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ALMOST α-CONTINUOUS MULTIFUNCTIONS

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Abstract. The purpose of the present paper is to introduce the notion of almost α-continuous multifunctions. We obtain several characterizations and properties of such multifunctions.

1. Introduction

In 1965, Njastad [10] introduced a weak form of open sets called α-sets. The authors [14, 18] of the present paper investigated a class of functions called almost α-continuous or almost feebly continuous. In 1993, the authors [20] introduced the notion of α-continuous multifunctions. The purpse of the present paper is to define almost α-continuous multifunctions and to obtain several characterizations and some properties of almost α-continuous multifunctions.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be *α-open* [10] (resp. *semi–open* [5], *preopen* [8], *β-open* [1], or *semi–preopen* [2]) if $A \subset Int(Cl(Int(A)))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$, $A \subset Cl(Int(Cl(A)))$). The family of all semi–open (resp. α-open) sets of X containing a point $x \in X$ is denoted by $SO(X, x)$ (resp $\alpha(X, x)$). The family of all α-open (resp. semi–open, preopen, semi–preopen) sets in X is denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$, $SPO(X)$). For these four families, it is shown in [13, Lemma 3.1] that $SO(X) \cap PO(X) = \alpha(X)$ and it is obvious that $SO(X) \cup P(X) \subset SPO(X)$. Since $\alpha(X)$ is a topology for X [10, Prop. 2], by $\alpha Cl(A)$ (resp. $\alpha Int(A)$) we denote the closure (resp. interior) of A with respect to $\alpha(X)$. The complement of a semi–open (resp. α-open) set is said to be *semi-closed* (resp. *α-closed*).

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The intersection of all semi–closed sets of X containing A is called the *semi–closure* [3] of A and is denoted by $sCl(A)$. The union of all semi–open sets of X contained in A is called the *semi–interior* of A and is denoted by $sInt(A)$. A subset A is said to be *feebly open* [6] if there exists an open set U such that $U \subset A \subset sCl(U)$. It is shown in [13, Lemma 4.12] that the notion of feebly open sets is equivalent to that of α-opens sets. A subset A of a space X is said to be *regular open* (resp. *regular closed*) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). The family of regular open (resp. regular closed) sets of X is denoted by $RO(X)$ (resp. $RC(X)$). Maheshwari at al. [7] defined a function to be *almost feebly continuous* if the inverse image of every regular open set is feebly open. Noiri [14] defined a function $f : X \to Y$ to be *almost α-continuous* if $f^{-1}(V) \in \alpha(X)$ for every $V \in RO(Y)$ and pointed out that almost feeble continuity is equivalent to almost α-continuity.

Throughout this paper, spaces (X, τ) and (X, σ) (or simply X and Y) always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multi alued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set G of Y by $F^+(G)$ and $F^-(G)$, respectively, that is

$$F^+(G) = \{x \in X : F(x) \subset G\} \quad \text{and} \quad F^-(G) = \{x \in X : F(x) \cap G \neq \varnothing\}.$$

Let $\mathbf{A}(Y)$ be the collection of all nonempty subsets of Y. For an open set V of Y, we denote $V^+(A \in \mathbf{A}(Y) : A\ V\}$ and $V^- = \{A \in \mathbf{A}(Y) : A \cap V \neq \varnothing\}$ [24].

3. Characterizations

Definition 1. A multifunction $F : X \to Y$ is said to be *almost α-continuous* (briefly a.α.c.) at a point $x \in X$, if for any open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$ and $U \in SO(X, x)$, there exists a nonempty open set G_U of X such that $G_U \subset U$, $F(G_U) \subset sCl(G_1)$ and $F(g) \cap sCl(G_2) \neq \varnothing$ for every $g \in G_U$.

A multifunction $F : X \to Y$ is said to be *almost α-continuous* if it has this properties at each point of X.

Theorem 1. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is a.α.c. at a point $x \in X$;*

(2) *for any open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset sCl(G_1)$ and $F(u) \cap sCl(G_2) \neq \varnothing$ for every $u \in U$;*

(3) *$x \in \alpha Int[F^+(sCl(G_1)) \cap F^-(sCl(G_2))]$ for any open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$;*

(4) $x \in Int(Cl(Int[F^+(sCl(G_1)) \cap F^-(sCl(G_2))]))$ *for any open sets* G_1, G_2 *of* Y *such that* $F(x) \in G_1^+ \cap G_2^-$.

