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RPP SEMIGROUPS SATISFYING PERMUTATION IDENTITIES

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Abstract. In this paper we study rpp semigroups satisfying permutation identities. In particular, we determine when a rpp semigroup will satisfy the permutation identities.

1. Introduction and preliminaries

Throughout this paper, we use the terminologies and notions given in [1, 2, 6]. A semigroup S is called a *rpp semigroup* if all of its principal right ideals aS^1 $(a \in S)$, regarded as right S^1-systems, are projective. Equivalently, a semigroup is a rpp semigroup if and only if each $\mathcal{L}^*$-class contains at least one idempotent (see [5]). The lpp semigroup can be dually defined. According to Fountain [2], a semigroup is abundant if and only if it is both rpp and lpp. We call a rpp semigroup a *strongly rpp semigroup* [5] if for each $a \in S$, there exists a unique idempotent e, $\mathcal{L}^*$-related to a, such that $ea = a$.

A semigroup S in which $x_1x_2\cdots x_n = x_{p_1}x_{p_2}\cdots x_{p_n}$ (1) for all $x_1, x_2, \dots,$ $x_n \in S$, where $(p_1p_2\dots p_n)$ is a nontrivial permutation of $(12\dots n)$, is termed *to satisfy the permutation identity* (1) (or a *PI-semigroup* for short). Yamada [7] investigated the PI-regular semigroups and discussed the structure of such semigroups. In [3], the author generalized the results of Yamada [7] and investigated the structure of PI-abundant semigroups. Recently, he [4] also the classification of PI-strongly rpp semigroups. The aim of this note is to study the PI-rpp semigroups, and to determine when a rpp semigroup will satisfy the permutation identities.

We first recall some known concepts and results (we only list the results for $\mathcal{L}^*$ because the result for $\mathcal{R}^*$ is a dual result for $\mathcal{L}^*$.

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Lemma 1.1. [2] *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*b$;
(2) *for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.*

As an easy but useful consequence, we have

Corollary 1.2. [2] *Let $a \in S$ and e be an idempotent of S. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*e$;
(2) *$ae = e$ and for all $x, y \in S^1$, $ax = ay$ implies $ex = ey$.*

Evidently, $\mathcal{L}^*$ is a right congruence on S. In general, $\mathcal{L} \subset \mathcal{L}^*$ and when a and b are regular elements, $a\mathcal{L}b$ if and only if $a\mathcal{L}^*b$. Let E be the set of idempotents of S. We write a^* to denote the typical idempotents $\mathcal{L}^*$-related to a.

A band B is called a *(left;right) normal band* if it satisfies the identity: $(xyz = xzy; xyz = yxz)$ $xyzw = xzyw$. By [6, Theorem IV 3.1], a band B is a semilattice Y of rectangular bands B_α $(\alpha \in Y)$. Such a decomposition of B is unique. Accordingly Y is unique up to isomorphism and so are the $B_\alpha' s$. Hereafter, we call Y the *structure semilattice* of B and also this decomposition is called the *structure decomposition* of B. For simplicity, if $e \in B_\alpha$, then we write $E(e) = B_\alpha$ and for $e, f \in B$, if $E(e) = E(e)E(f)$, we denote $E(e) \leq E(f)$. Obviously, $E(e) = E(f)$ if and only if $E(e) \leq E(f)$ and $E(f) \leq E(e)$.

The following result will be used in the sequel.

Lemma 1.3. [3] *Let S be an abundant semigroup. Then the following statements are equivalent:*

(1) *S is a PI-semigroup and E a right normal band;*
(2) *S satisfies the identity $xyz = yxz$.*

2. Some characterizations for PI-rpp semigroups

In this section, we always assume that S is a rpp semigroup satisfying the permutation identity: $x_1x_2 \cdots x_n = x_{p_1}x_{p_2} \cdots x_{p_n}$.

Lemma 2.1. *E is a normal band.*

Proof. This is straightforwards. In fact, it follows directly from the proof in [7]. □

Lemma 2.2. *For all $e, f \in E$ and $a \in S$, we have*

(1) $efa = eafa^*$;
(2) $eaf = eaef$.

Proof. Because $(p_1p_2\cdots p_n)$ is a nontrivial permutation of $(12\cdots n)$, there exists a positive integer $k(< n)$ such that $p_i = i$ when $1 \leq i < k$ but $p_k \neq k$. Obviously, $k < p_k$.

