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HOLOMORPHIC BESOV SPACES B^p, $0 < p < 1$, ON BOUNDED SYMMETRIC DOMAINS

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Abstract. We define and study a class of holomorphic Besov type spaces B^p, $0 < p < 1$, on bounded symmetric domains Ω. A description of these Besov spaces is given in terms of differential operators. It is shown that B^p, $0 < p < 1$, can be naturally embedded as a complemented subspace of the space $L^{1,p}(\Omega, d\tau)$.

1. Introduction

Let Ω be an irreducible symmetric domain in C^n in its Harish-Chandra realization. In [11] and [13] K. Zhu defined and studied a class of holomorphic Besov-type spaces B^p on Ω for $1 \leq p \leq \infty$. The purpose of the present paper is to define analogous spaces B^p for $0 < p < 1$ and to extend some of the results presented in [11] and [13] to the case $0 < p < 1$.

It is well known [5] that the domain Ω is uniquely determined (up to a biholomorphic mapping among standard irreducible bonded symmetric domains) by three analytic invariants; r, a and b, all of which are nonnegative integers. The invariant r is called the rank of Ω, which is of course always positive. See [5] for the definition of a and b. We shall make extensive use of the following invariant of Ω: $N = a(r-1) + b + 2$.

Let ν be Lebesgue measure on Ω normalized so that $\nu(\Omega) = 1$. For $0 < p < \infty$ the Bergman space $L^p_a(\Omega)$ is the closed subspace of $L^p(\Omega, d\nu)$ consisting of holomorphic functions. The Bergman projection P (namely, the orthogonal projection from $L^2(\Omega, d\nu)$ onto $L^2_a(\Omega)$ is an integral operator

$$Pf(z) = \int_\Omega K(z,w) f(w)\, d\nu(w), \qquad z \in \Omega, \quad f \in L^2(\Omega, d\nu).$$

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By [3] there exists a polynomial $h(z,w)$ in z and $\bar{w}$ such that the Bergman kernel of Ω is given by

$$K(z,w) = h(z,w)^{-N}, \qquad z, w \in \Omega.$$

Throughout this paper we assume α is a real number satisfying $\alpha > -1$. Let c_α be a positive normalizing constant such that the measure $d\nu_\alpha(z) = c_\alpha h(z,z)^\alpha d\nu(z)$ has total mass 1 on Ω.

Let $H(\Omega)$ be the space of all holomorphic functions in Ω. We equip $H(\Omega)$ with the topology of uniform convergence on compact sets. In [13] it is shown that the operator

$$D^{m,\alpha} : H(\Omega) \to H(\Omega), \qquad m \geq 0, \quad \alpha > -1,$$

defined by

$$D^{m,\alpha} f(z) = \lim_{r \to 1} \int_\Omega \frac{f(rw) d\nu_\alpha(w)}{h(z,w)^{N+\alpha+m}}, \quad f \in H(\Omega),$$

is continuous and invertible on $H(\Omega)$. The inverse of $D^{m,\alpha}$ admits the following integral representation:

$$D_{m,\alpha} f(z) = c_{m+\alpha} \lim_{r \to 1} \int_\Omega \frac{h(w,w)^{m+\alpha} f(rw) d\nu(w)}{h(z,w)^{N+\alpha}}, \; f \in H(\Omega), \; z \in \Omega.$$

We note that if

$$f \in L_a^{1,\alpha}(\Omega) = L^1(\Omega, d\nu_\alpha) \cap H(\Omega)$$

then

$$D^{m,\alpha} f(z) = \int_\Omega \frac{f(w) d\nu_\alpha(w)}{h(z,w)^{N+\alpha+m}}.$$

The above formula extends the domain of $D^{m,\alpha}$ to $L^1(\Omega, d\nu_\alpha)$. We write

$$V_{m,\alpha} f(z) = h(z,z)^m D^{m,\alpha} f(z), \text{for} \quad f \in L^1(\Omega, d\nu_\alpha), \quad m \geq 0, \quad \alpha > -1,$$

and

$$E_{m,\alpha} f(z) = h(z,z)^m D^{m,\alpha} f(z), \quad m \geq 0, \quad \alpha > -1, \quad \text{for} \quad f \in H(\Omega).$$

Thus, if $f \in L_a^{1,\alpha}(\Omega)$,then $E_{m,\alpha} f = V_{m,\alpha} f$.

