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WHEN IS KURATOWSKI CONVERGENCE TOPOLOGICAL?

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Abstract. It is known that if a topological space X is locally compact then the Kuratowski convergence on the closed subsets of X is topological. We show that the converse is true, provided that X is quasi-sober.

We also show that the Kuratowski convergence is topological if and only if it is pretopological.

1. Introduction

Let X be a topological space. It is not difficult to show that if X is locally compact, in the sense that every point has a neighborhood base consisting of compact sets, then the upper Kuratowski convergence on the closed subsets of X agrees with the co-compact topology [4;p. 353] and therefore both the upper Kuratowski and the Kuratowski convergences are topological.

Conversely, suppose that the Kuratowski (or the upper Kuratowski) convergence is topological. The question is: must then X be locally compact?

One can easily see [12;Prop. 4] that the answer is yes, if we also assume that X is regular (not necessarily T_0). On the other hand, we will give an example (due to Hofmann and Lawson [6]) which shows that the answer is no in general.

On the contrary, some authors, begining with Choquet [2], believed that the answer was yes in general: for example, see the paper of Mrówka [9], who probably was not aware of the work Choquet did ten years before, and the book [8], where the theorem of Mrówka is reported, with more or less the same proof (which works only if the space is regular).

In 1970, Mrówka [10] showed that the question has a positive answer assuming T_2.

In this paper we show, inspired by the ideas of the book [5], that the answer is yes with the only assumption that X is quasi-sober.

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Moreover we show that the Kuratowski and upper Kuratowski convergences are topological whenever they are pretopological.

Most of the results presented here were obtained in the author Ph.D. thesis [11] (which was written in Italian and not published).

2. Convergences

The appoach to abstract convergences given here has been influenced by the systematic paper of Choquet [2], and also by the more recent article [3].

Let E be a set and denote by $\Phi(E)$ the set of all filters on E: a *convergence* on E is a mapping $\vartheta\colon E \longrightarrow 2^{\Phi(E)}$. Given a filter $\mathcal{F}$ on E and an element $x \in E$, we say that $\mathcal{F}$ is ϑ-convergent to x if $\mathcal{F} \in \vartheta(x)$.

If a convergence ϑ has been defined on E we say that a net $(a_j)_{j\in J}$ in E is ϑ-convergent to a point $x \in E$ if the filter generated by the family $\{\, a_j \mid j \geq k \,\}_{k\in J}$ belongs to $\vartheta(x)$.

We will denote by $\Xi(E)$ the set of all convergences on E.

Given a topology τ on E, we can define a convergence in the usual way. This (topological) convergence will be identified with τ, so that the set $\Theta(E)$ of all topologies on E is regarded as a subset of $\Xi(E)$.

Thus, for every $x \in E$, $\tau(x)$ is the set of all filters which τ-converge to x, and therefore $\bigcap \tau(x)$ coincides with the filter $\mathcal{N}_\tau(x)$ of all τ-neighborhoods of x.

We say that the convergence ϑ' is *coarser* than ϑ'' if $\vartheta'(x) \supset \vartheta''(x)$ for every $x \in E$. If both ϑ' and ϑ'' are topologies this definition agrees with the usual one.

This ordering makes $\Xi(E)$ a complete lattice (in fact a Boolean algebra): one can easily see that, given any subset Σ of $\Xi(E)$, the supremum [respectively, the infimum] of Σ is the convergence which maps every $x \in E$ into $\left(\bigcap_{\theta\in\Sigma}\theta(x)\right)$ [resp. $\left(\bigcup_{\theta\in\Sigma}\theta(x)\right)$].

Proposition 2.1. *Let E be a set. Then $\Theta(E)$ is closed under sups in $\Xi(E)$.*

Proof. Let Σ be a subset of $\Theta(E)$ and let τ be the topology generated by the union of Σ, that is the sup of Σ in Θ: we clearly have $\tau(x) \subset \bigcap_{\theta\in\Sigma}\theta(x)$ for every $x \in E$. We show the opposite inclusion.

Let $\mathcal{U}(x)$ be the filter generated by $\bigcup_{\theta\in\Sigma}\mathcal{N}_\theta(x)$. For every τ-neighborhood U of x, there exist $V_1, V_2, \ldots, V_n$, where V_i is τ_i-open and $\tau_i \in \Sigma$ for each $i = 1, 2, \ldots, n$, such that $x \in V_1 \cap V_2 \cap \cdots \cap V_n \subset U$; since V_i is a τ_i-neighborhood of x, we have $V_i \in \bigcup_{\theta\in\Sigma}\mathcal{N}_\theta(x)$ for each i: therefore $V_1 \cap V_2 \cap \cdots \cap V_n \in \mathcal{U}(x)$ so that $U \in \mathcal{U}(x)$, too. As U was arbitrary we conclude that $\mathcal{N}_\tau(x) \subset \mathcal{U}(x)$.

Now if $\mathcal{F} \in \bigcap_{\theta\in\Sigma} \theta(x)$ then $\mathcal{F} \supset \mathcal{N}_\theta(x)$ for every $\theta \in \Sigma$, thus $\mathcal{F} \supset \bigcup_{\theta\in\Sigma} \mathcal{N}_\theta(x)$. It follows that $\mathcal{F}$ contains $\mathcal{U}(x)$: hence it contains $\mathcal{N}_\tau(x)$ and we have $\mathcal{F} \in \tau(x)$, which completes the proof. □

On the other hand, $\Theta(E)$ is not closed under infs in $\Xi(E)$, not even finite infs, except for trivial cases.

Proposition 2.2. *Let E be a set. The following are equivalent:*
(1) *$\Theta(E)$ is closed under finite infs in $\Xi(E)$;*
(2) *$\Theta(E)$ is a sublattice of $\Xi(E)$;*
(3) *$\Theta(E)$ is a distributive lattice;*
(4) *E has at most two elements.*

Proof. From the previous proposition it follows that (1) implies (2) and, since $\Xi(E)$ is distributive we have (2) ⇒ (3); since it is easy to check that (4) implies (1), it remains to prove that (3) implies (4).

Suppose that a, b and c are distinct elements of E and consider the following topologies on E:

$$\begin{aligned}\tau_1 &= \{\varnothing, \{a\}, E\}\,,\\ \tau_2 &= \{\varnothing, \{a,b\}, E\}\,,\\ \tau_3 &= \{\varnothing, \{a,c\}, E\}\,;\end{aligned}$$

then

$$\tau_2 \vee \tau_3 = \{\varnothing, E\} \cup \{\,\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\,\}\,,$$

so that $\tau_1 \wedge (\tau_2 \vee \tau_3) = \tau_1$; on the other hand,

$$\tau_1 \wedge \tau_2 = \{\varnothing, E\} = \tau_1 \wedge \tau_3:,$$

therefore $\tau_1 \wedge (\tau_2 \vee \tau_3) \neq (\tau_1 \wedge \tau_2) \vee (\tau_1 \wedge \tau_3)$.

In view of Proposition 2.1, we can define, for every convergence ϑ, a topology $T\vartheta$ (called the *topologization* of ϑ) as the supremum of all topologies which are coarser than ϑ. Thus for a convergence ϑ to be *topological* (i.e. to coincide with a topology) it is necessary and sufficient that $T\vartheta = \vartheta$ or, equivalently, $T\vartheta \geq \vartheta$.