(1) Take $x_i = e$ when $i < k$, $x_i = f$ if $k \leq i < p_k$, $x_{p_k} = a$ and $x_i = a^*$ otherwise. Then $e(x_1x_2\cdots x_n)a^* = efa$. On the other hand, by Lemma 2.1, $e(x_{p_1}x_{p_2}\cdots x_{p_n})a^* = eafa^*$. Thus by hypothesis, we obtain $efa = eafa^*$.

(2) Now, let $x_i = e$ when $1 \leq i < p_k$, $x_{p_k} = a$ otherwise $x_i = f$. Then $e(x_1x_2\cdots x_n)f = eaf$. But $e(x_{p_1}x_{p_2}\cdots x_{p_n})f = eaef$ or $eafef$. By Lemma 2.1, we deduce that

$$eafef = eaa^*fef = eaa^*eff = eaef.$$

Hence, $eaf = eaef$. □

Lemma 2.3. *For all $a, b \in S$ and $f \in E$, if $a = bf$, then $E(a^*) \leq E(f)$.*

Proof. If $a = bf$ then $a = af$. Now, by $a\mathcal{L}^*a^*$, we have $a^* = a^*f$ and thus $E(a^*) \leq E(f)$. □

We now define a relation η on S as follows: $a, b \in S$:

$$a\eta b \text{ if and only if for some } f \in E(b^*),\ a = bf.$$

Lemma 2.4. (1) *η is a congruence on S preserving $\mathcal{L}^*$-classes;*
(2) *$\eta \cap \mathcal{L}^* = i_s$ (the identical mapping on S).*

Proof. (1) We first show that η is an equivalence relation. Certainly, $x\eta x$ for every $x \in S$, since $x = xx^*$. Let $a, b \in S$ with $a\eta b$. Then for some $f \in E(b^*)$, $a = bf$. By Lemma 2.3, $E(a^*) \leq E(f) = E(b^*)$. It follows that $a^*b^* \in E(a^*)$ and that $a\eta b$ implies $E(a^*) \leq E(b^*)$. Since

$$a(a^*b^*) = ab^* = bfb^* = bb^*fb^* = bb^* = b,$$

we have $b\eta a$, and thus η satisfies the symmetric relation. On the other hand, by the proof above, $b\eta a$ permits $E(b^*) \leq E(a^*)$. Now $E(a^*) = E(b^*)$. Therefore, from $a\eta b$, we can deduce that $E(a^*) = E(b^*)$. To prove the transitivity, we let $x, y, z \in S$ with $x\eta y, y\eta z$. Then we have $E(x^*) = E(y^*) = E(z^*)$. By the hypothesis, there are $f, g \in E(z^*)$ such that $x = yf$ and $y = zg$. Accordingly, $x = zgf$. Notice that $gf \in E(z^*)$, we hence get $x\eta z$.

Now let $x, y, z \in S$ and $x\eta y$. Then there exists $f \in E(y^*)$ such that $x = yf$. Obviously, $zx = zyf = zy(zy)^*f$. It follows from Lemma 2.3 that $E((zx)^*) \leq E((zy)^*)$ and $E((zx)^*) \leq E(f)$, and so $(zx)^*y^* \in E((zx)^*)$. But $x = yf$, this implies that $y = xy^*$. Now $zy = zxy^* = zx(zx)^*y^*$. Hence $zx\eta zy$. On the other hand, by

$$\begin{aligned} xz &= yfz = yy^*fz = yy^*zfz^* \quad \text{(by Lemma 2.2)} \\ &= yz(fz^*) = yz(yz)^*fz^*, \end{aligned}$$

we can deduce that $E((xz)^*) \leq E((yz)^*)$ and $E((xz)^*) \leq E(fz^*)$ by Lemma 2.2. A similar argument for $y = xy^*$, we can show that $E((yz)^*) \leq E((xz)^*)$. Thus $E((xz)^*) = E((yz)^*)$. This means that $(yz)^*fz^* \in E((yz)^*)$. Therefore $xz\eta yz$. Consequently, η is a congruence on S.