We are now ready to state our first result.

Theorem 1.1. *Let* $k, m > \frac{N-1}{p}$ *and* $\alpha, \beta > max\{\frac{a(r-1)}{2p} - N, -1\}$. *If* $0 < p \leq 1$ *and* $f \in H(\Omega)$ *then* $\int_\Omega |E_{m,\alpha}f(z)|^p d\tau(z) < \infty$ *if and only if* $\int_\Omega |E_{k,\beta}f(z)|^p d\tau(z) < \infty$, *where* $d\tau(z) = h(z,z)^{-N} d\nu(z)$ *is the Möbious invariant measure on* Ω.

Recall that the holomorphic Besov space B^p, $1 \leq p \leq \infty$ consists of functions in $H(\Omega)$ such that $E_{N,0}f$ is in $L^p(\Omega, d\tau)$ (see [11] and [13]). For any irreducible bounded symmetric domain Ω we have $\frac{a(r-1)}{2} - N \leq -2$. Thus, a holomorphic function f in Ω belongs to B^1 if and only if $E_{m,\alpha}f$ is in $L^1(\Omega, d\tau)$ for same(any) $m > N-1$ and some (any) $\alpha > -1$. This is also proved in [13], Theorem 4, by a different method.

Definition 1.2. *For* $0 < p \leq 1$ *the holomorphic Besov space* $B^p = B^p(\Omega)$ *consists of holomorphic functions* $f \in H(\Omega)$ *such that*

$$\|f\|_{B^p} = |f(0)| + \|E_{m,\alpha}f\|_{L^p(\Omega,d\tau)} < \infty,$$

for some (any) $\quad m > \frac{N-1}{p}$ *and* $\alpha > max\{\frac{a(r-1)}{2p} - N, -1\}$.

For every z in Ω let $E_r(z)$ be the closed Bergman metric ball with center z and radius $r > 0$, i.e.,

$$E_r(z) = \{w : \beta(z,w) \leq r\},$$

where $\beta(\cdot,\cdot)$ is the Bergman metric on Ω.

For a complex measurable function f on B we define

$$M_{\infty,r} = esssup \quad \{ \quad |f(w)| : \quad w \in E_r(z) \quad \}$$

and

$$M_{p,r}f(z) = \left[\frac{1}{\tau(r)} \int_{E_r(z)} |f(w)|^p \, d\tau(w)\right]^{\frac{1}{p}}, \qquad 0 < p < \infty,$$

where $\tau(r) = \tau(E_r(z))$.

For $0 < p, q \leq \infty$, we define $L^{p,q}_r(\Omega, d\tau)$ to be the space of all measurable functions f on Ω for which

$$\|f\|_{L^{p,q}_r(\Omega,d\tau)} \quad = \quad \|M_{p,r}f\|_{L^q(\Omega,d\tau)} < \infty.$$

Since the definition is independent of r, $0 < r < 1$, we will write $L^{p,q}(\Omega, d\tau)$ instead of $L^{p,q}_r(\Omega, d\tau)$ (see [1]).

We let P_α denote the orthogonal projection from $L^2(\Omega, d\nu_\alpha)$ onto $L_a^{2,\alpha}(\Omega) = L^2(\Omega, d\nu_\alpha) \cap H(\Omega)$. It can be shown that (see [9],for instance)

$$P_\alpha f(z) = \int_\Omega \frac{f(w) d\nu_\alpha(w)}{h(z,w)^{N+\alpha}}, \quad z \in \Omega,\ f \in L^2(\Omega, d\nu_\alpha).$$

The above formula extends the domain of P_α to $L^1(\Omega, d\nu_\alpha)$. Note that $P_\alpha f = D^{0,\alpha} f$ for $f \in L^1(\Omega, d\nu_\alpha)$.