The topologization of a convergence can also be described by means of open sets.

Proposition 2.3. *Let ϑ be a convergence on E. A subset A of E is $T\vartheta$-open if and only if*

$$\forall a \in A \quad \bigcap \vartheta(a) \ni A. \tag{2.1}$$

Proof. It is easily seen that the subsets A of E satisfying (2.1) form the collection of open sets of a topology. Call this topology τ: we first show that $\tau \leq \vartheta$.

Let $x \in E$ and consider a filter $\mathcal{F}$ which ϑ-converges to x: from the definition of τ it follows that every τ-neighborhood of x is in the filter $\bigcap \vartheta(x)$ and hence in $\mathcal{F}$, so that $\mathcal{F}$ is τ-convergent to x. This proves the claim.

It remains to show that τ is finer than every topology $\tau' \leq \vartheta$. So, let $\tau' \in \Theta(E)$ be coarser than ϑ: for every τ'-open set A and every $a \in A$ we have $\tau'(a) \supset \vartheta(a)$, whence

$$A \in \mathcal{N}_{\tau'}(a) = \bigcap \tau'(a) \subset \bigcap \vartheta(a):,$$

therefore A satisfies (2.1), i.e. A is τ-open. As A was arbitrary, the proof is complete. □

Among all convergences, topologies can be characterized as follows.

Theorem 2.4. *A convergence ϑ on a set E is a topology if and only if it satisfies the following conditions:*

(1) *$\forall x \in E$, the principal filter generated by $\{x\}$ is ϑ-convergent to x;*
(2) *$\forall x \in E$, if $\mathcal{F} \in \vartheta(x)$ and $\mathcal{G}$ is a filter finer than $\mathcal{F}$, then $\mathcal{G} \in \vartheta(x)$;*
(3) *$\forall x \in E$, the family $\vartheta(x)$ is closed under intersections;*
(4) *$\forall x \in E$: $\forall U \in \bigcap \vartheta(x)$: $\exists V \in \bigcap \vartheta(x)$ such that $U \in \bigcap \vartheta(v)$ for each $v \in V$.*

Proof. If ϑ is a topology then all the conditions are clearly satisfied (to verify (4) take as V the ϑ-interior of U).

Conversely suppose that ϑ satisfy the four conditions: we have to prove that $T\vartheta \geq \vartheta$. To this end, take a point $x \in E$ and a filter $\mathcal{F}$ not ϑ-converging to x: we will show that $\mathcal{F}$ is not $T\vartheta$-convergent to x.

The first three conditions imply that a filter ϑ converges to x if and only if it is finer than (i.e. contains) the filter $\bigcap \vartheta(x)$: hence there exists $U \in \bigcap \vartheta(x)$ with $U \notin \mathcal{F}$. Denote by $\mathcal{V}(x, U)$ the collection of all sets of the form $U \cap V$, where $V \in \bigcap \vartheta(x)$ and $U \in \bigcap \vartheta(v)$ for each $v \in V$: it follows from (4) that $\mathcal{V}(x, U)$ is nonempty.

Let $A = \bigcup \mathcal{V}(x, U)$: we claim that satisfies (2.1) that is A is $T\vartheta$-open. Indeed, if $a \in A$, then $U \in \bigcap \vartheta(a)$; so, let $A' = A \cup W$, where $W \in \mathcal{V}(a, U)$: we have $W \subset U$ and $W \in \bigcap \vartheta(a)$ (since W is the intersection of U with another member of the filter $\bigcap \vartheta(a)$); thus $A' \subset U$ and $A' \in \bigcap \vartheta(a)$. Moreover $A' \in \bigcap \vartheta(x)$, as $A' \supset A$; now, since $U \in \bigcap \vartheta(z)$ for every $z \in A'$, it follows that $A' \in \mathcal{V}(x, U)$ and therefore $A' = A$: hence $A \in \bigcap \vartheta(a)$, as claimed.

Thus A is $T\vartheta$-open; also $A \in \bigcap \vartheta(x)$, whence $x \in A$ by (1). It follows that $\mathcal{F}$ cannot $T\vartheta$-converge to x: otherwise we should have $A \in \mathcal{F}$, which is impossible because $A \subset U$. □

We say that ϑ is a *pseudotopology* (or *Choquet convergence*) on E if it satisfies (1) and (2) in Theorem 2.4 and the condition:
(2.2)

$$\forall x \in E\colon\ \forall \mathcal{F} \in \Phi(E)\ [\forall \mathcal{F}' \supset \mathcal{F}\colon\ \exists \mathcal{F}'' \supset \mathcal{F}'\colon\ \mathcal{F}'' \in \vartheta(x)\ \Rightarrow\ \mathcal{F} \in \vartheta(x)].$$

Equivalently ϑ is a pseudotopology if, besides (1) and (2) in Theorem 2.4, it has the following property (where $\Upsilon(E)$ denotes the set of all ultrafilters on E):
(2.3)

$$\forall x \in E\colon\ \forall \mathcal{F} \in \Phi(E)\ [\forall \mathcal{U} \in \Upsilon(E)\ [\mathcal{U} \supset \mathcal{F} \Rightarrow\ \mathcal{U} \in \vartheta(x)]\ \Leftrightarrow\ \mathcal{F} \in \vartheta(x)].$$

Hence, a pseudotopology can be viewed as a mapping which associate to every $x \in E$ a set of ultrafilters.

We are going to see that the set of all pseudotopologies on E is closed under sups.

Lemma 2.5. *The properties* (1) *and* (2) *of Theorem* 2.4 *are stable under sups and infs.*

Proof. Consider a collection $\{\vartheta_j \mid j \in J\}$ of convergences on E, whose supremum and infimum are denoted by σ and η, repsectively, and let x be any point of E.

If the principal filter generated by $\{x\}$ belongs to $\vartheta_j(x)$ for every $j \in J$, then it clearly belongs to $\sigma(x)$ and to $\eta(x)$. Hence (1) of Theorem 2.4 is stable.

Now suppose that every ϑ_j satisfies (2) of Theorem 2.4. If the filter $\mathcal{F}$ belongs to $\sigma(x)$ [resp. to $\eta(x)$] and $\mathcal{F}' \supset \mathcal{F}$, then we have $\mathcal{F} \in \vartheta_j(x)$, hence $\mathcal{F}' \in \vartheta_j(x)$, for all [resp. for some] $j \in J$ and therefore $\mathcal{F}'$ also is in $\sigma(x)$ [resp. $\eta(x)$]. □

Proposition 2.6. *The supremum σ of a collection $\{\vartheta_j \mid j \in J\}$ of pseudotopologies on E is again a pseudotopology.*

Proof. In view of Lemma 2.5 we have only to prove that σ satisfies (2.2).