It remains to prove that η preserves $\mathcal{L}^*$-classes. To see this, we let $a, b \in S$ with $a\mathcal{L}^*b$. For $x\eta, y\eta \in (S/\eta)^1$ (where $x, y \in S^1$), if $(ax)\eta = (ay)\eta$, then there exists $f \in E((ay)^*)$ such that $ax = ayf$. By $a\mathcal{L}^*b$, we have $bx = byf$. Notice that $\mathcal{L}^*$ is a right congruence, $ay\mathcal{L}^*by$ and so $E((ay)^*) = E((by)^*)$. Now $(bx)\eta = (by)\eta$. From this equality and its dual, we can deduce that $a\eta\mathcal{L}^*(S/\eta)b\eta$, as required.

(2) Let $(a, b) \in \eta \cap \mathcal{L}^*$. Then for some $f \in E(b^*)$, $a = bf$. Since $a\mathcal{L}^*b$, we have $a^*\mathcal{L}b^*$. Hence

$$a = ab^* = bfb^* = bb^*fb^* = bb^* = b.$$

Therefore $\eta \cap \mathcal{L}^* = i_s$. □

Lemma 2.5. *S/η is a PI-rpp semigroup and $E(S/\eta)$ a left normal band.*

Proof. By Lemma 2.4, S/η is a rpp semigroup. Since S is a PI-semigroup, we easily check that S/η is a PI-semigroup. Notice that $a\eta \in E(S/\eta)$ implies $a \in E(S)$ and that $\eta \cap (E \times E) = \mathcal{R}$, $E(S/\eta) = E/\mathcal{R}$ and furthermore, it is a left normal band. □

Lemma 2.6. *If E is a left normal band, then S satisfies the identity $xyz = xzy$.*

Proof. Let k have the same meaning as that in the proof of Lemma 2.2. For all $x, y, z \in S$, take $x_k = y$, $x_{p_k} = z$ and $x_i = x^*$ otherwise. Then $x^*(x_1x_2\cdots x_n)z^* = x^*yx^*zx^*z^*$, $x^*yzx^*z^*$, x^*yx^*z or x^*yz. Since by hypothesis and Lemma 2.2, we have

$$\begin{aligned} x^*yx^*zx^*z^* &= x^*yx^*z = x^*yy^*x^*z = x^*yy^*x^*y^*z \\ &= x^*yy^*z = x^*yzz^* = x^*yzx^*z^*, \end{aligned}$$

Thus, we obtain that $x^*(x_1x_2\cdots x_n)z^* = x^*yz$. On the other hand, we have $x^*(x_{p_1}x_{p_2}\cdots x_{p_n})z^* = x^*zx^*yx^*z^*$, $x^*zyx^*z^*$, $x^*zx^*yz^*$ or x^*zyz^*. But by the hypothesis and Lemma 2.2, we have

$$\begin{aligned} x^*zx^*yz^* &= x^*zx^*yx^*z^* = x^*zz^*x^*yx^*z^* \\ &= x^*zz^*x^*z^*yx^*z^* = x^*zx^*z^*yx^*z^* \\ &= x^*zz^*yx^*z^* = x^*zyx^*z^* = x^*zyz^* \\ &= x^*zz^*yy^*z^* = x^*zz^*yy^*z^*y^* = x^*zz^*yy^* = x^*zy. \end{aligned}$$

Hence, $x^*(x_{p_1}x_{p_2}\cdots x_{p_n})z^* = x^*zy$. Thus, we obtain $x^*yz = x^*zy$ and thereby, we have

$$xyz = xx^*yz = xx^*zy = xzy. \quad \square$$

Now we arrive at the main result of this section.

Theorem 2.7. *Let T be a rpp semigroup with set E of idempotents. Denote by λ_a the inner left translation of T associated with $a(\in T)$. Then the following statements are equivalent:*

(1) *T satisfies permutation identities;*
(2) *T satisfies the identity: $xyzw = xzyw$;*
(3) *for all $e \in E$, eTe is a commutative semigroup and λ_e is a homomorphism;*
(4) *for all $e \in E$, eT satisfies the identity: $xyz = yxz$, and λ_e is a homomorphism.*

Proof. (1) $\Rightarrow$ (2) Let T be a PI-semigroup. For all $x, y, z, w \in T$, by Lemmas 2.4-2.6 $(xyzw)\eta = (xzyw)\eta$. Then for some $f \in E((xzyw)^*)$, $xyzw = xzywf$. Furthermore, we also have

$$\begin{aligned} xyzw &= xyzww^* = xzywfw^* \\ &= xzyw(xzyw)^*fw^* \\ &= xzyw(xzyw)^*f(xzyw)^*w^* \quad \text{(by Lemma 2.1)} \\ &= xzyw(xzyw)^*w^* = xzyw \end{aligned}$$

(2) $\Rightarrow$ (3) Suppose that (2) holds. Let $e \in E$. Then, for all $x, y \in eTe$, we have $x = ex = xe$ and $y = ey = ye$. Thus

$$xy = exye = eyxe = yx$$

This means that eTe is a commutative semigroup. On the other hand, since

$$\lambda_e(x) \cdot \lambda_e(y) = exey = eexy = exy = \lambda_e(xy),$$

λ_e is a homomorphism of T into itself. Therefore (3) holds.