Proof. (1) $\Rightarrow$ (2) Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. For each $H \in SO(X,x)$, there exists a nonempty open set $G_H \subset H$ such that $F(G_H) \subset sCl(G_1)$ and $F(g) \cap sCl(G_2) \neq \varnothing$ for every $g \in G_H$. Let $W = \bigcup\{G_H : H \in SO(X,x)\}$. Then W is open in X, $x \in sCl(W)$, $F(W) \subset sCl(G_1)$ and $F(w) \cap sCl(G_2) \neq \varnothing$ for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subset U \subset sCl(W) = Int(Cl(W))$. Therefore, we obtain $U \in \alpha(X,x)$ [13, Lemma 4.12], $F(U) \subset sCl(G_1)$ and $F(u) \cap sCl(G_2) \neq \varnothing$ for every $u \in U$.

(2) $\Rightarrow$ (3) Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then there exists $U \in \alpha(X,x)$ such that $F(U) \subset sCl(G_1)$ and $F(u) \cap sCl(G_2) \neq \varnothing$ for every $u \in U$. Thus we have $x \in U \subset F^+(sCl(G_1)) \cap F^-(sCl(G_2))$. Since $U \in \alpha(X)$, we obtain

$$x \in U = \alpha Int(U) \subset \alpha Int[F^+(sCl(G_1)) \cap F^-(sCl(G_2))].$$

(3) $\Rightarrow$ (4) Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Now put $U = \alpha Int[F^+(sCl(G_1)) \cap F^-(sCl(G_2))]$. Then $U \in \alpha(X)$ and

$$x \in U \subset F^+(sCl(G_1)) \cap F^-(sCl(G_2)).$$

Thus we have

$$x \in U \subset Int(Cl(Int(U))) \subset Int(Cl(Int[F^+(sCl(G_1)) \cap F^-(sCl(G_2))])).$$

(4) $\Rightarrow$ (1) Let $U \in SO(X,x)$ and G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. The we have

$$\begin{aligned} x &\in Int(Cl(Int[F^+(sCl(G_1)) \cap F^-(sCl(G_2))])) = \\ &= sCl(Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2)))). \end{aligned}$$

It follows from [12, Lemma 3] and [11, Lemma 1] that

$$\varnothing \neq U \cap Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2))) \in SO(X,x).$$

Put $G_U = Int[U \cap Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2)))]$, then G_U is a nonempty open set of X [11, Lemma 4] such that $G_U \subset U$, $F(G_U) \subset sCl(G_1)$ and $F(g) \cap sCl(G_2) \neq \varnothing$ for every $g \in G_U$. Therefore, F is a.α.c. at x.

Theorem 2. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is a.α.c;*

(2) *for each $x \in X$ and any open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset sCl(G_1)$ and $F(u) \cap sCl(G_2) \neq \varnothing$ for every $u \in U$;*

(3) *for each $x \in X$ and any regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U$;*

(4) *$F^+(G_1) \cap F^-(G_2) \in \alpha(X)$ for every $G_1, G_2 \in RO(Y)$;*

(5) *$F^+(V_1) \cup F^-(V_2)$ is α-closed in X for every $V_1, V_2 \in RC(Y)$;*

(6) *$F^+(G_1) \cup F^-(G_2) \subset \alpha Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2)))$ for any open sets G_1, G_2 of Y;*

(7) *$\alpha Cl(F^-(sInt(K_1)) \cup F^+(sInt(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ for any closed sets K_1, K_2 of Y;*

(8) *$\alpha Cl(F^-(Cl(Int(K_1))) \cup F^+(Cl(Int(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ for any closed sets K_1, K_2 of Y;*

(9) *$\alpha Cl(F^-(Cl(Int(Cl(B_1)))) \cup F^+(Cl(Int(Cl(B_2))))) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2))$ for any subsets B_1, B_2 of Y;*

(10) *$Cl(Int(Cl(F^-(Cl(Int(K_1))) \cup F^+(Cl(Int(K_2)))))) \subset F^-(K_1) \cup F^+(K_2)$ for any closed sets K_1, K_2 of Y;*

(11) *$Cl(Int(Cl(F^-(sInt(K_1)) \cup F^+(sInt(K_2))))) \subset F^-(K_1) \cup F^+(K_2)$ for any closed sets K_1, K_2 of Y;*

(12) *$F^+(G_1) \cap F^-(G_2) \subset Int(Cl(Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2)))))$ for any open sets G_1, G_2 of Y.*

Proof. (1) $\Rightarrow$ (2) The proof follows immediately from Theorem 1.

(2) $\Rightarrow$ (3) The proof is obvious.

(3) $\Rightarrow$ (4) Let $G_1, G_2 \in RO(Y)$ and $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and there exists $U_x \in \alpha(X, x)$ suh that $F(U_x) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U_x$. Therefore, we have $x \in U_x \subset F^+(G_1) \cap F^-(G_2)$ and hence $F^+(G_1) \cap F^-(G_2) \in \alpha(X)$.