Let x be any point of E and let $\mathcal{F}$ be any filter on E. If for every $\mathcal{F}' \supset \mathcal{F}$ there exists $\mathcal{F}'' \supset \mathcal{F}'$ such that $\mathcal{F}'' \in \sigma(x)$, then $\mathcal{F}'' \in \vartheta_j(x)$, so that $\mathcal{F} \in \vartheta_j(x)$, for all $j \in J$: thus $\mathcal{F} \in \sigma(x)$. □

On the other hand, if E has at least three distinct points a, b and c, there exist even two topologies ϑ' and ϑ'' on E whose infimum in $\Xi(E)$ is not a pseudotopology. Indeed, let the open sets of ϑ' [resp. ϑ''] be $\varnothing$, E, $\{a, c\}$ [resp. $\{b, c\}$] and all singletons $\{x\}$ with $x \neq c$; let ϑ denote the infimum of ϑ' and ϑ'' as convergences, and let $\mathcal{F}$ be the principal filter generated by $\{a, b, c\}$: all the ultrafilters finer than $\mathcal{F}$ are ϑ-convergent to c, while $\mathcal{F}$ is not.

A convergence ϑ on E satisfying the first three conditions of Theorem 2.4 is called a *pretopology*. To define a pretopology ϑ on E one can assign to every point x of E a single filter $\mathcal{N}_\vartheta(x)$ (of "neighborhoods" of x): it turns out that $\mathcal{F} \in \vartheta(x)$ iff $\mathcal{N}_\vartheta(x) \subset \mathcal{F}$ and $\mathcal{N}_\vartheta(x) = \bigcap \vartheta(x)$.

It is easy to see that every pretopology is a pseudotopology.

Proposition 2.7. *The supremum of a collection of pretopologies is a pretopology.*

Proof. Let $\{ \vartheta_j \mid j \in J \}$ be a collection of pretopologies on E and denote its supremum by σ. In view of Lemma 2.5 we have only to show that σ satisfies (3) in Theorem 2.4.

Let $x \in E$; consider a collection Λ of filters σ-converging to x, and denote by $\mathcal{F}$ the intersection of Λ. For each $\mathcal{G} \in \Lambda$, as $\mathcal{G} \in \sigma(x)$, we have $\mathcal{G} \in \vartheta_j(x)$ for every $j \in J$. Therefore $\mathcal{F}$ also is ϑ_j-convergent to x for every $j \in J$, and hence σ-convergent. □

A similar result for infima does not hold in general (see the remark following Proposition 2.6).

We say that a convergence ϑ on E is *Hausdorff* (or *separated*) if $\vartheta(x') \cap \vartheta(x'') = \varnothing$ whenever x' and x'' are distinct. A convergence is *compact* if every ultrafilter converges to some point. When we are considering topologies these terms are obviously consistent with the usual ones.

Clearly every convergence finer than a Hausdorff convergence is Hausdorff and every convergence coarser than a compact convergence is compact.

Proposition 2.8. *Let ϑ' and ϑ'' be pseudotopologies on E, with $\vartheta' \leq \vartheta''$. If ϑ' is Hausdorff and ϑ'' is compact then $\vartheta' = \vartheta''$.*

Proof. Let $x \in E$ and consider a filter $\mathcal{F}$ in $\vartheta'(x)$. We have to prove that $\mathcal{F} \in \vartheta''(x)$ and, since ϑ'' is a pseudotopology, it suffices to show that every ultrafilter $\mathcal{U} \supset \mathcal{F}$ converges to x.

Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}$. By compactness we have $\mathcal{U} \in \vartheta''(z)$ for some $z \in E$, hence $\mathcal{U} \in \vartheta'(z)$, too, as ϑ' is coarser. Now $\mathcal{F} \in \vartheta'(x)$ implies that $\mathcal{U}$ also is ϑ'-convergent to x. Since ϑ' is Hausdorff, we conclude that $z = x$. □

From the above proof it follows that ϑ' and ϑ'' are not required to be pseudotopologies, *a priori*: it suffices for ϑ' to satisfy (2) of Theorem 2.4 and for ϑ'' to satisfy (2.2) (or (2.3)).

3. Lim-inf convergence and continuous lattices

This section and the next one are greatly indebted to [5], where some of the material has been taken from. Anyway, we have preferred to give as many details as possible, in order to make the exposition reasonably self-contained.

In the sequel we will denote by L a complete lattice. If U is a subset of L, the *upper set* of U is

$$\uparrow U = \{\, x \in L \mid \exists u \in U\colon\ u \le x \,\}.$$

A subset T of L is *upper* if $T = \uparrow U$ for some $U \subset L$ or, equivalently, if $\uparrow T = T$. Similarly one defines the *lower sets*.

Let L be a complete lattice and $\mathcal{F}$ be a filter on L. The *lim inf* of $\mathcal{F}$ is defined as

$$\liminf \mathcal{F} = \sup\{\, \inf F \mid F \in \mathcal{F} \,\}.$$

The *lim-inf convergence* on L, denoted by ξ, maps each $x \in L$ to to

$$\xi(x) = \{\, \mathcal{F} \in \Phi(L) \mid x \le \liminf \mathcal{F} \,\}.$$

Proposition 3.1. *The lim-inf convergence satisfies* (1) *and* (2) *of Theorem* 2.4.

Proof. For every $x \in L$, the lim-inf of the principal filter generated by $\{x\}$ is x, and this gives (1) of Theorem 2.4. To verify (2) from Theorem 2.4 it suffices to observe that, whenever $\mathcal{F}'$ and $\mathcal{F}''$ are filters on L with $\mathcal{F}' \subset \mathcal{F}''$, we have $\liminf \mathcal{F}' \le \liminf \mathcal{F}''$.

The topologization of ξ will be called the *Scott topology* of the complete lattice L, and the $T\xi$-open ($T\xi$-closed) sets are usually called *Scott-open* (resp. *Scott-closed*).

From Proposition 2.3 we can deduce a characterization of Scott-open sets.

Recall that subset D of a lattice L is *directed* if it is nonempty and for every $x, y \in D$ there exists $z \in D$ such that both $x \le z$ and $y \le z$.

Proposition 3.2. *A subset U of a complete lattice L is Scott-open if and only if*

(1) $\uparrow U = U$, *i.e. U is upper;*

(2) *for every directed subset D of L such that* $\sup D \in U$, *we have* $D \cap U \neq \varnothing$.

Proof. Suppose that U is Scott-open. For every $u \in U$ and every $v \geq u$, since the principal filter $\mathcal{P}_v$ generated by $\{v\}$ is ξ-convergent to u, we have $U \in \bigcap \xi(u) \subset \mathcal{P}_v$ so that $v \in U$: hence U satisfies (1). Now let D be a directed subset of L and suppose that $s = \sup D$ belongs to U; denote by $\mathcal{F}_D$ the filter generated by all sets of the form $\uparrow\{d\}$ with $d \in D$; since $\liminf \mathcal{F}_D = s$, we have $U \in \bigcap \xi(s) \subset \mathcal{F}_D$, hence U contains $\uparrow\{d\}$ for a suitable $d \in D$ and in particular $d \in U$: therefore U satisifies (2).

Conversely suppose that (1) and (2) hold, and take any $x \in U$: we will show that $U \in \bigcap \xi(x)$, i.e. $U \in \mathcal{F}$ for every $\mathcal{F} \in \xi(x)$. So, let $\mathcal{F}$ be a filter ξ-converging to x; we have $\liminf \mathcal{F} \in U$ by (1), and therefore, by (2), U intersects $\{\inf F \mid F \in \mathcal{F}\}$ because this is a directed set: as U is upper, it follows that $U \in \mathcal{F}$, which completes the proof. □

Observe that, since complements of upper set are lower and vice versa, it follows from the previous proposition that a subset C of a complete lattice L is Scott-closed if and only if it is a lower set which is closed under sups of directed subsets.