(3) $\Rightarrow$ (4) Suppose that (3) holds. It remains to prove the first part. For all $x, y, z \in eT$, we have $x = ex$, $y = ey$ and $z = ez$. Thus, since λ_e is a homomorphism, we have

$$\begin{aligned} xyz &= exeyez = (exe)(eye)z \\ &= (eye)(exe)z = (ey)(ex)(ez) = yxz. \end{aligned}$$

(4) $\Rightarrow$ (2) Let $x, y, z, w \in T$. Then

$$\begin{aligned} xyzw &= x(x^*yzw) = x(x^*y)(x^*z)(x^*w) \\ &= x(x^*z)(x^*y)(x^*w) = x(x^*zyw) = xzyw. \end{aligned}$$

(2) $\Rightarrow$ (1) This part is trivial. □

3. Weak-spined product

In this section, we introduce the concept of weak-spined product. A characterization of PI-rpp semigroups in terms of weak-spined product will be hence given.

Let T be a rpp semigroup whose idempotents form a subsemigroup E. Let Y be the structure semilattice of E such that $E = \bigcup_{\alpha \in Y} E_\alpha$ is a structure decomposition of E. Now let B be a right normal band with structure semilattice Y, having the structure decomposition $\mathcal{S}(Y; B_\alpha; \varphi_{\alpha,\beta})$. In this case, each B_α is a right zero semigroup (see [6, Corollary IV 5.18]). For $a \in T$, if $a^* \in E_\alpha$ we denote $a^+ = \alpha$. Take $M = \{(a, x) \in T \times B : x \in B_{a^+}\}$. Define a multiplication on M as follows:

$$(a, x) \circ (b, y) = (ab, y\varphi_{b^+,(ab)^+}), \text{ i.e. } = (ab, zy),$$

where $z \in B_{(ab)^+}$. Notice that $ab = abb^*$ implies $(ab)^* = (ab)^* b^*$, we have $(ab)^+ = (ab)^+ b^+$, i.e. $(ab)^+ \leq b^+$ ($\leq$ is the natural order) in Y. This means that $y\varphi_{b^+,(ab)^+} \in B_{(ab)^+}$. Accordingly, $(ab, y\varphi_{b^+,(ab)^+}) \in M$. Thus "$\circ$" is well-defined, and with respect to "$\circ$", M is closed. In addition, we have the following result.

Lemma 3.1. *$(M, \circ)$ is a rpp semigroup.*

Proof. At first, we shall show that $(M, \circ)$ satisfies the associative law. Let $(a, x), (b, y), (c, z) \in M$. Then, by the above statement, we can show that, $(abc)^+ \leq (bc)^+ \leq c^+$. Hence

$$\begin{aligned}((a,x)\circ(b,y))\circ(c,z) &= (ab, y\varphi_{b^+,(ab)^+})\circ(c,z)\\ &= (abc, z\varphi_{c^+,(abc)^+})\\ &= (a,x)\circ(bc, z\varphi_{c^+,(bc)^+})\\ &= (a,x)\circ((b,y)\circ(c,z)).\end{aligned}$$

Thus $(M, \circ)$ is indeed a semigroup.

It remains to prove that for all $(a, x) \in M$, $(a, x)\mathcal{L}^*(a^*, x)$. To see this, let $(b, y), (c, z) \in M^1$. If $(a, x)(b, y) = (a, x)(c, z)$, i.e. $(ab, y\varphi_{b^+,(ab)^+}) = (ac, z\varphi_{c^+,(ac)^+})$, then $ab = ac$ and $y\varphi_{b^+,(ab)^+} = z\varphi_{c^+,(ac)^+}$. From the above equality, we deduce that $a^*b = a^*c$. Consider that $ab\mathcal{L}^*a^*b$. It is easy to see that $(ab)^*\mathcal{L}(a^*b)^*$ so that $(ab)^+ = (a^*b)^+$. But, $(a^*b)^+ = (a^*c)^+$. Thus $y\varphi_{b^+,(a^*b)^+} = z\varphi_{c^+,(a^*c)^+}$. Accordingly, we have

$$\begin{aligned}(a^*, x)(b, y) = (a^*b, y\varphi_{b^+,(a^*b)^+}) &= (a^*c, z\varphi_{c^+,(a^*c)^+})\\ &= (a^*, x)(c, z).\end{aligned}$$

