the evaluation function on X usually defined by $\pi_g \circ e_X = g$ for any $g \in C^*(X)$.

Lemma 2.4. *The function Φ_X is an embedding.*

Proof. It is clear that Φ_X is continuous and 1-1. Moreover, it is open (respect to $\Phi_X(X)$) as for each $x \in U \in \tau(X)$, said $y = f(x)$, by complete regularity of f, there are $O \in \mathcal{N}_y$ and $g \in C^*(X)$ such that $g(x) = 1$ and $g\left((X \backslash U) \cap f^{-1}(O)\right) \subseteq \{0\}$. Let $T = \pi_g^{-1}\left(\left]\frac{1}{2}, 1\right]\right) \times O$, it is easy to verify that $\Phi_X(x) \in T \in \tau\left(\left(\prod_{g\in C^*(X)} I_g\right) \times Y\right)$ and that $T \cap \Phi_X(X) \subseteq \Phi_X(U)$. This proves that Φ_X is an embedding. □

Now, we define

$$\beta(X, f) = cl_{\left(\prod_{g\in C^*(X)} I_g\right)\times Y}\left(\Phi_X(X)\right)$$

and $\beta f : \beta(X, f) \to Y$ by setting $\beta f = \pi_Y|_{\beta(X,f)}$.

Clearly, $\Phi_X(X)$ is dense in $\beta(X, f)$ and, as Φ_X is an embedding, we can identify $\Phi_X(X)$ with X (relabelling the points $\Phi(x) = (e_X(x), f(x))$ simply with x) to say that X is dense in $\beta(X, f)$. Further, it results $\beta f \circ \Phi_X = f$ and, identifying X with its homeomorphics image $\Phi_X(X)$, we can say that $\beta f|_X = f$.

The next easy Lemma extends to non-Hausdorff space the Theorem 3.7.1 [E].

Lemma 2.5. *Let K be a compact space, then the projection $\pi_Y : K \times Y \to Y$ is compact.*

Proof. Obviously, the function π_Y is fibrewise compact. Moreover, it is a closed function. In fact, fixed $F \in \sigma(K \times Y)$, for any $y \in Y \backslash \pi_Y(F)$ and $k \in K$, it results $(k, y) \in (K \times Y) \backslash F \in \tau(K \times Y)$. So, there are $U_k \in \tau(K)$ and $V_k \in \tau(Y)$ such that $k \in U_k$, $y \in V_k$ and $(U_k \times V_k) \cap F = \emptyset$. Thus $\{U_k\}_{k\in K}$ is an open cover of the compact set K and it admits an open subcover $\{U_k\}_{k\in A}$. Then $V = \bigcap_{k\in A} V_k$ is an open neighbourhood of y in Y such that $(K \times U) \cap F = \emptyset$. Hence, $V \subseteq Y \backslash \pi_Y(F)$. This proves that $Y \backslash \pi_Y(F) \in \tau(Y)$ and so that the function $\pi_Y : K \times Y$ is closed and hence perfect. □

The following useful property can be found in [E].

Lemma 2.6. *A restriction of a compact function to a closed set, is compact too.*

Proposition 2.7. *The function $\beta f : \beta(X,f) \to Y$ is compact Tychonoff.*

Proof. Clearly, βf is continuous and, by Lemmas 2.5 and 2.6, it is also compact.

Moreover, βf is Hausdorff, as for each $p_1 = (z_1, y)$, $p_2 = (z_2, y)$ such that $p_1 \neq p_2$ and $\beta f(p_1) = \beta f(p_2)$ it follows that $z_1 \neq z_2$ i.e. $\langle z_{1g} \rangle_{g \in C^*(X)} \neq \langle z_{2g} \rangle_{g \in C^*(X)}$ and so there is some $g \in C^*(X,I)$ such that $z_{1g} \neq z_{2g}$. Since the unit interval $I = [0,1]$ is Hausdorff, there are disjoint $U, V \in \tau(I)$ such that $z_{1g} \in U$, $z_{2g} \in V$ and $U \cap V = \emptyset$. Hence $U' = \left(\pi_g^{-1}(U) \times Y\right) \cap \beta(X,f)$ and $V' = \left(\pi_g^{-1}(V) \times Y\right) \cap \beta(X,f)$ are two disjoint open sets of $\beta(X,f)$ containing respectively p_1 and p_2.

