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UNKNOTTING NUMBER AND ∞-UNKNOTTING NUMBER OF A KNOT

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Abstract. In this paper are proposed Conjectures about unknotting number and ∞-unknotting number. According to them, both numbers can be calculated from the minimal projections of knots, if in every step is permitted an ambient isotropy. Using Conjectures, they are calculated unknotting numbers and ∞-unknotting numbers for all knots with $n \leq 10$ crossings.

For every crossing point of a regular knot projection of a knot k, they are two possible changes transforming k to another knot: the crossing-change (or sign-change) $L_+ \to L_-$ or $L_- \to L_+$ and the change $L_+ \to L_\infty$ or $L_- \to L_\infty$ (Fig. 1). We will call them, respectively, 2- and 1-change.

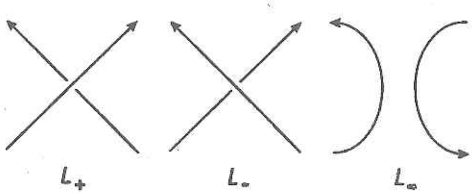

Figure 1

The unknotting number $u(k)$ of a knot k is the smallest number of 2-changes required to obtain the unknot, the minimum taken over all the regular projections of k [1,2,3,4]. According to this definition, we perform all the 2-changes in a single projection of k.

Traditionally, the unknotting number is defined to be the least number of 2-changes necessary to transform a knot to the unknot, where we can perform some 2-changes in one projection of the knot, then do an ambient isotropy of the changed projection to a new projection, perform next 2-changes in the

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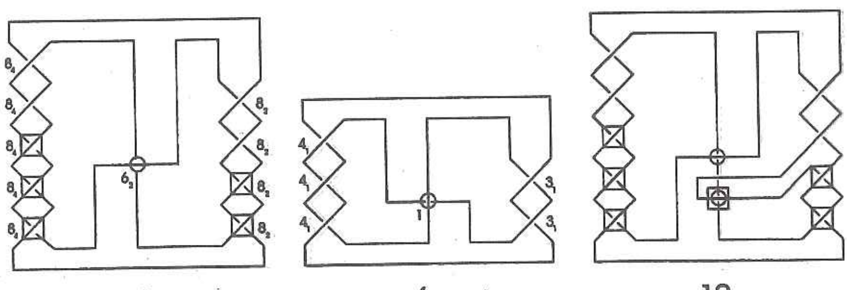
new projection... and continue in this manner until the unknot is obtained [4, pp. 57]. These two definitions are equivalent [4, pp. 58].

If in the first definition we restrict "all the regular projections" to "all the minimal regular projections", we cannot obtain the unknotting number [2,3]. This shows the well known example of the knot 10_8 (or 514 in Conway notation), given by Y. Nakanishi and S. Bleiler: the alternating knot with $u(10_8) = 2$ and with the only one minimal projection needing at least three 2-changes to be transformed to the unknot [2,3,4] (Fig. 2 (a)).

Let us reconsider this example, with regard to the traditional definition and the use of minimal projections. The first 2-change in the minimal projection of 10_8 can result in the three knots: 8_2 occurring four times, 8_4 occurring five times, and 6_2 occurring once. Their unknotting numbers are: $u(8_2) = 2$, $u(8_4) = 2$, $u(6_2) = 1$. There is the only one minimal projection of 8_4 given by Dowker sequence 6 10 12 16 14 4 2 8, which by one 2-change cannot be unknotted. If we use the minimal projections, in order to obtain the correct unknotting number $u(10_8) = 2$ the only strategy is to make the first 2-change in the central point of 10_8, then to transform the obtained projection of 6_2 to it's minimal projection, and finaly to derive from it the unknot by the second 2-change in the central point of 6_2 (see Fig.2 (a,b), where to every point is assigned the corresponding knot obtained by the 2-change).



(a) (b) (c)

Figure 2

If we accept the first definition, this cannot be done. The central point of 6_2 is not a point preserved from the minimal projection of 10_8 by the reduction (Fig. 2, where such points are denoted by squares). This is the reason to introduce the non-minimal projection of 10_8 with the new crossing point (Fig. 2 (c)), which will be preserved by the reduction as the central point of the minimal projection of 6_2. Hence, the non-minimal projection can be directly unknotted by the two 2-changes.

Therefore, we will calculate unknotting numbers by the recursive method, using minimal projections and the traditional definition.

Definition 1.