(4) $\Rightarrow$ (5) This follows from the fact $F^+(Y - B) = X - F^-(B)$ and $F^-(X - B) = X - F^+(B)$ for every subset B of Y.

(5) $\Rightarrow$ (6) Let G_1, G_2 be any open sets of Y and $x \in F^+(G_1) \cap F^-(G_2)$. Then we have $F(x) \subset G_1 \subset sCl(G_1)$ and $\varnothing \neq F(x) \cap G_2 \subset F(x) \cap sCl(G_2)$ and hence

$$x \in F^+(sCl(G_1)) = X - F^-(Y - sCl(G_1))$$

and

$$x \in F^-(sCl(G_2)) = X - F^+(Y - sCl(G_2)).$$

Since $Y - sCl(G_1)$ and $Y - sCl(G_2)$ are regular closed, $F^-(Y - sCl(G_1)) \cup F^+(Y - sCl(G_2))$ is α-closed in X. Since

$$F^-(Y - sCl(G_1)) \cup F^+(Y - sCl(G_2)) =$$
$$= (X - F^+(sCl(G_1))) \cup (X - F^-(sCl(G_2))) =$$
$$= X - (F^+(sCl(G_1)) \cap F^-(sCl(G_2))),$$

we have $F^+(sCl(G_1)) \cap F^-(sCl(G_2)) \in \alpha(X)$ and $x \in \alpha Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2)))$. Cosequently, we obtain

$$F^+(G_1) \cap F^-(G_2) \subset \alpha Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2))).$$

(6) $\Rightarrow$ (7) Let K_1, K_2 be any closed sets of Y. Then since $Y - K_1$ and $Y - K_2$ are open sets, we have

$$\begin{aligned} X - (F^-(K_1) \cup F^+(K_2)) &= (X - F^-(K_1)) \cap (X - F^+(K_2)) \\ &= F^+(Y - K_1) \cap F^-(Y - K_2) \\ &\subset \alpha Int(F^+(sCl(Y - K_1)) \cap F^-(sCl(Y - K_2))) \\ &= \alpha Int(F^+(Y - sInt(K_1)) \cap F^-(Y - sInt(K_2))) \\ &= \alpha Int(X - F^-(sInt(K_1)) \cap (X - F^+(sInt(K_2))) \\ &= X - \alpha Cl(F^-(sInt(K - 1)) \cup F^+(sInt(K_2))). \end{aligned}$$

Therefore, we obtain $\alpha Cl(F^-(sInt(K_1)) \cup F^+(sInt(K_2))) \subset F^-(K - 1) \cup F^+(K_2)$.

(7) $\Rightarrow$ (8) The proof is obvious since $sInt(K) = Cl(Int(K))$ for every closed set K.

(8) $\Rightarrow$ (9) The proof is obiouvs.

(9) $\Rightarrow$ (10) It follows from [21, Lemma 2.2] that $Cl(Int(Cl(S))) \subset \alpha Cl(S)$ for every subset S. Thus, for any closed sets K_1, K_2 of Y we have

$$Cl(Int(Cl(F^-(Cl(Int(K_1))) \cup F^+(Cl(Int(K_2)))))) \subset$$
$$\subset \alpha Cl(F^-(Cl(Int(K_1))) \cup F^+(Cl(Int(K_2))))$$
$$= \alpha Cl(F^-(Cl(Int(Cl(K_1)))) \cup F^+(Cl(Int(Cl(K_2))))) \subset$$
$$\subset F^-(K_1) \cup F^+(K_2).$$

(10) $\Rightarrow$ (11) The proof is obious since $sInt(K) = Cl(Int(K))$ for every closed set K.

(11) $\Rightarrow$ (12) Let G_1, G_2 be any open set of Y. Then $Y - G_1$ and $Y - G_2$ are closed sets of Y and we have

$$Cl(Int(Cl(F^-(sInt(Y - G_!)) \cup F^+(sInt(Y - G_2))))) \subset$$
$$\subset F^-(Y - G_1) \cup F^+(Y - G_2)$$
$$= (X - F^+(G_1)) \cup (X - F^-(G_2)) = X - (F^+(G_1) \cap F^-(G_2)).$$

Moreover, we have

$$Cl(Int(Cl(F^-(sInt(Y - G_1)) \cup F^+(sInt(Y - G_2)))))$$
$$= Cl(Int(Cl(F^-(Y - sCl(G_1)) \cup F^+(Y - sCl(G_2)))))$$
$$= Cl(Int(Cl(X - F^+(sCl(G_1))) \cup (X - F^-(sCl(G_2)))))$$
$$= Cl(Int(Cl(X - (F^+(sCl(G_1)) \cap F^-(sCl(G_2)))))$$
$$= X - Int(Cl(Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2))))).$$

Therefore, we obtain

$$F^+(G_1) \cap F^-(G_2) \subset Int(Cl(Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2))))).$$

(12) $\Rightarrow$ (1) Let x be any point of X and G_1, G_2 be any open set of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then

$$x \in F^+(G_1) \cap F^-(G_2) \subset Int(Cl(Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2)))))$$

and hence F is a.α.c at x by Theorem 1. Therefore, F is a.α.c.