To show that βf is completely regular, let $(z,y) \in W \in \tau\left(\beta(X,f)\right)$ with $z \in \prod_{g \in C^*(X)} I_g$ and $y \in Y$. There are $U \in \tau\left(\prod_{g \in C^*(X)} I_g\right)$ and $V \in \tau(Y)$ such that $(z,y) \in (U \times V) \cap \beta(X,f) \subseteq W$. Since $\prod_{g \in C^*(X)} I_g$ is completely regular, there is some $h \in C^*\left(\prod_{g \in C^*(X)} I_g\right)$ such that $h(x) = 1$ and $h\left(\prod_{g \in C^*(X)} I_g \backslash U\right) \subseteq \{0\}$.
Let $\widetilde{h} : \left(\prod_{g \in C^*(X)} I_g\right) \times Y \to I$ be the mapping defined by $\widetilde{h}\left((\xi,\eta)\right) = h(\xi)$ for each $(\xi,\eta) \in \left(\prod_{g \in C^*(X)} I_g\right) \times Y$. Since for each $A \in \tau(I)$, $\widetilde{h}^{-1}(A) = h^{-1}(A) \times Y$ is open in $\left(\prod_{g \in C^*(X)} I_g\right) \times Y$, by the continuity of h it follows that $\widetilde{h}$ is continuous.
Let $H = \widetilde{h}|_{\beta(X,f)} \in C^*\left(\beta(X,f)\right)$. Observed that $V \in \mathcal{N}_y$, we have that $H\left((z,y)\right) = \widetilde{h}\left((z,y)\right) = h(z) = 1$ and that:

$$
\begin{aligned}
& H\left((\beta(X,f)\backslash W) \cap \beta f^{-1}(V)\right) \\
& \quad \subseteq \; H\left(\beta(X,f)\backslash\left((U \times V) \cap \beta(X,f)\right) \cap \beta f^{-1}(V)\right) \\
& \quad = \; \widetilde{h}\left(\beta(X,f)\backslash(U \times V) \cap \left(\pi_Y|_{\beta(X,f)}\right)^{-1}(V)\right) \\
& \quad \subseteq \; h\left(\prod_{g \in C^*(X)} I_g \backslash U\right) \subseteq \{0\}
\end{aligned}
$$

and this shows that βf is completely regular. □

So, we have proved that each Tychonoff function $f \in C(X,Y)$ has a

Tychonoff compactification $\beta f \in C(\beta(X,f),Y)$ that we will call the *Stone–Čech compactification of f*.

Proposition 2.8. *If $g \in C(D,Y)$ is a compact function and $G \in C(X,Y)$ is a Hausdorff function such that $G|_D = g$ and D is dense in X (i.e. G is a Hausdorff extension of g) then $X = D$ and so $G = g$.*

Proof. First we observe that

$$G(X) = G(cl_X(D)) \subseteq cl_Y(G(D)) = cl_Y(g(D)) = g(D) \subseteq G(X)$$

and so that $g(D) = G(D) = G(X)$.

Now, to prove that $X \subseteq D$, we start observing that for each $y \in g(D)$ it results $g^{-1}(\{y\}) = G^{-1}(\{y\})$. In fact, it is clear that $g^{-1}(\{y\}) \subseteq G^{-1}(\{y\})$ and we have also that $G^{-1}(\{y\}) \subseteq g^{-1}(\{y\})$ as if, by contradiction, there exists some $\xi \in G^{-1}(\{y\}) \backslash g^{-1}(\{y\})$, for any $x \in g^{-1}(\{y\})$ we have that $x \neq \xi$ and $G(x) = G(\xi)$. Since G is Hausdorff, there are $U_x, V_x \in \tau(X)$ such that $x \in U_x$, $\xi \in V_x$ and $U_x \cap V_x = \emptyset$. Then $\{U_x\}_{x \in g^{-1}(\{y\})}$ is an open cover of the compact set $g^{-1}(\{y\})$ and so there are $x_1, \ldots x_n \in g^{-1}(\{y\})$ such that $g^{-1}(\{y\}) \subseteq \bigcup_{i=1}^{n} U_{x_i} = U \in \tau(X)$ and, said $V = \bigcap_{i=1}^{n} V_{x_i}$, we also have that $\xi \in V \in \tau(X)$ with $U \cap V = \emptyset$. Note that

$$g(D\backslash U) \subseteq G(cl_X(D\backslash U)) \subseteq cl_Y(G(D\backslash U)) = cl_Y(g(D\backslash U)) = g(D\backslash U)$$