Now we are going to introduce an auxiliary relation, by means of which we will define the concept of continuous lattice.

Recall that an *ideal* of a lattice L is a subset I which is both lower and directed; equivalently I is the lower set of some nonempty sublattice of L.

Proposition 3.3.. *Let a, b be elements of the complete lattice L. The following are equivalent:*

(1) *for every $S \subset L$ with* $\sup S \geq b$ *there exists a finite set $F \subset S$ such that* $\sup F \geq a$;

(2) *every directed $D \subset L$ with* $\sup D \geq b$ *contains some $d \geq a$;*

(3) *every ideal I of L with* $\sup I \geq b$ *contains a.*

Proof. It suffices to to show that (3) implies (1), and we may also assume that b is not the smallest element of L. Let $S \subset L$ such that $\sup S \geq b$ (note that S is nonempty), and denote by I the lower set of the sublattice generated by S; as I is an ideal and $\sup I \geq b$, we have $a \in I$ by (3), and therefore $a \leq c$ for a suitable c of the form $c_1 \vee c_2 \vee \ldots \vee c_n$, with $c_i \in S$: thus we get (1), where $F = \{c_1, c_2, \ldots, c_n\}$. □

If a and b are elements of a complete lattice L satisfying one of the equivalent conditions above, we say that a is *way below* b, and write $a \ll b$.

Proposition 3.4. *Let L be a complete lattice and denote by 0 the smallest element of L. The way-below relation on L satisfies the following properties:*

(1) $\forall a, b \in L\colon\ [a \ll b \ \Rightarrow\ a \leq b]$;
(2) $\forall a \in L\ \ 0 \ll a$;
(3) $\forall a, b, c, d \in L\colon\ [a \leq b,\ b \ll c$ *and* $c \leq d \ \Rightarrow\ a \ll d]$;
(4) $\forall a, b, c \in L\colon\ [a \ll c$ *and* $b \ll c \ \Rightarrow\ a \vee b \ll c]$.

Proof. We prove the fourth statement. Let a, b and c be such that $a \ll c$ and $b \ll c$: for every ideal I such that $\sup I \geq c$, we have both $a \in I$ and $b \in I$: hence $a \vee b \in I$.

The other properties can be proved in a similar way. ☐

Denote by $\Uparrow x$ [respectively $\Downarrow x$] the set of all $u \in L$ such that $x \ll u$ [resp. $u \ll x$]: if $\sup \Downarrow x = x$ for every $x \in L$ we say that L is a *continuous lattice.*

On a continuous lattice the Scott topology has some additional properties.

Lemma 3.5. *Let L be a continuous lattice. Then*

$$\forall x, z \in L\colon\ [x \ll z \ \Rightarrow\ \exists y \in L\colon\ x \ll y \ \text{and}\ y \ll z].$$

Proof. Given $x, z \in L$ with $x \ll z$, let $I = \{v \in L \mid \exists y \in L\colon\ v \ll y$ and $y \ll z\}$: then I is an ideal of L, as one can easily check by applying the properties listed in Proposition 3.4. We have to prove that $x \in I$ and, by Proposition 3.3, it suffices to show that $\sup I \geq z$.

Suppose not: as L is a continuous lattice, there should exist $t \in L$ such that $t \ll z$ and $\sup I \not\geq t$; by continuity again, since $\sup \Downarrow t = t$ there should be $u \ll t$ with $\sup I \not\geq u$; but now $u \in I$ and we get a contradiction. ☐

Proposition 3.6. *If L is a continuous lattice then, for every $x \in L$, the subsets of the form $\Uparrow u$, where $u \ll x$, form a base of Scott-open neighborhoods at x.*

Proof. Let U be a Scott-open neighborhood of x; since $\sup \Downarrow x = x$ and $\Downarrow x$ is directed, by Proposition 3.4, it follows from Proposition 3.2 that there exists some $u \in \Downarrow x \cap U$ and therefore

$$x \in \Uparrow u \subset \uparrow\{u\} \subset \uparrow U = U.$$

It remains to prove that $\Uparrow u$ is Scott-open.

First observe that $\Uparrow u$ is an upper set, by (3) of Proposition 3.4. Now let D be a directed subset of L with $\sup D = s \in \Uparrow u$, i.e. $u \ll s$: by Lemma 3.5 there exists $t \in L$ such that $u \ll t$ and $t \ll s$; it follows from (1) in Proposition 3.4 and Proposition 3.3 that D contains a $d \geq t$: consequently we have $u \ll d$ (by (3) in Proposition 3.4 again) so that $D \cap \Uparrow u \neq \varnothing$ and the conclusion follows from Proposition 3.2. $\square$

Continuous lattices are precisely the ones on which lim-inf convergence and Scott topology coincide.

Theorem 3.7. *Let L be a complete lattice. The following are equivalent:*
(1) *lim-inf convergence on L is topological;*
(2) *lim-inf convergence on L is a pretopology;*
(3) *L is a continuous lattice.*

Proof. Since (1)⇒(2) is trivial, we first prove that (2) implies (3).

For every $a \in L$ let $\mathcal{J}(a)$ be the set of all ideals J of L with $\sup J \geq a$. Since $\Downarrow a = \bigcap \mathcal{J}(a)$, we have to show that $\bigcap \mathcal{J}(a) \in \mathcal{J}(a)$.

We associate to every ideal I of L the filter $\mathcal{F}_I$ on L generated by all sets of the form $\uparrow\{i\}$ with $i \in I$. Then a filter $\mathcal{G}$ on L is ξ-convergent to an element a of L if and only if there exists $J \in \mathcal{J}(a)$ such that $\mathcal{F}_J \subset \mathcal{G}$: indeed, letting $J = \{\inf G \mid G \in \mathcal{G}\}$ defines an ideal such that $\mathcal{F}_J \subset \mathcal{G}$ and, since $\sup J = \liminf \mathcal{G} \geq a$, we have $J \in \mathcal{J}(a)$; conversely, if $J \in \mathcal{J}(a)$ and $\mathcal{F}_J \subset \mathcal{G}$, then

$$\begin{aligned}\liminf \mathcal{G} &= \sup\{\inf G \mid G \in \mathcal{G}\} \geq \sup\{\inf F \mid F \in \mathcal{F}_J\} \\ &\geq \sup\{\inf \uparrow\{j\} \mid j \in J\} = \sup J \geq a,\end{aligned}$$

so that $\mathcal{G} \in \xi(a)$.

Now, as ξ is a pretopology, we have $\bigcap \xi(a) \in \xi(a)$ and hence there exists $H \in \mathcal{J}(a)$ such that $\mathcal{F}_H \subset \bigcap \xi(a)$; on the other hand we must have $\mathcal{F}_H \in \xi(a)$ and therefore $\mathcal{F}_H = \bigcap \xi(a)$. Since $\mathcal{F}_I \subset \mathcal{F}_J$ if and only if $I \subset J$ we have $H \subset J$ for every $J \in \mathcal{J}(a)$, that is $H \subset \bigcap \mathcal{J}(a)$, and it follows that $\bigcap \mathcal{J}(a) = H \in \mathcal{J}(a)$.