If $1 \leq p \leq \infty$ and $\alpha > -1$ then $B^p = P_\alpha L^p(\Omega, d\tau)$. See ([13]). In this note we show that the analytic Besov space B^p, $0 < p < 1$,can be naturally embedded as a complemented subspace of $L^{1,p}(\Omega, d\tau)$ by a topological embedding

$$E_{m,\alpha}: \quad B^p \to L^{1,p}(\Omega, d\tau).$$

We show that $E_{m,\alpha} \circ P_\alpha$ is projection on this embedded copy and that $B^p = P_\alpha L^{1,p}(\Omega, d\tau)$.

More precicely we prove

Theorem 1.3. *Let $0 < p < 1$.Then for any $\alpha > max \quad \{\frac{a(r-1)}{2p} - N, -1\}$,*

$$P_\alpha : \quad L^{1,p}(\Omega, d\tau) \to B^p$$

is a continuous linear map. Moreover if $m > \frac{N-1}{p}$ and $\alpha > max\{\frac{a(r-1)}{2p} - N, -1\}$ then

$$E_{m,\alpha}: \quad B^p \to L^{1,p}(\Omega, d\tau)$$

is a topological embedding.

This theorem was proved in [4] for the open unit ball.

Finally we apply Theorem 1.3. to obtain a result about duality.

Theorem 1.4. *Let $0 < p < 1$,$m > \frac{N-1}{p}$ and $\alpha = m - N$.The integral pairing*

$$< f, g >_\tau = \int_\Omega E_{m,\alpha} f(z)\, \overline{E_{m,\alpha} g(z)}\, d\tau(z)$$

induces the following duality $(B^p)^\star \ = B^\infty$

2. Preliminaries

Throughout this paper we will use the conventions of denoting by C any positive constant which is independent of the relevant parameters in the expression in which it occurs. The value of C may change from one occurence to the next. In addition, we will use the notation $A \cong B$ to mean $C^{-1} A \leq B \leq CA$ for some positive constant C.

We begin with a version of Theorem 3 in [13].

Lemma 2.1. *Suppose* $0 \leq m$ *and* $\alpha, \beta > -1$. *Then*

$$D_z^{m,\alpha}(h(z,w)^{-N-\beta}) = F(z,w)h(z,w)^{-N-\beta-m},$$

where F *is holomorphic in* z, *conjugate holomorphic in* w, *and bounded in* $\Omega \times \Omega$.

Proof. From the integral representation of the operator $D^{m,\alpha}$ we obtain

$$D_z^{m,\alpha}\left(h(z,w)^{-N-\beta}\right) = c_\alpha \int_\Omega \frac{h(u,u)^\alpha d\nu(u)}{h(z,u)^{N+\alpha+m}h(u,w)^{N+\beta}}.$$

Denote the integral above by $f(z,w)$ (forget about the constant). Then f is holomorphic in z and conjugate holomorphic in w. Fix $z \in \Omega$ and let φ_z be the involutive authomorphism interchanging z and 0; see [2]. A change of variables $u = \varphi_z(\xi)$ gives

$$f(z,z) = \int_\Omega \frac{h(\varphi_z(\xi),\varphi_z(\xi))^{\alpha+N} d\nu(\xi)}{h(z,\varphi_z(\xi))^{N+\alpha+m}h(\varphi_z(\xi),z)^{N+\beta}h(\xi,\xi)^N}.$$

Since,

$$h(\varphi_z(\xi),\varphi_z(\xi)) = \frac{h(z,z)h(\xi,\xi)}{|h(z,\xi)|^2}$$

and

$$h(z,\varphi_z(\xi)) = \frac{h(z,z)}{h(z,\xi)},$$

we have

$$f(z,z) = \frac{1}{h(z,z)^{N+\beta+m}} \int_\Omega \frac{h(\xi,\xi)^\alpha h(z,\xi)^m \, d\nu(\xi)}{h(\xi,z)^{\alpha-\beta}}$$

Let

$$F(z,w) = \int_\Omega \frac{h(z,u)^m \, h(u,u)^\alpha \, d\nu(u)}{h(u,w)^{\alpha-\beta}}, \quad z,w \in \Omega.$$

Then F is holomorphic in z, conjugate holomorphic in w, and

$$f(z,z) = \frac{F(z,z)}{h(z,z)^{N+\beta+m}}, \quad \text{for all} \quad z \in \Omega.$$

By a well known uniqueness theorem in several complex variables (see [6]) we must have

$$f(z,w) = \frac{F(z,w)}{h(z,w)^{N+\beta+m}}, \text{ for all } \quad z,w \in \Omega.$$

It remains to show that F is bounded in $\Omega \times \Omega$.