This equality, together with the fact that

$$(a, x)(a^*, x) = (aa^*, x\varphi_{(a^*)^+,(aa^*)^+}) = (a, x),$$

implies that $(a, x)\mathcal{L}^*(a^*, x)$, as required. □

Definition 3.2. We call $(M, \circ)$ above the *weak-spined product* of T and B, and denote it by $WS(T, B)$.

Lemma 3.3. *If T satisfies the identity $xyz = xzy$, then $WS(T, B)$ satisfies the identity $xyzw = xzyw$.*

Proof. Let $(a, i), (b, j), (c, k), (d, l) \in WS(T, B)$. Then

$$\begin{aligned}(a,i)(b,j)(c,k)(d,l) &= (abcd, l\varphi_{d^+,(abcd)^+})\\ &= (acbd, l\varphi_{d^+,(acbd)^+})\\ &= (a,i)(c,k)(b,j)(d,l).\end{aligned}$$

Thus $WS(T, B)$ satisfies the identity $xyzw = xzyw$. □

In virtue of the weak-spined product, the PI-rpp semigroups can be described as follows:

Theorem 3.4. *A rpp semigroup is a PI-semigroup if and only if it is isomorphic to the weak-spined product of a rpp semigroup satisfying the identity $xyz = xzy$, and a right normal band.*

Proof. By Lemma 3.3, it suffices to prove the "only if" part. Suppose that S is a PI-rpp semigroup with normal band E. Then by Lemma 2.5 and 2.6, S/η is a rpp semigroup satisfying the identity: $xyz = xzy$. Let Y be the structure semilattice of E and $E = \bigcup_{\alpha \in Y} E_\alpha$ be the structure decomposition of E. Then $E/\mathcal{L} = \bigcup_{\alpha \in Y} E_\alpha/\mathcal{L}$ is a right normal band and the structure decomposition of $E/\mathcal{L}$. Since $a\eta \in E(S/\eta)$ implies $a \in E$ and $\eta \cap (E \times E) = \mathcal{R}$, we can easily know that $E(S/\eta) = E/\mathcal{R} = \bigcup_{\alpha \in Y} E_\alpha/\mathcal{R}$ and further Y is the structure semilattice of $E(S/\eta)$. Thus we can consider the weak-spined product $WS(S/\eta, E/\mathcal{L})$.

Now, we define a mapping θ:

$$S \to WS(S/\eta, E/\mathcal{L}), x \to (x\eta, \overline{x^*}),$$

where $\overline{x^*}$ is the congruence class of E containing x mode $\mathcal{L}$. In order to prove the theorem, we only need to show that θ is an isomorphism. By Lemma 2.4, θ is well-defined and injective. Take any element $(a, \overline{x}) \in WS(S/\eta, E/\mathcal{L})$, where $x \in E$. Since $a \in S/\eta$, there exists $s \in S$ such that $a = s\eta$. By the definition of $WS(S/\eta, E/\mathcal{L})$, $s^*\mathcal{D}^E x$, that is, $x \in E(s^*)$. Hence $a = (sx)\eta$ and $sx\mathcal{L}^* s^* x\mathcal{L}x$. Thus $(sx)\theta = (a, \overline{x})$. This means that θ is onto. For all $s, t \in S$, by $st = (st)t^*$, $(st)^* = (st)^* t^*$, and $\overline{(st)^*} \in B_{((st)\eta)^+}$ by Lemma 2.4. Now

$$\begin{aligned} \theta(st) &= ((st)\eta, \overline{(st)^*}) = ((st)\eta, \overline{(st)^* t^*}) \\ &= (s\eta, \overline{s^*})(t\eta, \overline{t^*}) = \theta(s)\theta(t). \end{aligned}$$

Hence θ is a homomorphism. Summing up, θ is an isomorphism of S onto $WS(S/\eta, E/\mathcal{L})$. The proof is completed. □

4. Special cases

Let S be a PI-rpp semigroup. By Lemma 2.1 E is a band. Obviously, E satisfies permutation identities. According to Yamada [7] and Yamada and Kimura [8], there are exactly four varieties of bands defined by permutation identities. They are

$\mathcal{SL} = [ab = ba]$ the variety of semilattices,

$\mathcal{LN} = [abc = acb]$ the variety of left normal bands,

$\mathcal{RN} = [abc = bac]$ the variety of right normal bands, and

$\mathcal{N} = [abcd = acbd]$ the variety of normal bands.