We conclude with the proof of the implication (3)⇒(1). To this end it suffices to show that $\xi \leq T\xi$.

So, let x be any point of L and let $\mathcal{F}$ be a filter $T\xi$-converging to x. If $u \ll x$ then $\Uparrow u$ is a $T\xi$-neighborhood of x, by Proposition 3.6; hence $\Uparrow u \in \mathcal{F}$ so that $\uparrow\{u\} \in \mathcal{F}$, too. Therefore

$$\liminf \mathcal{F} = \sup\{\inf F \mid F \in \mathcal{F}\} \geq \inf \uparrow\{u\} = u;$$

as u is arbitrary and L is continuous, we have $\liminf \mathcal{F} \geq x$. $\square$

4. The lattice of open sets

In this section we focus our attention on the (complete) lattice $\mathcal{A}(X)$ of all open subsets of a given topological space X ordered by inclusion. In the sequel we will consider also the lattice $\mathcal{C}(X)$ of all closed subsets of X ordered by reverse inclusion. These lattices are isomorphic via complementation.

Our aim is to find necessary and sufficient conditions for $\mathcal{A}(X)$ to be continuous. We begin with the way-below relation: first observe that $U \ll V$ in $\mathcal{A}(X)$ means that every open cover of V contains a finite open cover of U.

Proposition 4.1. *Let U and V be open subsets of a topological space X. Consider the following conditions:*

(1) *$U \ll V$ in $\mathcal{A}(X)$;*

(2) *every filter $\mathcal{F}$ on X, with $U \in \mathcal{F}$, has a cluster point in V;*

(3) *there exists a compact set $Q \subset X$ such that $U \subset Q \subset V$.*

Then the third one implies the first two, which are equivalent; moreover, if X is locally compact, all three conditions are equivalent.

Our definition of *locally compact* (not necessarily Hausdorff) topological space is: Each point has a neighborhood base consisting of compact sets.

Proof. (1) ⇒ (2): Let $\mathcal{F}$ be a filter with $U \in \mathcal{F}$ and suppose that $\mathcal{F}$ has no cluster point in V; then, for every $x \in V$, there exist an open set $W(x)$ containing x and a member $F(x)$ of $\mathcal{F}$ such that $W(x) \cap F(x) = \varnothing$. Since the collection $\{ W(x) \mid x \in V \}$ is an open cover of V, we can find x_1, x_2, ..., x_n in V such that $\bigcup_{i=1}^{n} W(x_i) \supset U$; now letting $F = \bigcap_{i=1}^{n} F(x_i)$ we have $F \in \mathcal{F}$ and $F \cap U = \varnothing$, in contrast with the assumption that $U \in \mathcal{F}$.

(2) ⇒ (1): Let $\mathcal{V}$ be an open cover of V and suppose that no finite subcollection of $\mathcal{V}$ covers U. The sets of the form $U - W$ with $W \in \mathcal{V}$ generate a filter $\mathcal{F}$, and we have $U \in \mathcal{F}$. Hence there exists some $x \in V$ which is a cluster point of $\mathcal{F}$. Thus, if $W(x)$ is any member of $\mathcal{V}$ containing x then $W(x)$ meets every $F \in \mathcal{F}$, but this is impossible because the complement of $W(x)$ is in $\mathcal{F}$.

(3) ⇒ (1): Let $\mathcal{V}$ be an open cover of V; then it is an open cover of Q, so there exists a finite subcollection $\mathcal{U}$ of $\mathcal{V}$ which covers Q and hence U.

Finally, let X be a locally compact topological space.

(1) ⇒ (3): For each $v \in V$, denote by $Q(v)$ a compact neighborhood of v contained in V and let $W(v)$ be the interior of $Q(v)$. The collection $\mathcal{W} = \{ W(v) \mid v \in V \}$ is an open cover of V and hence there exist v_1, v_2, ..., v_n in V such that $\bigcup_{i=1}^{n} W(v_i) \supset U$. Letting $Q = \bigcup_{i=1}^{n} Q(v_i)$ defines a compact set such that $U \subset Q \subset V$. □

As a consequence we can readily establish a sufficient condition for the lattice of open sets to be continuous.

Corollary 4.2. *If X is a locally compact topological space, then $\mathcal{A}(X)$ is a continuous lattice.*

Proof. Let V be an open subset of X and, for each $x \in V$, let $Q(x)$ be a compact neighborhood of x contained in V. Denoting by $U(x)$ the interior of $Q(x)$, we have $U(x) \ll V$ for every $x \in V$ and $\bigcup_{x\in V} U(x) = V$. □

The converse does not hold, in general, as we are going to see.

Example 4.3. Consider the set $E = [0,1] \times [0,1[$ endowed with the topology whose open sets have the form $A_f = \{(x,y) \mid y < f(x)\}$, where f runs through the lower semicontinuous functions of $[0,1]$ into itself. Let D be a (dense) subset of $[0,1]$ which intersects every nonempty open subinterval of $[0,1]$ in a non-Borel set. Define X as the set of all $(x,y) \in E$ such that either $x \in D$ and y is rational or $x \notin D$ and y is irrational, with the topology inherited as a subspace of E. Then X is a T_0-space in which every compact set has empty interior, yet $\mathcal{A}(X)$ is a continuous lattice (see [6;Sect. 7]).

Nevertheless, it is easy to establish the following result.

Proposition 4.4. *Let X be a regular topological space. If $\mathcal{A}(X)$ is a continuous lattice then X is locally compact.*

Proof. Let $x \in X$ and let V be an open neighborhood of x. Since $\mathcal{A}(X)$ is continuous there exists $U \ll V$ such that $x \in U$ and, by regularity, we can find another open neighborhood W of x whose closure is contained in U. Now consider an open cover $\mathcal{V}$ of $\overline{W}$; the open cover $\mathcal{V} \cup \{V - \overline{W}\}$ of V contains a finite subcollection $\mathcal{U}$ which covers U and therefore $\mathcal{U} - \{V - \overline{W}\}$ is a finite open cover of $\overline{W}$ contained in $\mathcal{V}$. Hence $\overline{W}$ is compact.

As x and V were arbitrary, we conclude that X is locally compact. □

Example 4.3 shows that in the above propostion we cannot drop the assumption of regularity. But such assumption can be weakened, and this is what the remainder of this section is concerned in.

Let L be a complete lattice, whose greatest element is denoted by 1. We say that $p \in L$ is *prime* if $p \neq 1$ and, for every $x, y \in L$ with $x \wedge y \leq p$, we have either $x \leq p$ or $y \leq p$.

The *spectrum* of L, denoted by Spec$L$, is the topological space whose points are the prime elements of L and whose open sets have the form

$$\Omega(a) = \{p \in L \mid p \text{ is prime and } p \not\geq a\},$$

where a runs through the elements of L. It is easy to check that they just are the open sets of a topology: indeed $\bigcup_{s\in S} \Omega(s) = \Omega(\sup S)$ for every $S \subset L$ and $\Omega(x) \cap \Omega(y) = \Omega(x \wedge y)$ for every $x, y \in L$.