It is well known that there exists a universal constant $C > 0$ such that $h(z,z) \leq C|h(z,w)|$, for all z and w in Ω. If $\alpha \geq \beta$,then

$$|F(z,w)| \leq C \int_\Omega |h(z,w)|^m h(u,u)^\beta \, d\nu(u), \quad z,w \in \Omega,$$

which shows that F is bounded, since h is polynomial and $\int_\Omega h(u,u)^\beta d\nu(u) < \infty$. If $\alpha < \beta$, then

$$|F(z,w)| \leq \int_\Omega |h(z,u)|^m |h(u,w)|^{\beta-\alpha} h(u,u)^\alpha d\nu(u), z, w \in \Omega,$$

and so F is bounded.

Note that if $\alpha = \beta$, then $F(z,w) \equiv 1$, and therefore

$$D_z^{m,\alpha}\left(\frac{1}{h(z,w)^{N+\alpha}}\right) = \frac{1}{h(z,w)^{N+\alpha+m}}.$$

Lemma 2.2. *([2]). For fixed $r > 0$, there is a sequence $\{\xi_j\}$ in Ω such that*

(i) $\cup_{j=1}^\infty E_r(\xi_j) = \Omega$, and

(ii) there is a positive integer $M = M(r)$ such that for any $z \in \Omega$, z is contained in at most M of the sets $E_{2r}(\xi_j)$.

Lemma 2.3. *Let $k, m \geq 0$ and $\alpha > -1$. If $f \in L^1(\Omega, d\nu_\alpha)$ then*

$$D^{m,k+\alpha}(D^{k,\alpha} f) = D^{m+k,\alpha} f.$$

Proof. It follows easily from the reproducing property of the Bergman projections $P_{\alpha+k}$ and Fubini's theorem that

$$\begin{aligned} D^{m,k+\alpha}(D^{k,\alpha} f)(z) &= D_z^{m,k+\alpha}\left(c_{k+\alpha}\int_\Omega \frac{h(u,u)^{\alpha+k} D^{k,\alpha} f(u)\, d\nu(u)}{h(z,u)^{N+\alpha+k}}\right) \\ &= c_{k+\alpha}\int_\Omega h(u,u)^{\alpha+k} D_z^{m,k+\alpha}(h(z,u)^{-N-\alpha-k}) D^{k,\alpha} f(u) d\nu(u) \\ &= c_{k+\alpha}\int_\Omega \frac{h(u,u)^{k+\alpha} D^{k,\alpha} f(u) d\nu(u)}{h(z,u)^{N+k+\alpha+m}} \\ &= c_{k+\alpha}\int_\Omega \frac{h(u,u)^{k+\alpha}\, d\nu(u)}{h(z,u)^{N+k+\alpha+m}} c_\alpha \int_\Omega \frac{h(\xi,\xi)^\alpha f(\xi) d\nu(\xi)}{h(u,\xi)^{N+k+\alpha}} \\ &= c_\alpha \int_\Omega h(\xi,\xi)^\alpha f(\xi) d\nu(\xi) \int_\Omega \frac{c_{k+\alpha} h(u,u)^{k+\alpha}\, d\nu(u)}{h(u,\xi)^{N+k+\alpha} h(z,u)^{N+\alpha+k+m}} \\ &= c_\alpha \int_\Omega \frac{h(\xi,\xi)^\alpha f(\xi) d\nu(\xi)}{h(z,\xi)^{N+k+m+\alpha}} = D^{k+m,\alpha} f(z) \end{aligned}$$

Lemma 2.4. *([3]). For $t > -1$ and c real let*

$$I_{t,c}(z) = \int_\Omega \frac{h(w,w)^t d\nu(w)}{|h(z,w)|^{N+t+c}}, \quad z \in \Omega.$$