and so that $g(D\backslash U) = G(cl_X(D\backslash U))$.
Now $y \notin g(D\backslash U)$ as otherwise from $g^{-1}(\{y\}) \subseteq U$ follows $D\backslash U \subseteq D\backslash g^{-1}(\{y\}) = g^{-1}(Y\backslash\{y\})$ and so it should be $y \in g(D\backslash U) \subseteq g(g^{-1}(Y\backslash\{y\}))$ i.e. $y \in Y\backslash\{y\}$ that is a contradiction. So, $y \in Y\backslash g(D\backslash U) = Y\backslash G(D\backslash U)$ and

$$\begin{aligned} G^{-1}(\{y\}) &\subseteq G^{-1}(Y\backslash G(D\backslash U)) = X\backslash G^{-1}(G(D\backslash U)) \\ &= X\backslash G^{-1}(G(cl_X(D\backslash U))) \subseteq X\backslash cl_X(D\backslash U) \end{aligned}$$

i.e. that $G^{-1}(\{y\}) \subseteq X\backslash cl_X(D\backslash U)$ with $X\backslash cl_X(D\backslash U) \in \tau(X)$.
Hence, $D \cap (X\backslash cl_X(D\backslash U)) \subseteq U$ and it results

$$\begin{aligned} cl_X(D \cap (X\backslash cl_X(D\backslash U))) &= cl_X(cl_X(D) \cap (X\backslash cl_X(D\backslash U))) \\ &= cl_X(X\backslash cl_X(D\backslash U)) \end{aligned}$$

and thus

$$\begin{aligned} G^{-1}(\{y\}) &\subseteq X\backslash cl_X(D\backslash U) \subseteq cl_X(X\backslash cl_X(D\backslash U)) \\ &= cl_X(D \cap (X\backslash cl_X(D\backslash U))) \subseteq cl_X(U) \end{aligned}$$

Since $\xi \in G^{-1}(\{y\})$ it follows that $\xi \in cl_X(U)$. A contradiction as $\xi \in V$ and $cl_X(U) \cap V = \emptyset$. This proves that $g^{-1}(\{y\}) = G^{-1}(\{y\})$ for each $y \in g(D) = G(X)$.

Finally, we have that

$$\begin{aligned} X &= G^{-1}(G(X)) = G^{-1}(g(D)) = G^{-1}\left(\bigcup_{y\in g(D)}\{y\}\right) = \bigcup_{y\in g(D)} G^{-1}(\{y\}) \\ &= \bigcup_{y\in g(D)} g^{-1}(\{y\}) = g^{-1}\left(\bigcup_{y\in g(D)}\{y\}\right) = g^{-1}(g(D)) = D \end{aligned}$$

i.e. that $X = D$ and, consequently, that $G = g$. □

Corollary 2.9. *For every compact Tychonoff function f, it results $\beta f = f$.*

The following two properties are well-known (see for example [PW]).

Lemma 2.10. *Every continuous function from a compact space to a Hausdorff space is compact.*

Lemma 2.11. *The product of compact functions is compact.*

Lemma 2.12. *Let $g \in C(X,Y)$ be a function on a Tychonoff space and Z be a space, then the product function $g \times id_Z : X \times Z \to Y \times Z$ is Tychonoff.*

Proof. Since the space X is Hausdorff, g is a Hausdorff function (see Remark 2.1). Since id_Z is clearly a Hausdorff function, by Theorem 3.2 [N_1], $g \times id_Z$ is Hausdorff too.
Now, let $(x,z) \in U \times V$ with $U \in \tau(X)$ and $V \in \tau(Z)$. Since X is completely regular, there is $\varphi \in C^*(X)$ such that $\varphi(x) = 1$ and $\varphi(X \backslash U) \subseteq \{0\}$. So, defined $\widetilde{\varphi} \in C^*(X \times Z)$ by setting $\widetilde{\varphi} \circ \pi_X = \varphi$ and $O = Y \times V$, it is clear that $(x,z) \in O \in \tau(Y \times Z)$, $\widetilde{\varphi}((x,z)) = 1$ and $\widetilde{\varphi}((X \times Z \backslash U \times V)(g \times id_Z)^{-1}(O)) \subseteq \{0\}$. This proves that the function $g \times id_Z$ is completely regular. □

Proposition 2.13. *Let $f \in C(X,Y)$ be a Tychonoff function and $F \in C(Z,Y)$ be a Tychonoff compactification of f, then there exists a compact Tychonoff function $\varphi \in C(\beta(X,f), Z)$ such that $F \circ \varphi = \beta f$ and $\varphi|_X = id_X$.*