Corollary 1. *(Maheshwari et al.* [7], *Noiri* [14], *Popa* [18], *Thakur and Paik* [26]*). For a function $f : X \to Y$, the following are equivalent:*

(1) *f is almost α-continuous;*
(2) *for each $x \in X$ and any opens set G of Y containing $f(x)$, there exists $U \in \alpha(X, x)$ such that $f(U) \subset sCl(G)$;*
(3) *for each $x \in X$ and any regular open set G of Y containing $f(x)$, there exists $U \in \alpha(X, x)$ such that $f(U) \subset G$;*
(4) *$f^{-1}(G) \in \alpha(X)$ for every $G \in RO(Y)$;*
(5) *$f^{-1}(V)$ is α-closed in X for every $V \in RC(Y)$;*
(6) *$f^{-1}(G) \subset \alpha Int(f^{-1}(sCl(G)))$ for any open set G of Y;*
(7) *$\alpha Cl(f^{-1}(sInt(K)))) \subset f^{-1}(K)$ for any closed set K of Y;*
(8) *$\alpha Cl(f^{-1}(Cl(Int(K)))) \subset f^{-1}(K)$ for any closed set K of Y;*
(9) *$\alpha Cl(f^{-1}(Cl(Int(Cl(B)))) \subset f^{-1}(Cl(B))$ for any subset B of Y;*
(10) *$Cl(Int(Cl(f^{-1}(Cl(Int(K)))) \subset f^{-1}(K)$ for any closed set K of Y;*
(11) *$Cl(Int(Cl(f^{-1}(sInt(K))) \subset f^{-1}(K)$ for any closed set K of Y;*
(12) *$f^{-1}(G) \subset Int(Cl(Int(f^{-1}(sCl(G))))$ for any open set G of Y.*

Theorem 3. *The following are eqivalent for a multifunction $F: X \to Y$:*

(1) *F is a.α.c.*
(2) *$\alpha Cl(F^-(G_1) \cup F^+(G_2)) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2))$ for any $G_1, G_2 \in SPO(Y)$;*
(3) *$\alpha Cl(F^-(G_1) \cup F^+(G_2)) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2))$ for any $G_1, G_2 \in SO(Y)$;*
(4) *$F^+(G_1) \cap F^-(G_2) \subset \alpha Int(F^+(sCl(G_2)) \cap F^-(sCl(G_1)))$ for any $G_1, G_2 \in PO(Y)$.*

Proof. (1) $\Rightarrow$ (2) Let G_1, G_2 be any semi–preopen sets of Y. Since $Cl(G_1)$ and $Cl(G_2)$ are regular closed, by Theorem 2 $F^-(Cl(G_1)) \cup F^+(Cl(G_2))$ is α-closed in X and $F^-(G_1) \cup F^+(G_2) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2))$. Therefore, we have

$$\alpha Cl(F^-(G_1) \cup F^+(G_2)) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2)).$$

(2) $\Rightarrow$ (3) This is obvious since $SO(Y) \subset SPO(Y)$.

(3) $\Rightarrow$ (1) Let $K_1, K_2 \in RC(Y)$. Then $K_1, K_2 \in SO(Y)$ and hence $\alpha Cl(F^-(K_1) \cup F^+(K_2)) \subset F^-(K_1) \cup F^+(K_2)$. Therefore, $F^-(K_1) \cup F^+(K_2)$ is α-closed in X and hence F is a.α.c by Theorem 2.

(1) $\Rightarrow$ (4) Let G_1, G_2 be any preopen sets of Y. Since $Int(Cl(G_1))$ and $Int(Cl(G_2))$ are regular open in Y, $Int(Cl(G_1)) = sCl(G_1)$ and $Int(Cl(G_2)) = sCl(G_2))$ [14, Lemma 3.1], by Theorem 2 we have

$$F^+(sCl(G_1)) \cap F^-(sCl(G_2)) \in \alpha(X)$$

and hence

$$\begin{aligned} &F^+(G_1) \cap F^-(G_2) \subset F^+(sCl(G_1)) \cap F^-(sCl(G_2)) = \\ &= \alpha Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2))). \end{aligned}$$

(4) $\Rightarrow$ (1) Let G_1, G_2 be any regular open sets of Y. Since $G_1, G_2 \in PO(Y)$, we have $F^+(G_1) \cap F^-(G_2) \subset \alpha Int(F^+(sCl(G_1)) \cap F^-(sCl(G_2))) = \alpha Int(F^+(G_1) \cap F^-(G_2))$ and hence $F^+(G_1) \cap F^-(G_2) \in \alpha(X)$. It follows from Theorem 2 that F is a.α.c.