We have

(i) *If $c < -\frac{a(r-1)}{2}$, then $I_{t,c}(z)$ is bounded on Ω.*

(ii) *If $c > \frac{a(r-1)}{2}$, then $I_{t,c} \cong h(z,z)^{-c}$.*

Lemma 2.5. *Let $0 < p < \infty$ and let $r > 0$. There exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{C}{\nu(E_r(z))} \int_{E_r(z)} |f(z)|^p d\nu(w), \text{for all} \quad z \in \Omega \text{ and } \quad f \in H(\Omega).$$

As a final preliminary result we need the following lemma.

Lemma 2.6. *Let $r > 0$. There exists $C = C(r) > 0$ such that for $a, b \in \Omega$ with $\beta(a,b) \leq r$ and arbitrary $z \in \Omega$ we have*

$$C^{-1} \leq \frac{|h(z,a)|}{|h(z,b)|} \leq C.$$

Proof. Obviously it is sufficient to show that

$$C^{-1} \leq \frac{|K(z,b)|}{|K(z,a)|} \leq C.$$

For the Bergman reproducing kernel we have the well-known transformation law

$$K(\varphi_a(z), \varphi_a(w))(J_c\varphi_a)(z)\overline{(J_c\varphi_a)(w)} = K(z,w), \quad z, w \in \Omega,$$

where $(J_c\varphi_a)(z)$ denotes the determinant of the complex Jacobian of φ_a. Note that

$$|J_c\varphi_a(z)|^2 = \frac{|K(z,a)|^2}{K(a,a)},$$

(see [2], Proposition 2).

Thus,

$$\begin{aligned}\frac{|K(z,b)|}{|K(z,a)|} &= \frac{|K(\varphi_a(z), \varphi_a(b))||(J_c\varphi_a)(b)|}{|K(\varphi_a(z), \varphi_a(a))||(J_c\varphi_a)(a)|} \\ &= \frac{|K(\varphi_a(z), \varphi_a(b))||K(b,a)|}{|K(a,a)|}\end{aligned}$$

Since $\beta(a,b) \le r$, $\varphi_a(b) \in E_r(0)$. Using the fact that $K(\xi, w)$ is continuous and nonvanishing on the compact $\overline{\Omega} \times E_r(0)$ and that

$$C^{-1} \le \frac{|K(b,a)|}{|K(a,a)|} \le C, \qquad \text{for } b \in E_r(a),$$

we see that there is a constant $C(r) > 0$ with

$$C(r)^{-1} \le \frac{|K(z,b)|}{|K(z,a)|} \le C(r).$$

3. Analytic Besov space B^p, $0 < p \le 1$

Proof of Theorem1.1. Let $r > 0$ and $\{\xi_j\}$ be the same as those in Lemma 2.2 and assume that $E_{m,\alpha} f \in L^p(\Omega, d\tau)$. Since $f = D_{m,\alpha} D^{m,\alpha} f$, we have

$$\begin{aligned} D^{k,\beta} f(z) &= D_z^{k,\beta}\left[c_{m+\alpha} \int_\Omega \frac{h(\xi,\xi)^{m+\alpha} D^{m,\alpha} f(\xi)\, d\nu(\xi)}{h(z,\xi)^{N+\alpha}}\right] \\ &= c_{m+\alpha} \int_\Omega \frac{h(\xi,\xi)^{m+\alpha} F(z,\xi) D^{m,\alpha} f(\xi) d\nu(\xi)}{h(z,\xi)^{N+\alpha+k}}, \end{aligned}$$

by Lemma 2.1.