Lemma 4.5. *Let G be an open subset of SpecL. Then*

$$G = \Omega\,(\inf\,((\mathit{SpecL}) - G)),$$

where the inf is computed in L.

Proof. Let $G' = \Omega\,(\inf((\mathrm{Spec}L) - G))$. If $p \in G'$ then p is prime and $p \not\geq \inf((\mathrm{Spec}L) - G)$, so that $p \in G$.

Conversely, let $x \in L$ such that $G = \Omega(x)$. For every $q \in G$ we have $q \not\geq x$; on the other hand,

$$x \leq \inf\{\, p \in \mathrm{Spec}L \mid p \geq x \,\} = \inf\,((\mathrm{Spec}L) - \Omega(x)) = \inf\,((\mathrm{Spec}L) - G)$$

and therefore $q \not\geq \inf\,((\mathrm{Spec}L) - G)$, i.e. $q \in G'$. □

Now we can establish a compactness criterion for subspaces of Spec$L$.

Proposition 4.6. *Let L be a complete lattice and Q a subspace of SpecL. If the set $\downarrow Q = \{\, x \in L \mid \exists q \in Q\colon\ x \leq q \,\}$ is Scott-closed in L, then Q is compact.*

Proof. Let $\mathcal{G}$ be a collection of open subsets of Spec$L$ whose union covers Q; we may assume that $\mathcal{G}$ is closed under finite unions, thus we have to show that Q is contained in some member of $\mathcal{G}$.

Suppose on the contrary that, for every $G \in \mathcal{G}$, we have $Q - G \neq \varnothing$ and hence $\inf\,((\mathrm{Spec}L) - G) \in \downarrow Q$. Then the set

$$D = \{\, \inf\,((\mathrm{Spec}L) - G) \mid G \in \mathcal{G} \,\}$$

is contained in $\downarrow Q$; moreover the above lemma implies that

$$\bigcup \mathcal{G} = \bigcup \{\, \Omega\,(\inf\,((\mathrm{Spec}L) - G)) \mid G \in \mathcal{G} \,\} = \bigcup_{d \in D} \Omega(d) = \Omega(\sup D);$$

finally D is a directed set: indeed if $G', G'' \in \mathcal{G}$ then $G' \cup G'' \in \mathcal{G}$ and

$$\inf\,((\mathrm{Spec}L) - G') \vee \inf\,((\mathrm{Spec}L) - G'') \leq \inf\,((\mathrm{Spec}L) - (G' \cup G''))\,.$$

As $\downarrow Q$ is closed under sups of directed sets (see the remark following Proposition 3.2), we have $\sup D \in \downarrow Q$ i.e. there exists $q \in Q$ such that $\sup D \leq q$; it follows that $q \notin \Omega(\sup D) = \bigcup \mathcal{G}$, which is impossible. □

Now we are going to show that Spec$L$ is a locally compact topological space, provided that L is a distributive continuous lattice.

Recall that a *dual ideal* of L is a nonempty upper set $F \subset L$ such that for every $x, y \in F$ there exists $z \in F$ with $z \leq x$ and $z \leq y$ (i.e. F is an ideal in the reversed ordering).

Lemma 4.7. *Let K be a Scott-closed subset of a complete distributive lattice L. If $L - K$ is a dual ideal then $\downarrow((SpecL) \cap K) = K$.*

Proof. Take any $k \in K$; since K is closed under sups of directed subsets, we may apply Zorn's lemma and find a maximal $m \in K$ such that $m \geq k$. Observe that $m \neq 1$ because $1 \notin K$: otherwise $K = L$ and $L - K = \varnothing$ could not be a dual ideal. We complete the proof by showing that m is prime in L.

Indeed, if m were not prime we would have $x \wedge y \leq m$ for suitable $x, y \in L$ with $x \not\leq m$ and $y \not\leq m$, so that $x \vee m > m$ and $y \vee m > m$; hence $x \vee m$ and $y \vee m$ would belong to $L - K$, because m is maximal. Since $L - K$ is a dual ideal it follows that $(x \vee m) \wedge (y \vee m) \in L - K$, too, but applying distributivity we get $(x \vee m) \wedge (y \vee m) = (x \wedge y) \vee m = m$ so that $m \in L - K$, a contradiction. □

Proposition 4.8. *Let L be a distributive continuous lattice. Then SpecL is a locally compact topological space.*

Proof. Consider a point q in Spec$L$ and an open neighborhood U of q; let $a \in L$ be such that $U = \Omega(a)$, so that $a \not\leq q$; as L is continuous, there exists $b \ll a$ with $b \not\leq q$, so that $q \in \Omega(b)$.

By Proposition 3.5, we can construct inductively a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_1 = a$ and $b \ll x_n \ll x_{n-1}$ for every $n > 1$. Let $F = \bigcup_{n=1}^{\infty} \uparrow\{x_n\}$; observe that F is a dual ideal: it is clearly an upper set and, given $y', y'' \in F$, there exists a positive integer k such that $y' \wedge y'' \geq x_k \in F$.

Denote by K the complement of F. We claim that K is Scott-closed: indeed let D be a directed subset of K and let $s = \sup D$; if $s \notin K$ then $s \in \uparrow\{x_n\}$ for some $n \in \mathbb{N}$; consequently $x_{n+1} \ll s$ and hence $x_{n+1} \leq d$ for a suitable $d \in D$; but this means that $d \in \uparrow\{x_{n+1}\} \subset F$, which is impossible.

Now if we put $Q = (\text{Spec}L) \cap K$, the above lemma implies that $\downarrow Q = K$; thus Q is a compact subspace of Spec$L$, by Proposition 4.6.

Given $p \in \Omega(b)$, since $p \not\geq b$, we have $p \not\geq x_n$ for each $n \in \mathbb{N}$ so that $p \in K$; hence $\Omega(b) \subset Q$. On the other hand if $p \in Q$ then p is prime and $p \not\geq a = x_1$; hence $Q \subset \Omega(a) = U$. Therefore Q is a compact neighborhood of q contained in U. As q and U were arbitrary, the proof is complete. □

Let C be a closed subset of a topological space X. We say that C is *irreducible* if $C \neq \varnothing$ and whenever A and B are closed sets with $A \cup B = C$ we have either $A = C$ or $B = C$. The space X is *quasi-sober* [7;Def. 2.1] if every irreducible closed subset of X is the closure of a singleton.

One easily sees that regular (not necessarily T_0) spaces are quasi-sober and that Hausdorff spaces are *sober*, i.e. quasi-sober and T_0.

Proposition 4.9. *Let X be a quasi-sober space. The mapping*

$$\varphi\colon X \to \mathit{Spec}\mathcal{A}(X), \quad x \mapsto X - \overline{\{x\}}$$

is continuous open and onto; moreover $\varphi^{-1}(\varphi(U)) = U$ for every open set $U \subset X$. If X is sober then φ is a homeomorphism.

Proof. One immediately sees that a closed set $C \subset X$ is irreducible if and only if the complement is a prime element of the lattice $\mathcal{A}(X)$. Since closures of singletons are always irreducible, the mapping φ is well defined. Moreover φ is onto, as X is quasi-sober.