(a) $u(1) = 0$, *where* 1 *is the unknot;*

(b) $u(k) = min\ u(k") + 1$, *where the minimum is taken over all the knots* $k"$, *obtained from a minimal projection of* k *by a 2-change.*

Conjecture 1. *Definition 1 is equivalent to the previously given definitions.*

For the knots with $3 \leq n \leq 9$ crossings, the results obtained according to this definition completely coincide with the unknotting numbers or their estimated values given by Y. Nakanishi [1], except for the knot 9_{29}. According to Y. Nakanishi $u(9_{29}) = 1$, but we obtained $u(9_{29}) = 2$. The knot 9_{29} has the only one minimal projection given by Dowker sequence 6 10 14 18 4 16 8 2 12, which by one 2-change cannot be unknotted, but maybe there is some other it's not-minimal projection, that can be unknotted by one 2-change. If the first result is correct, it is the negative answer to the *Unsolved question 3* by C. Adams [4, pp. 61]: 9_{29} is the alternating knot with the unknotting number 1, but there is not a crossing in it's minimal projection that we can change to make it the unknot. Anyway, we believe that $u(9_{29}) = 2$, and than this question will still remain unsolved. Also, if $u(9_{29}) = 1$, then Conjecture 1 not holds for all the knots, and our Definition 1 requires some additional conditions.

If a knot k is given by it's minimal projection K, it is not possible to make any conclusion about the number $u(k")$, where $k"$ is the knot obtained from K by one 2-change: it is possible that $u(k) > u(k")$ (e.g. $u(10_8) = 2$, $u(6_2) = 1$), $u(k) = u(k")$ (e.g. $u(6_2) = u(4_1) = 1$), or $u(k) < u(k")$ (e.g. $u(9_{26}) = 1$, $u(5_1) = 2$) (see *Problem* by Y. Nakanishi [2]).

For all the calculations, the complete list of different alternating knot projections, obtained by the algorithm of Dowker&Thislethwaite [5], has been used. Anyway, because all the minimal projections of an alternating knot give the same series of knots obtained by a 2-change, for every alternating knot it is sufficient to use only one minimal projection. This property not holds for non-alternating knots, so for them we are not sure that the calculation is independent from the choice of the minimal projection. Because of a very large number of their minimal projections, for non-alternating knots we only have used the results [1] without their independent control, in all the cases where some 2-change of an alternating knot projection resulted in a non-alternating knot.

Since our calculation of the unknotting number is recursive, we are giving the complete table of results for knots $3 \leq n \leq 9$ and for alternating knots with $n = 10$ (Table 1).

In the same way as we defined the unknotting number, it is possible to describe the ∞-unknotting number $u^\infty(k)$ by the three definitions, and make a conjecture that they are equivalent.

For our calculation is used Definition 2:

Definition 2.

(a) $u^\infty(1) = 0$, *where 1 is the unknot;*

(b) $u^\infty(k) = min\ u^\infty(k') + 1$, *where the minimum is taken over all the knots k', obtained from a minimal projection of k by a 1-change.*

Every 1-change transforms an alternating knot to an alternating knot, so the set of all the alternating knots is closed with regard to 1-changes. All the minimal projections of an alternating knot give the same series of knots obtained by a 1-change, so for every alternating knot it is sufficient to use only one minimal projection. Anyway, wishing to have a double control of our results, for all the calculations we have used the complete set of projections, as well as the projections of composite knots ($6 \leq n \leq 9$), for which holds the property $u^\infty(k_1\#k_2) = u^\infty(k_1) + u^\infty(k_2)$.

Conjecture 2. $u^\infty(k_1\#k_2) = u^\infty(k_1) + u^\infty(k_2)$.

The results of the calculation of $u^\infty(k)$ for all the alternating knots with $3 \leq n \leq 10$ are given in Table 1.

Table 1.

k	$u(k)$	$u^\infty(k)$	k	$u(k)$	$u^\infty(k)$
3_1	1	1	10_1	1	2
			10_2	3	2
4_1	1	2	10_3	2	3
			10_4	2	2
5_1	2	1	10_5	2	3
5_2	1	2	10_6	3	3
			10_7	1	3
6_1	1	2	10_8	2	2
6_2	1	2	10_9	1	3
6_3	1	3	10_{10}	1	3
			10_{11}	3	3
7_1	3	1	10_{12}	2	3
7_2	1	2	10_{13}	2	4