By Lemmas 6 and 8 in [2] and by Lemma 2.6 there exists a constant C, $C > 0$, such that

$$C^{-1} \le \frac{h(\xi,\xi)}{h(\xi_j,\xi_j)} \le C, \quad C^{-1} \le \frac{|h(z,\xi)|}{|h(z,\xi_j)|} \le C, \quad C^{-1} \le \frac{\nu(E_r(\xi_j))}{h(\xi_j,\xi_j)^N} \le C,$$

for all $z \in \Omega$, for all $\xi_j, j \ge 1$, and $\xi \in E_r(\xi_j)$. Thus,

$$\begin{aligned} &\|E_{k,\beta} f\|^p_{L^p(\Omega,d\tau)} \\ &\le C \int_\Omega d\tau(z) \left(\int_\Omega \frac{h(\xi,\xi)^{m+\alpha} |D^{m,\alpha} f(\xi)| h(z,z)^k d\nu(\xi)}{|h(z,\xi)|^{N+\alpha+k}} \right)^p \\ &\le C \int_\Omega \left(\sum_{j=1}^\infty \int_{E_r(\xi_j)} \frac{h(\xi,\xi)^{m+\alpha} |D^{m,\alpha} f(\xi)| h(z,z)^k d\nu(\xi)}{|h(z,\xi|^{N+\alpha+k}} \right)^p d\tau(z) \\ &\le C \int_\Omega \left(\sum_{j=1}^\infty \frac{h(\xi_j,\xi_j)^{m+\alpha} sup_{\xi\in E_r(\xi_j)} |D^{m,\alpha} f(\xi)| h(z,z)^k \nu(E_r(\xi_j))}{|h(z,\xi_j)|^{N+\alpha+k}} \right)^p d\tau(z) \\ &\le C \sum_{j=1}^\infty h(\xi_j,\xi_j)^{(N+m+\alpha)p} \sup_{\xi\in E_r(\xi_j)} |D^{m,\alpha} f(\xi)|^p \int_\Omega \frac{h(z,z)^{kp-N} d\nu(z)}{|h(z,\xi_j)|^{p(N+\alpha+k)}} \end{aligned}$$

By Lemma 2.5

$$\sup_{\xi \in E_r(\xi_j)} |D^{m,\alpha} f(\xi)|^p \leq \frac{C}{\nu(E_{2r}(\xi_j))} \int_{E_{2r}(\xi_j)} |D^{m,\alpha} f(w)|^p d\nu(w)$$

and by Lemma 2.4 we have

$$\int_\Omega \frac{h(z,z)^{kp-N}}{|h(z,\xi_j)|^{p(N+\alpha+k)}} \leq \frac{C}{|h(\xi_j,\xi_j)|^{p(N+\alpha)}}.$$

Thus,

$$\begin{aligned} \|E_{k,\beta} f\|^p_{L^p(\Omega,d\tau)} &\leq C \sum_{j=1}^{\infty} \int_{E_{2r}(\xi_j)} h(\xi,\xi)^{pm} |D^{m,\alpha} f(\xi)|^p d\tau(\xi) \\ &\leq C \|E_{m,\alpha} f\|^p_{L^p(\Omega,d\tau)} \end{aligned}$$

This finishes the proof of Theorem 1.1.

4. Embedding of Besov spaces B^p, $0 < p < 1$

To prove Theorem 1.3 the following lemma will be needed.

Lemma 4.1. *If $0 < p < 1$, then $L^{1,p}(\Omega, d\tau) \subset L^1(\Omega, d\tau)$ and the inclusion map is continuous.*

Proof. Let $0 < \delta < \frac{1}{4}$ and let $f \in L^{1,p}_{2\delta}(\Omega, d\tau)$. The invariance of the measure τ and Fubini's theorem show that $L^q(\Omega, d\tau) = L^{q,q}(\Omega, d\tau)$, for any q, $0 < q < \infty$. Thus, we have

$$\begin{aligned} \|f\|_{L^{1,1}_\delta(\Omega,d\tau)} &\leq C \int_\Omega \left[\int_{E_\delta(z)} |f(\xi)| d\tau(\xi) \right] d\tau(z) \\ &\leq C \operatorname{ess\,sup}_{z\in\Omega} \left[\int_{E_\delta(z)} |f(\xi)| d\tau(\xi) \right]^{1-p} \int_\Omega \left[\int_{E_\delta(z)} |f(\xi)| d\tau(\xi) \right]^p d\tau(z). \end{aligned}$$