Corollary 2. *The following are equivalent for a function $f: X \to Y$:*

(1) *f is almost α-continuous;*
(2) *$\alpha Cl(f^{-1}(G)) \subset f^{-1}(Cl(G))$ for any $G \in SPO(Y)$;*
(3) *$\alpha Cl(f^{-1}(G)) \subset f^{-1}(Cl(G))$ for any $G \in SO(Y)$;*
(4) *$f^{-1}(V) \subset \alpha Int(f^{-1}(sCl(V)))$ for any $V \in PO(Y)$.*

Definition 2. A subset A of a topological space X is said to be *α-regular* [4] for any point $x \in A$ and any open set U of X containing x, there exsists an open set G of X such that $x \in G \subset Cl(G) \subset U$.

Definition 3. A subset A of a topological space X is said to be *α-paracompact* [27] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X.

For a multifunction $F : X \to Y$, a multifunction $\alpha ClF : X \to Y$ is defined as follows: $(\alpha ClF)(x) = \alpha Cl(F(x))$ for each $x \in X$.

Lemma 1. *(Popa and Noiri [21]) If $F : X \to Y$ is a multifunction such that $F(x)$ is α-regular α-paracompact for each $x \in X$, then $(\alpha ClF)^+(V) = F^+(V)$ for each open set V of Y.*

Lemma 2. *(Popa and Noiri [21]) For a multifunction $F : X \to Y$, it follows that $(\alpha ClF)^-(V) = F^-(V)$ for every open set V of Y.*

Theorem 4. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$. Then F is a.α.c. if and only if $\alpha ClF : X \to Y$ is a.α.c.*

Proof. Necessity: Suppose that F is a.α.c. Let $x \in X$ and V_1, V_2 be any regular open sets of Y such that $(\alpha ClF)(x) \in V_1^+ \cap V_2^-$; hence $(\alpha ClF)(x) \subset V_1$ and $(\alpha ClF)(x) \cap V_2 \neq \varnothing$. By Lemmas 1 and 2, we have $x \in (\alpha ClF)^+(V_1) = F^+(V_1)$ and $x \in (\alpha ClF)^-(V_2) = F^-(V_2)$ and hence $F(x) \in V_1^+ \cap V_2^-$. Since F is a.α.c., by Theorem 1 we obtain $x \in \alpha Int(F^+(V_1) \cap F^-(V_2))$ and hence $x \in \alpha Int((\alpha ClF)^+(V_1) \cap (\alpha ClF)(V_2))$. This shows that αClF is a.α.c.

Sufficiency: Suppose that αClF is a.α.c. Let $x \in X$ and V_1, V_2 be any regular open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By Lemmas 1 and 2, we have $x \in F^+(V_1) = (\alpha ClF)^+(V_1)$ and $x \in F^-(V_2) = (\alpha ClF)^-(V_2)$. Since αClF is a.α.c., by Theorem 1 we obtain $x \in \alpha Int((\alpha ClF)^+(V_1) \cap (\alpha ClF)^-(V_2)) = \alpha Int(F^+(V_1) \cap F^-(V_2))$. Thus, F is a.α.c.

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 3. *(Noiri and Popa [21]) For a multifunction $F : X \to Y$, the following hold:*

$$\text{(a)} \quad G_F^+(A \times B) = A \cap F^+(B) \quad \textit{and} \quad \text{(b)} \quad G_F^-(A \times B) = A \cap F^-(B)$$

for any subsets $A \subset X$ and $B \subset Y$.

Theorem 5. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is a.α.c. if and only if $G_F : X \to X \times X$ is a.α.c.*

Proof. Necessity: suppose that $F : X \to Y$ is a.α.c. Let $x \in X$ and W_1, W_2 be any open sets of $X \times Y$ such that $G_F(x) \in W_1^+ \cap W_2^-$. Then $G_F(x) \subset W_1$ and $G_F(x) \cap W_2 \neq \varnothing$. Since $G_F(x) \subset W_1$, for each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W_1$. The fasmily $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$ and there exist a finite number of points, says, $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$. Set

$$U_1 = \cap\{U(y_i) : 1 \leq i \leq n\} \quad \text{and} \quad V_1 = \cup\{V(y_i) : 1 \leq i \leq n\}.$$