In what follows, we consider PI-rpp semigroups whose idempotent bands contain in $\mathcal{SL}$, $\mathcal{LN}$ or $\mathcal{RN}$.

Proposition 4.1. *Let S be a PI-rpp semigroup. Then the following statements are equivalent:*

(1) *E is a left normal band;*
(2) *S satisfies the identity: $xyz = xzy$;*
(3) *for all $e \in E$, eS is a commutative subsemigroup of S.*

Proof. (1) $\Rightarrow$ (2) This follows from Lemma 2.6.

(2) $\Rightarrow$ (3) Suppose that (2) holds. Then, for $e \in E$ and $x, y \in eS$, we have $x = ex$ and $y = ey$. Thus

$$xy = exy = eyx = yx,$$

and this establishes (3).

(3) $\Rightarrow$ (1) Suppose that (3) holds. Then for all $e, f, g \in E$

$$efg = e(ef)g = (ef)(eg) = (eg)(ef) = egf$$

so that

$$(ef)^2 = e \cdot ef \cdot f = ef \in E.$$

Summing up the above, E is a left normal band. □

Lemma 4.2. *Let S be a PI-rpp semigroup. If E is a right normal band then $ES = \{es : e \in E, s \in S\}$ is an abundant subsemigroup of S.*

Proof. Obviously, ES is a subsemigroup of S. Notice that $E \subseteq ES$ and that S is a rpp semigroup, it is easy to see that ES is a rpp semigroup. It remains to prove that ES is an lpp semigroup. To see this, let $e \in E$ and $a = ea \in S$. Let $x, y \in (ES)^1$ with $xa = ya$. Since for some $f \in E$, $x = fx$, we have

$$\begin{aligned} xa &= fxaa^* = faxa^* \text{ (by Theorem 2.7) } = faa^*xa^* \\ &= fa^*axa^* = a^*fa^*axa^* \text{ (by hypothesis)} \\ &= a^*a^*afxa^* = a^*axa^*. \end{aligned}$$

Similarly, $ya = a^*aya^*$. Now $a^*axa^* = a^*aya^*$. Thus by Lemma 1.1, we have $(a^*a)^*xa^* = (a^*a)^*ya^*$. But as $a^*aa^* = a^*a$, $(a^*a)^*a^* = (a^*a)^*$. Accordingly, by the hypothesis, we have

$$\begin{aligned} (a^*a)^*xa^* &= (a^*a)^*fxa^* = f(a^*a)^*fxa^* \\ &= ffx(a^*a)^*a^* = x(a^*a)^*. \end{aligned}$$

Similarly, $(a^*a)^*ya^* = y(a^*a)^*$. Thus, $x(a^*a)^* = y(a^*a)^*$. On the other hand, since E is a right normal band, we have

$$\begin{aligned}(a^*a)^*a &= (a^*a)^*ea = e(a^*a)^*ea = e(a^*a)^*aa^* \\ &= e(a^*a)^*a^*aa^* \text{ (by Theorem 2.7) } = ea^*a(a^*a)^*a^* \\ &= ea^*aa^* = eaa^*a^* = a.\end{aligned}$$

Summing up the above facts, by Corollary 1.2, we know that $a\mathcal{R}^*(a^*a)^*$. Therefore ES is an lpp semigroup. □

The following result is immediate from Lemmas 1.3 and 4.2 since $E \subseteq ES$.

Proposition 4.3. *Let S be a PI-rpp semigroup. Then the following statements are equivalent:*

(1) *E is a right normal band;*
(2) *ES is an abundant semigroup satisfying the identity: $xyz = yxz$.*

By using Theorem 4.1 and 4.3, we can easily obtain the following corollary.

Corollary 4.4. *Let S be a PI-rpp semigroup. Then the following statements are equivalent:*

(1) *E is a semilattice;*
(2) *ES is a commutative abundant semigroup.*

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