It is easy to see that if $w \in E_\delta(z)$, then $E_\delta(w) \subset E_{2\delta}(z)$.
Thus,

$$\int_\Omega \Big(M_{\infty,\delta}(M_{1,\delta}) f(z) \Big)^p d\tau(z) \leq C \int_\Omega \Big(M_{1,2\delta} f(z) \Big)^p d\tau(z).$$

On the other hand,

$$\begin{aligned}\underset{w\in\Omega}{ess\ sup}\big(M_{1,\delta}f(w)\big)^p\tau(\delta) &\le \underset{w\in\Omega}{ess\ sup}\int_\Omega \mathcal{X}_{E_\delta(w)}(z)\big(M_{1,\delta}f(w)\big)^p d\tau(z)\\ &\le \int_\Omega \underset{w\in\Omega}{ess\ sup}\mathcal{X}_{E_\delta(z)}(w)\big(M_{1,\delta}f(w)\big)^p d\tau(z)\\ &= \int_\Omega\Big(M_{\infty,\delta}(M_{1,\delta}f)(z)\Big)^p d\tau(z)\end{aligned}$$

Combining the above inequalities we find that $\|f\|_{L^1_\delta(\Omega,d\tau)} \le C\|f\|_{L^{1,p}_{2\delta}(\Omega,d\tau)}$. This finishes the proof of Lemma 4.1.

Proof of Theorem 1.3.

Let $f \in L^{1,p}(\Omega,d\tau)$ and let $m > \frac{N-1}{p}$ and $\alpha > max\{\frac{a(r-1)}{2p} - N, -1\}$. Using Lemma 4.1 we see that

$$\begin{aligned}|D^{m,\alpha}f(z)| &\le C\int_\Omega \frac{h(w,w)^\alpha|f(w)|d\nu(w)}{|h(z,w)|^{N+\alpha+m}}\\ &\le C\Big[\int_\Omega\Big[\int_{E_\epsilon(w)}\frac{h(\xi,\xi)^{\alpha+N}|f(\xi)|d\tau(\xi)}{|h(z,\xi)|^{N+\alpha+m}}\Big]^p d\tau(w)\Big]^{\frac{1}{p}}\end{aligned}$$

for some fixed ϵ, $0<\epsilon<\frac{1}{2}$.

Since $h(\xi,\xi) \cong h(w,w)$ and $|h(z,\xi)| \cong |h(z,w)|$ (Lemma 2.6), if $\xi \in E_\epsilon(w)$ we have

$$|D^{m,\alpha}f(z)|^p \le C\int_\Omega \frac{h(w,w)^{p(\alpha+N)}}{|h(z,w)|^{p(N+\alpha+m)}}\Big[\int_{E_\epsilon(w)}|f(\xi)|d\tau(\xi)\Big]^p d\tau(w).$$

Using Lemmas 2.3 and 2.4 we obtain

$$\begin{aligned}&\|h(z,z)^m D^{m,\alpha}P_\alpha f(z)\|^p_{L^p(\Omega,d\tau)}\\ &= \|h(z,z)^m D^{m,\alpha}(D^{0,\alpha}f)(z)\|^p_{L^p(\Omega,d\tau)}\\ &= \|h(z,z)^m D^{m,\alpha}f(z)\|^p_{L^p(\Omega,d\tau)}\\ &\le C\int_\Omega h(z,z)^{mp}d\tau(z)\int_\Omega\frac{h(w,w)^{p(\alpha+N)}}{|h(z,w)|^{p(\alpha+N+m)}}\Big[\int_{E_\epsilon(w)}|f(\xi)|d\tau(\xi)\Big]^p d\tau(w)\\ &\le C\int_\Omega h(w,w)^{p(\alpha+N)}\Big[\int_{E_\epsilon(w)}|f(\xi)|d\tau(\xi)\Big]^p\Big[\int_\Omega\frac{h(z,z)^{mp-N}d\nu(z)}{|h(z,w)|^{p(N+\alpha+m)}}\Big]d\tau(w)\\ &\le C\int_\Omega\Big[\int_{E_\epsilon(w)}|f(\xi)|d\tau(\xi)\Big]^p d\tau(w) \cong \|f\|^p_{L^{1,p}(\Omega,d\tau)}\end{aligned}$$

Using Lemma 4.1 we find that

$$|P_\alpha f(0)| \leq C\,||f||_{L^1(\Omega,d\nu_\alpha)} \leq C\,||f||_{L^1(\Omega,d\tau)} \leq C\,||f||_{L^{1,p}(\Omega,d\tau)}.$$

Thus, $||P_\alpha f||_{B^p} \leq C\,||f||_{L^{1,p}(\Omega,d\tau)}$.