Now let U be an open subset of X. We have

$$\begin{aligned}\varphi(U) &= \{\, \varphi(x) \mid x \in U \,\} = \left\{\, X - \overline{\{x\}} \mid X - \overline{\{x\}} \not\supset U \,\right\} \\ &= \{\, P \in \operatorname{Spec}\mathcal{A}(X) \mid P \not\supset U \,\} = \Omega(U)\end{aligned}$$

thus $\varphi(U)$ is open in $\operatorname{Spec}\mathcal{A}(X)$; conversely every open subset $\mathcal{V}$ of $\mathcal{A}(X)$ has the form $\Omega(U)$ for some $U \in \mathcal{A}(X)$, so that

$$\varphi^{-1}(\mathcal{V}) = \varphi^{-1}(\Omega(U)) = \varphi^{-1}(\varphi(U)) = \{\, x \in X \mid \exists u \in U\colon\ \overline{\{x\}} = \overline{\{u\}} \,\} = U.$$

Hence φ is continuous and open, and $\varphi^{-1}(\varphi(U)) = U$ for every open subset U of X.

Finally, if X is also a T_0-space then φ is clearly one-to-one and therefore it is a homeomorphism. □

We are now able to obtain the desired generalization of Proposition 4.4.

Theorem 4.10. *Let X be a quasi-sober space. The lattice $\mathcal{A}(X)$ is continuous if and only if X is locally compact.*

Proof. In view of Proposition 4.2 only necessity has to be proved. So suppose that $\mathcal{A}(X)$ is a continuous lattice; local compactess of $\operatorname{Spec}\mathcal{A}(X)$ follows from Proposition 4.8. Now, given $x \in X$ and an open neighborhood U of x, we have to find a compact neighborhood of x contained in U; to this end consider the mapping φ defined in the above proposition: $\varphi(U)$ is an open neighborhood of $\varphi(x)$ and therefore it contains a compact neighborhood $\mathcal{Q}$; the set $Q = \varphi^{-1}(\mathcal{Q})$ is a neighborhood of x by continuity, and we also have $Q \subset \varphi^{-1}(\varphi(U)) = U$.

It remains to show that Q is compact. If $\mathcal{G}$ be an open cover of Q then $\{\, \varphi(G) \mid G \in \mathcal{G} \,\}$ is an open cover of $\mathcal{Q}$, thus there exist $G_1, G_2, \ldots, G_n \in \mathcal{G}$ such that $\bigcup_{i=1}^{n} \varphi(G_i) \supset \mathcal{Q}$ and we have

$$Q \subset \varphi^{-1}\left(\bigcup_{i=1}^{n} \varphi(G_i)\right) = \bigcup_{i=1}^{n} \varphi^{-1}\varphi(G_i) = \bigcup_{i=1}^{n} G_i$$

which completes the proof. □

5. Kuratowski convergence

Let X be a topological space. We define the *upper Kuratowski convergence*, denoted by K^+, as the lim-inf convergence on the lattice $\mathcal{C}(X)$. If, for a subcollection Σ of $\mathcal{C}(X)$, we put $\mathrm{Ls}\Sigma = \bigcap_{\mathcal{S}\in\Sigma} \overline{\bigcup \mathcal{S}}$, then a necessary and sufficient condition for a filter Γ in $\mathcal{C}(X)$ to K^+-converge to $C \in \mathcal{C}(X)$ is that $\mathrm{Ls}\Gamma \subset C$.

For every open set $A \subset X$, consider the collection A^- of all $C \in \mathcal{C}(X)$ such that the intersection $C \cap A$ is nonempty: the topology V^- on $\mathcal{C}(X)$, having $\{ A^- \mid A \text{ is an open subset of } X \}$ as a subbase, is called *lower Vietoris topology.*

Finally, the *Kuratowski convergence* is, by definition, $K = K^+ \vee V^-$.

The reader can easily verify that the usual presentation of K^+ and K in terms of nets agrees with our definition.

It is useful to characterize also lower Vietoris topology in terms of filters. To this end recall that the *grill* [1] of a filter Γ is the collection Γ^* of all sets which intersect every member of Γ.

Proposition 5.1. *A filter Γ on $\mathcal{C}(X)$ is V^--convergent to $C \in \mathcal{C}(X)$ if and only if $C \subset Ls\Gamma^*$.*

Proof. Suppose that $C \subset \mathrm{Ls}\Gamma^*$: we have to show that every V^--neighborhood $\mathcal{U}$ of C is a member of the filter Γ; we may assume that C is nonempty and that $\mathcal{U} = A^-$, where A is an open set which meets C. Let $\mathcal{H}$ be the complement of $\mathcal{U}$ with respect to $\mathcal{C}(X)$ and let x be a point in $C \cap A$, which must exist by our assumptions: since A is a neighborhood of x disjoint from every member of $\mathcal{H}$, we have $x \notin \overline{\bigcup \mathcal{H}}$, so that $C \not\subset \overline{\bigcup \mathcal{H}}$. It follows that $\mathcal{H} \cap \mathcal{G} = \varnothing$ for some $\mathcal{G} \in \Gamma$: therefore $\mathcal{G} \subset \mathcal{U}$, whence $\mathcal{U} \in \Gamma$.

Conversely suppose that the filter Γ is V^--convergent to $C \in \mathcal{C}(X)$ and let $\mathcal{H} \in \Gamma^*$: we will show that $C \subset \overline{\bigcup \mathcal{H}}$. Take a point $x \in C$ (if C is empty there is nothing to prove) and an open neighborhood A of x. As A^- is a V^--neighborhood of C, we have $A^- \in \Gamma$, hence $\mathcal{H}$ and A^- have some members in common: therefore $\bigcup \mathcal{H}$ intersects A, so that $x \in \overline{\bigcup \mathcal{H}}$ and it follows that $C \subset \overline{\bigcup \mathcal{H}}$, as required. □

As a consequence we can characterize K-convergence.

Proposition 5.2. *A filter Γ on $\mathcal{C}(X)$ is K-convergent to $C \in \mathcal{C}(X)$ if and only if $Ls\Gamma = Ls\Gamma^* = C$.*

Proof. Since $\Gamma \subset \Gamma^*$, we have $\mathrm{Ls}\Gamma^* \subset \mathrm{Ls}\Gamma$, and the conclusion follows from the previous proposition. □

Corollary 5.3. *An ultrafilter Ψ on $\mathcal{C}(X)$ is K-convergent to $C \in \mathcal{C}(X)$ if and only if $C = Ls\Psi$.*

Proof. It suffices to observe that, since Ψ is an ultrafilter, we have $\Psi^* = \Psi$. □

In view of the preceding corollary one may ask if K is a pseudotopology. The answer is yes, as shown by Choquet [2], but even more is true.

Proposition 5.4. *The upper Kuratowski convergence is a pseudotopology.*

Proof. By Proposition 3.1 it suffices to show that, given a filter Γ on $\mathcal{C}(X)$ and a $C \in \mathcal{C}(X)$ with $\mathrm{Ls}\Gamma \not\subset C$, there exists an ultrafilter which refines Γ and is not K^+-convergent to C.