Proof. Let $i : X \to Z$ the usual embedding from X to Z defined by $i(x) = x$ for every $x \in X$. We consider the continuous functions $i^* : C^*(Z) \to C^*(X)$ defined by setting $i^*(h) = h \circ i$ for each $h \in C^*(Z)$ and $i^{**} : \prod_{g \in C^*(X)} I_g \to \prod_{h \in C^*(Z)} I_h$ defined by $i^{**}(\alpha) = \alpha \circ i^*$ for each $\alpha \in \prod_{g \in C^*(X)} I_g$. From 2.10 follows that i^{**} is compact and so, by 2.11,

the product function $i^{**} \times id_Y : \left(\prod_{g\in C^*(X)} I_g\right) \times Y \to \left(\prod_{h\in C^*(Z)} I_h\right) \times Y$ is compact. Moreover, by 2.12, $i^{**} \times id_Y$ is Tychonoff.
Now, for each $x \in X$ it results

$$\begin{aligned}(i^{**} \times id_Y)(\Phi_X(x)) &= (i^{**} \times id_Y)(e_X(x), f(x)) \\ &= (i^{**}(e_X(x)), id_Y(f(x))) \\ &= (e_Z(x), f(x)) \\ &= (e_Z(x), F(x)) \\ &= \Phi_Z(x).\end{aligned}$$

Hence, $(i^{**} \times id_Y)(\Phi_X(X)) \subseteq \Phi_Z(X)$ and it results

$$\begin{aligned}(i^{**} \times id_Y)(\beta(X,f)) &= (i^{**} \times id_Y)\left(cl_{\left(\prod_{g\in C^*(X)} I_g\right)\times Y}(\Phi_X(X))\right) \\ &\subseteq cl_{\left(\prod_{h\in C^*(Z)} I_h\right)\times Y}(i^{**} \times id_Y)(\Phi_X(X)) \\ &\subseteq cl_{\left(\prod_{h\in C^*(Z)} I_h\right)\times Y}(\Phi_Z(X)) \\ &= \beta(Z,F).\end{aligned}$$

So, we can consider the corestriction $\varphi = (i^{**} \times id_Y)|_{\beta(X,f)} : \beta(X,f) \to \beta(Z,F)$ which is still continuous, compact and Tychonoff.

As F is compact Tychonoff, by 2.9, it results $\beta F = F$ and $Z = \beta(Z,F)$, so that $\varphi \in C(\beta(X,f), Z)$.

Since $F \circ \varphi = \beta f$ and $\varphi(\Phi_X(x)) = \Phi_Z(x)$ for each $x \in X$, identifying both $\Phi_X(X)$ and $\Phi_Z(X)$ with X, we have that $\varphi|_X = id_X$. This completes the proof. □

Lemma 2.14. *Let $g \in C(K,L)$ be a compact Hausdorff function such that $g|_X = id_X$ and X be a dense subspace of K and L. Then g is onto and $g(K\setminus X) = L\setminus X$.*

Proof. Since $X \subseteq g(K) \subseteq L$ and $g(K)$ is closed, it follows that $g(K) = L$.

So, to show that $g(K\setminus X) = L\setminus X$ it suffices to show that $g(K\setminus X) \subseteq L\setminus X$. If not, there are $y \in K\setminus X$ and $x \in X$ such that $g(y) = x$. Since $g(x) = id_X(x) = x$ and g is Hausdorff, there are $U, V \in \tau(K)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Now, $x \in U \cap X \in \tau(X)$ and there is some $W \in \tau(L)$ such that $U \cap X = W \cap X$. By continuity of $g : K \to L$, we can assume without loss of generality that $g(V) \subseteq W$ and this implies that $\emptyset \neq V \cap X \subseteq W$. Hence, $\emptyset \neq V \cap X \subseteq W \cap X = U \cap X$ which is a contradiction. □

In the following will be useful to give a new definition for the functions which is similar to the properties $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{T}$ defined and studied in [U].

Definition 2.15. We say that $f \in C(X,Y)$ has the property $\mathcal{U}$ if for each $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ there are $U, V \in \tau(X)$ such that $x_1 \in U$, $x_2 \in V$ and $cl_X(U) \cap cl_X(V) = \emptyset$.

Lemma 2.16. *Each Tychonoff function has the property* $\mathcal{U}$.