7_3	2	2
7_4	2	3
7_5	2	3
7_6	1	3
7_7	1	3
8_1	1	2
8_2	2	2
8_3	2	3
8_4	2	2
8_5	2	2
8_6	2	3
8_7	1	3
8_8	2	3
8_9	1	3
8_{10}	2	3
8_{11}	1	3
8_{12}	2	4
8_{13}	1	3
8_{14}	1	4
8_{15}	2	4
8_{16}	2	3
8_{17}	1	4
8_{18}	2	4
8_{19}	3	
8_{20}	1	
8_{21}	1	
9_1	4	1
9_2	1	2
9_3	3	2
9_4	2	2
9_5	2	3
9_6	3	3
9_7	2	3
9_8	2	3
9_9	3	3
9_{10}	3	3
9_{11}	2	3
9_{12}	1	3
9_{13}	3	3
10_{14}	2	4
10_{15}	2	3
10_{16}	2	3
10_{17}	1	3
10_{18}	1	4
10_{19}	2	3
10_{20}	2	3
10_{21}	2	3
10_{22}	2	3
10_{23}	1	4
10_{24}	2	4
10_{25}	2	4
10_{26}	1	4
10_{27}	1	4
10_{28}	2	3
10_{29}	2	4
10_{30}	1	4
10_{31}	1	4
10_{32}	1	4
10_{33}	1	4
10_{34}	2	3
10_{35}	2	4
10_{36}	2	4
10_{37}	2	4
10_{38}	2	4
10_{39}	2	4
10_{40}	2	5
10_{41}	2	4
10_{42}	1	5
10_{43}	2	4
10_{44}	1	4
10_{45}	2	5
10_{46}	3	2
10_{47}	3	3
10_{48}	2	3
10_{49}	3	4
10_{50}	2	3
10_{51}	3	4
10_{52}	2	3
10_{53}	3	4
10_{54}	3	3

9_{14}	1	3	10_{55}	2	4
9_{15}	2	4	10_{56}	2	4
9_{16}	3	3	10_{57}	2	5
9_{17}	2	3	10_{58}	2	4
9_{18}	2	4	10_{59}	1	4
9_{19}	1	4	10_{60}	1	4
9_{20}	2	3	10_{61}	3	2
9_{21}	1	4	10_{62}	2	3
9_{22}	1	3	10_{63}	2	4
9_{23}	2	4	10_{64}	2	3
9_{24}	1	4	10_{65}	2	4
9_{25}	2	4	10_{66}	3	4
9_{26}	1	4	10_{67}	2	4
9_{27}	1	4	10_{68}	2	3
9_{28}	1	4	10_{69}	2	5
9_{29}	2	3	10_{70}	2	4
9_{30}	1	4	10_{71}	1	4
9_{31}	2	4	10_{72}	2	4
9_{32}	2	4	10_{73}	1	5
9_{33}	1	4	10_{74}	2	4
9_{34}	1	4	10_{75}	2	4
9_{35}	3	3	10_{76}	3	4
9_{36}	2	3	10_{77}	3	4
9_{37}	2	4	10_{78}	2	4
9_{38}	3	4	10_{79}	3	4
9_{39}	1	4	10_{80}	3	4
9_{40}	2	4	10_{81}	2	5
9_{41}	2	3	10_{82}	1	4
9_{42}	1		10_{83}	2	4
9_{43}	2		10_{84}	1	5
9_{44}	1		10_{85}	2	3
9_{45}	1		10_{86}	2	4
9_{46}	2		10_{87}	2	4
9_{47}	2		10_{88}	1	5
9_{48}	2		10_{89}	2	5
9_{49}	3		10_{90}	2	4
			10_{91}	1	4
			10_{92}	2	4
			10_{93}	2	3
			10_{94}	2	4
			10_{95}	1	5

10_{96}	2	4
10_{97}	2	4
10_{98}	2	4
10_{99}	2	4
10_{100}	3	3
10_{101}	3	4
10_{102}	1	4
10_{103}	3	4
10_{104}	1	4
10_{105}	2	4
10_{106}	2	4
10_{107}	1	5
10_{108}	2	3
10_{109}	2	5
10_{110}	2	4
10_{111}	2	4
10_{112}	2	4
10_{113}	1	5
10_{114}	1	4
10_{115}	2	5
10_{116}	2	4
10_{117}	2	5
10_{118}	1	5
10_{119}	1	4
10_{120}	3	5
10_{121}	2	5
10_{122}	2	4
10_{123}	2	5

Additional remark: The first Conjecture (about unknotting numbers) was presented by the author at the conference "Knots in Hellas'98", as one of the unsolved problems. In the discussion, it was mentioned that the same Conjecture was proposed by J. Bernhard in 1994 [6], and that the results obtained for all the knots with $n \leq 10$ crossings using the Conjecture completely coincide with the all exactly determined unknotting numbers from [7], as well with the estimated values if for each of them we take the maximum. The both Conjectures are still opened. In the meantime, there is some recent progress: A. Stiomenow [8] succeeded to prove that the Conjecture holds for a restricted class of knots: a rational knot of unknotting number one has an unknotting number one minimal diagram.

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