Then U_1 and V_1 are open in X and Y, respectively, and $\{x\} \times F(x) \subset U_1 \times V_1 \subset W_1$. Since $G_F(x) \cap W_2 \neq \varnothing$, there exists $y \in F(x)$ such that $(x, y) \in W_2$ and hence $(x, y) \in U_2 \times V_2 \subset W_2$ for some open sets $U_2 \subset X$ and $V_2 \subset Y$. Put $U = U_1 \cap U_2$. Then U is an open set containing x, $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$. Since F is a.α.c., by Theorem 2 there exists $U_0 \in \alpha(X, x)$ such that $U_0 \subset F^+(sCl(V_1))$ and $U_0 \subset F^-(sCl(V_2))$. Put $G = U \cap U_0$, then $G \in \alpha(X, x)$. By Lemma 3, we obtain

$$\begin{aligned} G = U \cap U_0 &\subset sCl(U_1) \cap F^+(sCl(V_1)) = G_F^+(sCl(U_1) \times sCl(V_1)) \\ &= G_F^+(sCl(U_1 \times V_1)) \subset G_F^+(sCl(W_1)). \end{aligned}$$

Therefore, we obtain $G_F(G) \subset sCl(W_1)$. By Lemma 3, we obtain

$$\begin{aligned} G = U \cap U_0 &\subset sCl(U_2) \cap F^-(sCl(V_2)) = G_{F^-}(sCl(U_2) \times sCl(V_2)) \\ &= G_{F^-}(sCl(U_2 \times V_2)) \subset G_{F^-}(sCl(W_2)). \end{aligned}$$

Therefore, we obtain $G_F(G) \cap sCl(W_2) \neq \varnothing$ for every $g \in G$. By Theorem 2, it follows that G_F is a.α.c.

Sufficiency: Suppose that $G_F : X \to X \times Y$ is a.α.c. Let $x \in X$ and G_1, G_2 be any open sets of Y, such that $F(x) \in G_1^+ \cap G_2^-$. Then $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \varnothing$. By $F(x) \subset G_1$, we have $G_F(x) \subset X \times G_1$ and $X \times G_1$ is open in $X \times Y$. Since $F(x) \cap G_2 \neq \varnothing$, we have

$$G_F(x) \cap (X \times G_2) = (\{x\} \times F(x)) \cap (X \times G_2) = \{x\} \times (F(x) \cap G_2) \neq \varnothing.$$

Since $X \times G_2$ is open in $X \times Y$, there exists $U \in \alpha(X, x)$ such that $G_F(U) \subset sCl(X \times G_1) = X \times sCl(G_1))$ and $G_F(u) \cap sCl(X \times G_2) \neq \varnothing$ for every $u \in U$. By Lemma 3, we obtain $U \subset G_F^+(X \times sCl(G_1)) = F^+(sCl(G_1))$ and hence $F(U) \subset sCl(G_1))$. Moreover, by Lemma 3 we obtain $U \subset G_F^-(sCl(X \times G_2)) = G_F^-(X \times sCl(G_2)) = F^-(sCl(G_2))$ and hence $F(u) \cap sCl(G_2) \neq \varnothing$ for every $u \in U$. By Theorem 2, it follows that F is a.α.s,

Corollary 3. *(Noiri [14]) Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. Then f is a.α.c. if and only if g is a.α.c*

4. Some Properties

Lemma 4. *(Mashhour et al. [9], Reilly and Vamanamurthy [25]). Let U and X_0 be subsets of a topological space X. The following properties hold:*

(1) *If $U \in \alpha(X)$ and $X_0 \in SO(X) \cup PO(X)$, then $U \cap X_0 \in \alpha(X_0)$,*
(2) *If $U \subset X_0 \subset X$, $U \in \alpha(X_0)$ and $X_0 \in \alpha(X)$, then $U \in \alpha(X)$.*

Theorem 6. *If a multifunction $F : X \to Y$ is a.α.c. and $X_0 \in SO(X) \cup PO(X)$, then the restriction $F/X_0 : X_0 \to Y$ is a.α.c.*

Proof. Let $x \in X_0$ and V_1, V_2 be any open sets of Y such that $(F/X_0)(x) \subset V_1$ and $(F/X_0)(x) \cap V_2 \neq \varnothing$. Since $(F/X_0)(x) = F(x)$ and F is a.α.c., by Theorem 2 there exists $U \in \alpha(X, x)$ such that $F(U) \subset sCl(V_1)$ and $F(u) \cap sCl(V_2) \neq \varnothing$ for each $u \in U$. Let $U_0 = U \cap X_0$, then $U_0 \in \alpha(X_0, x)$ by Lemma 4 and $(F/X_0)(U_0) = F(U_0) \subset sCl(V_1)$ and $(F/X_0)(u) = sCl(V_2) \neq \varnothing$ for each $u \in U_0$. This shows that F/X_0 is a.α.c.