Assume now that $f \in B^p$. Since $L^p(\Omega,d\tau) = L^{p,p}(\Omega,d\tau)$, we have

$$\int_\Omega h(z,z)^{mp}\left[M_{p,\epsilon}D^{m,\alpha}f(z)\right]^p d\tau(z) \leq C\|f\|^p_{B^p}, \quad \text{for some } \epsilon, 0 < \epsilon < \frac{1}{2}.$$

Let $\delta = \frac{\epsilon}{2}$. The function $D^{m,\alpha}f \in H(\Omega)$ and therefore

$$|D^{m,\alpha}f(w)|^p \leq C\int_{E_\delta(w)} |D^{m,\alpha}f(\xi)|^p d\tau(\xi),$$

by Lemma 2.5.

From this we find that

$$M_{1,\delta}D^{m,\alpha}f(z) \leq M_{\infty,\delta}D^{m,\alpha}f(z) \leq C\big(M_{p,\epsilon}D^{m,\alpha}f(z)\big), \quad z \in \Omega.$$

Thus,

$$\|h(z,z)^m M_{1,\delta}D^{m,\alpha}f(z)\|_{L^p(\Omega,d\tau)} \leq C\|f\|_{B^p}$$

Since

$$\|h(z,z)^m D^{m,\alpha}f\|_{L^{1,p}_\delta(\Omega,d\tau)} \cong \|h(z,z)^m M_{1,\delta}D^{m,\alpha}f\|_{L^p(\Omega,d\tau)},$$

we see that $\|E_{m,\alpha}f\|_{L^{1,p}(\tau)} \leq C\|f\|_{B^p}$.

Using again the fact that $h(z,z) \cong h(w,w)$, for $z \in E_\epsilon(w)$ we get

$$\begin{aligned}\|E_{m,\alpha}f\|_{L^{1,p}_\epsilon(\Omega,d\tau)} &= \left[\int_\Omega \left[\int_{E_\epsilon(w)} h(z,z)^m|D^{m,\alpha}f(z)|d\tau(z)\right]^p d\tau(w)\right]^{\frac{1}{p}}\\ &\geq C\left[\int_\Omega h(w,w)^{mp}\left[\int_{E_\epsilon(w)} |D^{m,\alpha}f(z)|d\tau(z)\right]^p d\tau(w)\right]^{\frac{1}{p}}\\ &\geq C\left[\int_\Omega h(w,w)^{mp}\left[\int_{E_\epsilon(w)} |D^{m,\alpha}f(z)|^p\ d\tau(z)\right] d\tau(w)\right]^{\frac{1}{p}}\end{aligned}$$

Here we have used the estimate

$$M_{p,\epsilon}D^{m,\alpha}f(w) \leq M_{1,\epsilon}D^{m,\alpha}f(w), \qquad w \in \Omega.$$

Using Lemma 4.1 we obtain

$$|f(0)| \leq C\|E_{m,\alpha}f\|_{L_\epsilon^{1,p}(\Omega,d\tau)}.$$

Thus

$$\|f\|_{B^p} \cong \|E_{m,\alpha}f\|_{L^{1,p}(\Omega,d\tau)}.$$

i.e. $E_{m,\alpha}$ is a topological embedding.

We note that $V_{m,\alpha} \circ P_\alpha$is projection from $L^{1,p}(\tau)$ onto $V_{m,\alpha}(B^p)$.

From $f = c_{m+\alpha}c_\alpha^{-1}P_\alpha E_{m,\alpha}f$, $f \in B^p$, it follows by Lemma 4.1 that P_α maps $L^{1,p}(\Omega, d\tau)$ onto B^p.

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