Let x be a point in $\mathrm{Ls}\Gamma$ and not in C. Given any $\mathcal{G} \in \Gamma$ and any open neighborhood V of x, denote by $\mathcal{I}(\mathcal{G}, V)$ the collection of all $G \in \mathcal{G}$ such that $V \cap G \neq \varnothing$: as $x \in \mathrm{Ls}\Gamma$, this collection is always nonempty; moreover $\mathcal{I}(\mathcal{G}', V') \cap \mathcal{I}(\mathcal{G}'', V'') \supset \mathcal{I}(\mathcal{G}' \cap \mathcal{G}'', V' \cap V'')$ so that the set of all $\mathcal{I}(\mathcal{G}, V)$, where $\mathcal{G}$ is a member of Γ and V is an open set containing x, generates a filter Γ_x.

Since $\mathcal{I}(\mathcal{G}, X) = \mathcal{G}$ for every $\mathcal{G} \in \Gamma$, the filter Γ_x is finer than Γ. Furthermore it is clear that Γ_x is V^--convergent to $\{x\}$.

Now let Ψ be any ultrafilter finer than Γ_x: then Ψ also V^--converges to $\{x\}$, so that $x \in \mathrm{Ls}\Psi^* = \mathrm{Ls}\Psi$ and therefore Ψ cannot K^+-converge to C. □

Corollary 5.5. *The Kuratowski convergence is a pseudotopology.*

Proof. It follows from Proposition 2.6 and the previous proposition. □

Now we can give a necessary and sufficient condition for TK to be a T_2 topology.

Proposition 5.6. *The topologization of K is always compact. Moreover it is Hausdorff if and only if it coincides with K.*

Proof. From Proposition 5.2 and Corollary 5.3 it follows that K is compact and Hausdorff and therefore TK is compact; now apply Proposition 2.8. □

We conclude with a theorem which summarizes the main results of the paper.

Theorem 5.7. *Let X be a topological space. The first of the following six statements implies the other five, which are equivalent:*

(1) *X is locally compact;*
(2) *$\mathcal{C}(X)$ is a continuous lattice (in the reverse inclusion ordering);*
(3) *K^+ is a topological convergence on $\mathcal{C}(X)$;*
(4) *K^+ is a pretopological convergence on $\mathcal{C}(X)$;*

(5) *K is a topological convergence on $\mathcal{C}(X)$;*
(6) *K is a pretopological convergence on $\mathcal{C}(X)$.*
Moreover all these statements are equivalent if we also assume that X is quasi-sober, but in general they are not.

Proof. As the lattices $\mathcal{A}(X)$ and $\mathcal{C}(X)$ are isomorphic, it follows from Corollary 4.2 that (1) implies (2), while Theorem 4.10 says that equivalence holds when the space X is quasi-sober, but not in general, as shown by Example 4.3. Since the equivalence of (2), (3) and (4) is a particular case of Theorem 3.7, and (3) $\Rightarrow$ (5) $\Rightarrow$ (6) is trivial, it remains to prove that (6) implies (2).

So, suppose that $\mathcal{C}(X)$ (equivalently $\mathcal{A}(X)$) is not a continuous lattice. There exists an open set V and a point $x \in V$ such that no open set containing x is way below V in $\mathcal{A}(X)$: thus, denoting by $\mathcal{V}$ the collection of all open neighborhoods of x contained in V, if we take any $U \in \mathcal{V}$, by Proposition 4.1 we can find a filter $\mathcal{F}_U$ on X having no cluster point in V and such that $U \in \mathcal{F}_U$.

Let A be the complement of V: for each $C \in \mathcal{C}(X)$, consider the collection $\mathcal{T}(C)$ of all sets of the form $A \cup B$, where B is a nonempty closed subset of C and let Γ_U be the filter on $\mathcal{C}(X)$ generated by $\{\, \mathcal{T}(\overline{F}) \mid F \in \mathcal{F}_U \,\}$. We claim that Γ_U is K-convergent to A. Indeed, V^--convergence follows from the obvious fact that, for each $\mathcal{H} \in {\Gamma_U}^*$, there is some superset of A which is a member of $\mathcal{H}$ and hence $\overline{\bigcup \mathcal{H}} \supset A$; to verify K^+-convergence observe that

$$\mathrm{Ls}\Gamma_U \subset \bigcap_{F \in \mathcal{F}_U} \overline{\bigcup \mathcal{T}(\overline{F})} = \bigcap_{F \in \mathcal{F}_U} (A \cup \overline{F}) = A \cup \left(\bigcap_{F \in \mathcal{F}_U} \overline{F} \right)$$

and the last set equals A, because $\mathcal{F}_U$ has no cluster point in V.

Now let $\Gamma = \bigcap_{U \in \mathcal{V}} \Gamma_U$, and consider any $\mathcal{G} \in \Gamma$. For every $U \in \mathcal{V}$, since $\mathcal{G} \in \Gamma_U$, we have $\mathcal{T}(C) \subset \mathcal{G}$ for a suitable $C \in \mathcal{F}_U$ and in particular $A \cup C \in \mathcal{G}$; as $U \cap C \neq \varnothing$, it follows that $U \cap (\bigcup \mathcal{G}) \neq \varnothing$, hence $x \in \overline{\bigcup \mathcal{G}}$ because $\mathcal{V}$ is a neighborhood base at x. As $\mathcal{G}$ was arbitrary, we get $x \in \mathrm{Ls}\Gamma$ and therefore Γ cannot be K-convergent to A.

We conclude that K does not satisfy (3) from Theorem 2.4, i.e. is not a pretopology. □

References

[1] G. Choquet, *Sur les notions de filtre et de grille*, C. R. Acad. Sci. Paris **224** (1947), 171–173.

[2] G. Choquet, *Convergences*, Annales Univ. Grenoble **23** (1948), 55–112.

[3] S. Dolecki and G.H. Greco, *Cyrtologies of convergences, I*, Math. Nachr. **126** (1986), 327–348.

[4] S. Francaviglia, A. Lechicki, S. Levi, *Quasi-uniformization of hyperspaces and convergence of nets of semicontinuous multifunctions*, J. Math. Analysis Appl. **112 (2)** (1985), 347–370.

[5] G. Gierz et al., *A Compendium of Continous Lattices*, Springer Verlag, New York, 1980.

[6] K.H. Hofmann, J.D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. Amer. Math. Soc. **246** (1978), 285–310.

[7] S.S. Hong, *Extensive subcategories of the category of T_0-spaces*, Canad. J. Math. **27** (1975), 311–318.

[8] E. Klein, A.C. Thompson, *Theory of Correspondences*, Wiley, New York, 1984.

[9] S. Mrówka, *On the convergence of nets of sets*, Fund. Math. **45** (1958), 416–428.

[10] S. Mrówka, *Some comments on the space of subsets*, In: Set-Valued Mappings, Selections and Topological Properties of 2^X (W.M. Fleischmann, eds.), Springer Verlag, New York (Lecture Notes in Mathematics, vol. 171), 1970.

[11] P. Vitolo, *Strutture d'ordine e topologie sugli iperspazi*, Ph.D. thesis, Università degli studi di Milano (Italy), Biblioteca Nazionale, Roma–Firenze, 1991.

[12] P. Vitolo, *Scott topology and Kuratowski convergence on the closed subsets of a topological space*, Rend. Circolo Mat. Palermo **29** (1992), 593–603.

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