Proof. In fact, if $f \in C(X,Y)$ is a Tychonoff function, for each $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$ since f is Hausdorff there is some $W \in \tau(X)$ such that $x_2 \in W$ and $x_1 \notin cl_X(W)$. Moreover, as f is completely regular, there are $O \in \mathcal{N}_{f(x_1)}$ and $g \in C^*(X)$ such that $g(x_2) = 1$ and $g\left((X \setminus cl_X(W)) \cap f^{-1}(O)\right) \subseteq \{0\}$. So, set $U = g^{-1}\left(\left[0, \frac{1}{3}\right[\right)$ and $V = g^{-1}\left(\left]\frac{2}{3}, 1\right]\right)$, it is clear that $U, V \in \tau(X)$, $x_1 \in U$, $x_2 \in V$ and $cl_X(U) \cap cl_X(V) = \emptyset$. □

In general the composition of two Tychonoff functions is not Tychonoff but the following weaker result holds.

Lemma 2.17. *The composition of two functions with the property* $\mathcal{U}$ *is still a function with the property* $\mathcal{U}$.

Proof. Let $f \in C(X,Y)$ and $g \in C(Y,Z)$ be two functions holding the property $\mathcal{U}$ and $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $g(f(x_1)) = g(f(x_2))$. If $f(x_1) = f(x_2)$, the assertion follows from the fact that f has the property $\mathcal{U}$. If $f(x_1) \neq f(x_2)$, since g has the property $\mathcal{U}$, there are $U', V' \in \tau(Y)$ such that $f(x_1) \in U'$, $f(x_2) \in V'$ and $cl_Y(U') \cap cl_Y(V') = \emptyset$. Hence, said $U = f^{-1}(U')$ and $V = f^{-1}(V')$, by continuity of f, it is clear that $x_1 \in U$, $x_2 \in V$ and $cl_X(U) \cap cl_X(V) = \emptyset$. This proves that $g \circ f : X \to Z$ has the property $\mathcal{U}$. □

Definition 2.18. Let $F \in C(K,Y)$ and $G \in C(L,Y)$ be two Tychonoff compactifications of $f \in C(X,Y)$. We say that

- F is *projectively larger than* G and we write that $F \geq_f G$ if there is some compact Tychonoff function $h : K \to L$ such that $G \circ h = F$ and $h|_X = id_X$.
- F and G are *equivalent* and we write $F \equiv_f G$ if there is a homeomorphism $h : K \to L$ such that $G \circ h = F$ and $h|_X = id_X$.

Proposition 2.19. *Let* $F \in C(K,Y)$ *and* $G \in C(L,Y)$ *be two Tychonoff compactifications of* $f \in C(X,Y)$. *Then* $F \equiv_f G$ *iff* $F \geq_f G$ *and* $G \geq_f F$.

Proof. ($\Longrightarrow$) It is manifest.

($\Longleftarrow$) Suppose $F \geq_f G$ and $G \geq_f F$. Then there are two compact Tychonoff functions $h : K \to L$ and $k : L \to K$ such that $G \circ h = F$, $F \circ k = G$ and $h|_X = k|_X = id_X$. Hence, $k \circ h : K \to K$ is compact. By 2.14, h is onto, while by 2.16 and 2.17, $k \circ h$ has the property $\mathcal{U}$. So, it suffices to show that $k \circ h$ is 1-1. If not, there are $y, z \in K$ such that $y \neq z$ and $(k \circ h)(y) = (k \circ h)(z)$. Since $k \circ h$ has the property $\mathcal{U}$, there are $U, V \in \tau(K)$ such that $y \in U$, $z \in V$ and $cl_K(U) \cap cl_K(V) = \emptyset$.
Hence, $y \in cl_K(U) = cl_K(U \cap X)$ implies

$$\begin{aligned}(k \circ h)(y) &\in (k \circ h)\left(cl_K(U \cap X)\right)\\ &\subseteq cl_K\left((k \circ h)(U \cap X)\right)\\ &\subseteq cl_K(U \cap X)\\ &\subseteq cl_K(U).\end{aligned}$$

Likewise, $(k \circ h)(z) \in cl_K(V)$. Since $cl_K(U) \cap cl_K(V) = \emptyset$, it follows that $(k \circ h)(y) \neq (k \circ h)(z)$. A contradiction. □

Definition 2.20. If $f \in C(X,Y)$ is a Tychonoff function, $K(X,f)$ will denote the set of all the Tychonoff compactifications of f which belong to different equivalence classes (respect to $\equiv_f$).