Corollary 4. *(Maheshwari et al. [7]) If $f : X \to Y$ is almost feebly continuous and X_0 is an open set of X, then the restriction $f/X_0 : X_0 \to Y$ is almost feebly continuous.*

Corollary 5. *(Noiri [14]) If $f : X \to Y$ is almost α-continuous and $X_0 \in SO(X) \cup PO(X)$, then the restriction $f/X_0 : X_0 \to Y$ is almost α-continuous.*

Theorem 7. *A multifunction $F : X \to Y$ is a.α.c. if for each $x \in X$ there exists $X_0 \in \alpha(X, x)$ such that the restriction $F/X_0 : X_0 \to Y$ is a.α.c.*

Proof. Let $x \in X$ and V_1, V_1 be any open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. There exists $X_0 \in \alpha(X, x)$ such that $F/X_0 : X_0 \to Y$ is a.α.c. Therefore, there exists $U_0 \in \alpha(X_0, x)$ such that $(F/X_0)(U_0) \subset sCl(V_1)$ and $(F/X_0)(u) \cap sCl(V_2) \neq \varnothing$ for each $u \in U_0$. By Lemma 4, $U_0 \in \alpha(X, x)$ and $F(u) = (F/X_0)(u)$ for each $u \in U_0$. This shows that F is a.α.c.

Corollary 6. *Let $\{U_\alpha : \alpha \in \nabla\}$ be a cover of X by α-open sets of X. Then, a multifunction $F : X \to Y$ is a.α.c. if and only if the restriction $F/U_\alpha : U_\alpha \to Y$ is a.α.c. for each $\alpha \in \nabla$.*

Proof. This is an immediate consequence of Theorems 6 and 7.

Corollary 7. *(Thakur and Paik [26]) Let $f : X \to Y$ be a function and $\{U_\alpha : \alpha \in \nabla\}$ be an open cover of X. If the restriction $f/U_\alpha : U_\alpha \to Y$ is a.α.c. for each $\alpha \in \nabla$, then f is a.α.c.*

Definition 4. A multifunction $F : X \to Y$ is said to be *almost precontinuous* if for each $x \in X$ and any regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in PO(X, x)$ such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U$.

Lemma 5. *A multifunction $F : X \to Y$ is almost precontinuous if and only if for any regular open sets G_1, G_2 of Y, $F^+(G_1) \cap F^-(G_2) \in PO(X)$.*

Proof. Necessity: Let F be almost precontinuous and $G_1, G_2 \in RO(Y)$. Let $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and hence there exists $U_x \in PO(X, x)$ such that $F(U_x) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U_x$. Therefore, we have $U_x \subset F^+(G_1) \cap F^-(G_2)$ and hence $x \in U_x \subset Int(Cl(U_x)) \subset Int(Cl(F^+(G_1)) \cap F^-(G_2)))$. Therefore, we obtain

$$F^+(G_1) \cap F^-(G_2) \subset Int(Cl(F^+(G_1) \cap F^-(G_2))).$$

This shows that $F^+(G_1) \cap F^-(G_2) \in PO(X)$.

Sufficiency: Let $x \in X$ and $G_1, G_2 \in RO(Y)$ such that $F(x) \in G_1^+ \cap G_2^-$. Put $U = F^+(G_1) \cap F^-(G_2)$; then $x \in U \in PO(X, x)$, $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U$. Therefore, F is almost precontinuous.

Definition 5. A multifunction $F : X \to Y$ is said to be *almost quasicontinuous* [23] if for each $x \in X$ and any regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in SO(X, x)$ such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U$.

Lemma 6. *(Popa and Noiri [23]) A multifunction $F : X \to Y$ is almost quasicontinuous if and only if for any regular open sets G_1, G_2 of Y, $F^+(G_1) \cap F^-(G_2) \in SO(X)$.*

Theorem 8. *A multifunction $F : X \to Y$ is a.α.c. if and only if it is almost precontinuous and almost quasicontinuous.*

Proof. It is shown in [13, Lemma 3.1] that $SO(X) \cap PO(X) = \alpha(X)$. Thus this follows from Theorem 2, Lemmas 5 and 6.

A function $f : X \to Y$ is said to be *precontinuous* [8] if $f^{-1}(V) \in PO(X)$ for each open set V of Y. A function $f : X \to Y$ is said to be *almost quasi continuous* [16] if for each point $x \in X$, each open set U of X containing x and each open set V of Y containing $f(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $f(G) \subset Int(Cl(V))$.