It can easily be seen that $\geq_f$ and $\equiv_f$ are respectively an order relation and an equivalence relation on the set $K(X,f)$.

Finally we prove the following main result.

Theorem 2.21. *$(K(X,f), \geq_f)$ is a complete upper senilattice and its maximum is βf.*

Proof. Let $\{F_\alpha\}_{\alpha \in \Lambda} \subseteq K(X,f)$ be a family of Tychonoff compactifications $F_\alpha \in C(Z_\alpha, Y)$ of the Tychonoff function $f \in C(X,Y)$.
For each $\alpha \in \Lambda$, let $C_\alpha = \{g \in C^*(X) : \exists G \in C^*(Z_\alpha) \quad G|_X = g\}$ and $e_\alpha : X \to \prod_{g \in C_\alpha} I_g$ defined by $\pi_g \circ e_\alpha = g$ for every $g \in C_\alpha$.

Let us consider the space $P = \left(\prod_{\alpha \in \Lambda} \left(\prod_{g \in C_\alpha} I_g\right)\right) \times Y$ and the function $\Phi : X \to P$ defined by $\Phi(x) = (\langle e_\alpha(x) \rangle_{\alpha \in \Lambda}, f(x))$ for each $x \in X$.
The same tecnique used in the proof of Lemma 2.4 shows that $\Phi : X \to P$ is an embedding.

Now, let $Z = cl_P\left(\Phi(X)\right)$ and, if $\widetilde{\pi}_Y : P \to Y$ is the projection of P onto Y, we may consider the function $F = \widetilde{\pi}_Y|_Z \in C(Z,Y)$.

LIkewise in the proof of Proposition 2.7, we can prove that F is a Tychonoff compactification of f, i.e. that $F \in K(X,f)$.

We claim that $\sup\{F_\alpha : \alpha \in \Lambda\} = F$.
In fact, if for every $\alpha \in \Lambda$, $\pi_\alpha : P \to \prod_{g \in C_\alpha} I_g$ denote the α–th projection of

$P = \left(\prod_{\alpha\in\Lambda}\left(\prod_{g\in C_\alpha} I_g\right)\right) \times Y$ onto $\prod_{g\in C_\alpha} I_g$, using similar reasonings of the proof of 2.13, it is possible to prove that the function $\varphi_\alpha = (\pi_\alpha \times id_Y)|_Z \in C(Z, Z_\alpha)$ is compact and Tychonoff and that it results $F \circ \varphi_\alpha = F$ and $\varphi_\alpha|_X = id_X$, i.e. that $F \geq_f F_\alpha$ for every $\alpha \in \Lambda$.
Now, we suppose that there is some some Tychonoff compactification $H \in C(W, Y)$ of f such that $H \geq F_\alpha$ for each $\alpha \in \Lambda$, i.e. that there are $h_\alpha \in C(W, Z_\alpha)$ compact and Tychonoff such that $F_\alpha \circ h_\alpha = H$ and $h_\alpha|_X = id_X$. Let $C_W = \{g \in C^*(X) : \exists G \in C^*(W) \quad G|_X = g\}$ and consider the function $\psi = (\prod_{\alpha\in\Lambda} h_\alpha^{**}) \times id_Y : \left(\prod_{g\in C_W} I_g\right) \to P$. Likewise in the proof of Proposition 2.13, we can see that $\psi(\Phi_W(W)) = \beta(Z, F) = Z$ and so that the function $h = \psi|_W \in C(W, Z)$ is compact Tychonoff and that it results $F \circ h = H$ and $h|_X = id_X$, i.e. that $H \geq_f F$.
Thus, it is proved that the poset $(K(X, f), \geq_f)$ is a complete upper semilattice.

From Proposition 2.13 and Definition 2.18, it is clear that for each $F \in K(X, f)$ it results $\beta f \geq_f F$, i.e. that the Stone–Čech compactification βf of the function f is the maximum of $(K(X, f), \geq_f)$.
This concludes the proof. □

Remark 2.22 The Stone–Čech compactification of a function is a generalization of the corresponding notion for spaces. In fact, if the space Y is a singleton, to say that a function $f : X \to Y$ is Tychonoff is equivalent to say that the space X is Tychonoff and, from Theorem 2.21 follows that the Stone–Čech compactification βX of X coincides with the domain $\beta(X, f)$ of the Stone–Čech compactification βf of the funtion f.

From Theorem 2.21 arises in a natural way the following

Question. Which property must have the function $f : X \to Y$ because $(K(X, f), \geq_f)$ be a complete lattice ?

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