Corollary 8. *(Popa [18]) If a function $f : X \to Y$ is precontinuous and almost quasi continuous, then f is almost α-continuous.*

Definition 6. A multifunction $F : X \to Y$ is said to be

(a) *upper almost α-continuous* [22] at a point $x \in X$ if for each $U \in SO(X, x)$ and each open set V containing $F(x)$, there exists a nonempty open set $G \subset U$ such that $F(G) \subset sCl(V)$;

(b) *lower almost α-continuous* [22] at a point $x \in X$ if for each $U \in SO(X, x)$ and each open set V such that $F(x) \cap V \neq \varnothing$, there exists a nonempty open set $G \subset U$ such that $F(g) \cap sCl(V) \neq \varnothing$ for every $g \in G$;

(c) *upper (lower) almost α-continuous* if F has this property at every point of X.

Remark 1. Since $\alpha(X)$ is a topoly [10], the intersection of two α-open sets is α-open. Therefore, a multifunction $F : X \to Y$ is a.α.c. if and only if F is upper almost α-continuous and lower almost α-continuous.

Definition 7. A multifunction $F; X \to Y$ is said to be

(a) *upper almost continuous* [17] (resp. *upper weakly continuous* [15]) if for each point $x \in X$ and each open set V containing $F(x)$, there exists an open set U containing x such that $F(G) \subset sCl(V)$ (resp. $F(G) \subset Cl(V)$),

(b) *lower almost continuous* [17] (resp. *lower weakly continuous* [15]) if for each $x \in X$ and each open set V such gthat $F(x) \cap V \neq \varnothing$, there exists an open set U containing x such that $F(u) \cap sCl(V) \neq \varnothing$ (resp. $F(u) \cap Cl(V) \neq \varnothing$) for every $u \in G$.

Definition 8. A multifunction $F : X \to Y$ is said to be *α-continuous* [20] if for each point $x \in X$ and any open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U$.

Lemma 7. *(Popa and Noiri [21]) Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction.*

(1) *F is upper α-continuous (resp. upper almost α-continuous) if and onky if $F : (X, \tau^\alpha) \to (Y, \sigma)$ is upepr continuous (resp. upper almost continuous).*

(2) *F is lower α-continuous (resp. lower almost α-continuous) if and only if $F : (X, \tau^\alpha) \to (Y, \sigma)$ is lower continuous (rsp. lower almost continuous).*

Theorem 9. *The following are equivalent for a multifunction $F : X \to Y$ such that $F(x)$ is α-regular α-paracompact for each $x \in X$;*

(1) *F is a.α.c.;*
(2) *F is α-continuous.*

Proof. Suppose that F is a.α.c. By Remark 1, F is upper almost α-continuous and lower almost α-continuous. By Lemma 7, $F : (X, \tau^\alpha) \to (Y, \sigma)$ is upper almost continuous and lower almost continuous. Since $F : (X, \tau^\alpha) \to (Y, \sigma)$ is upper almost continuous, $F : (X, \tau^\alpha) \to (Y, \sigma)$ is upper weakly continuous and hence upper continuous [19, Theorem 1]. Therefore, By Lemma 7 $F : (X, \tau) \to (Y, \sigma)$ is upper α-continuous. Since $F : (X, \tau^\alpha) \to (Y, \sigma)$ is lower almost semicontinuous, $F : (X, \tau^\alpha) \to (Y, \sigma)$ is lower weakly continuous and hence lower rcontinuous [19, Theorem 2]. Therefore, by Lemma 7 $F : (X, \tau) \to (Y, \sigma)$ is lower α-continuous. Since F is upper α-continuous and lower α-continuous, it follows from [20, Remark 1] that F is α-continuous.

Theorem 10. *If a closed valued multifunction $F : X \to Y$ is a.α.c. and Y is a normal T_1 space, then F is α-continuous.*

Proof. Let $x \in X$ and G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Since $F(x)$ is closed in Y, by the normality of Y there exists an open set D of Y such that $F(x) \subset D \subset Cl(D) \subset G_1$. Since every normal T_1 space is T_3 and $F(x) \cap G_2 \neq \varnothing$, there exists an open set E of Y such that $E \cap F(x) \neq \varnothing$ and $Cl(E) \subset G_2$. Since F is a.α.c. and $F(x) \in D^+ \cap E$, by Theorem 2 there exists $U \in \alpha(X, x)$ such that $F(U) \subset Int(Cl(D))$ and $F(u) \cap Int(Cl(E)) \neq \varnothing$ for every $u \in U$. Therefore, we obtain $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \varnothing$ for every $u \in U$. Tjhis shows that F is α-continuous.

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