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The Multiplicity of -2 as an Eigenvalue of the Distance Matrix of Graphs

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Abstract. Let $D(G)$ be the distance matrix and let $\lambda_1(D(G)) \geq \cdots \geq \lambda_n(D(G))$ be the corresponding eigenvalues of a connected graph G. Let $m_\lambda(D)$ denote the multiplicity of the eigenvalue λ of the distance matrix D of G. In this paper, we characterize the graphs with $m_{-2}(D(G)) = n - i$, where $i = 1, 2, 3, 4$. Furthermore, we show that both S_n^+ and $S_{a,b}$ $(a + b = n - 2)$ are determined by their D-spectrum..

1. Introduction

In this paper we consider simple and connected graphs. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. Let $A(G) = (a_{ij})_{n\times n}$ be the $(0, 1)$-adjacency matrix of G, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. The diameter of G, denoted by d or $d(G)$, is the maximum distance between any pair of vertices of G. The induced subgraph $G[X]$ is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X.

Let $D(G) = (d_{ij})_{n\times n}$ be the distance matrix of a connected graph G, where $d_{ij} = d_G(v_i, v_j)$ is defined to be the length of shortest path between v_i and v_j. The polynomial $P_D(\lambda) = det(\lambda I - D(G))$ is defined as the distance characteristic polynomial of the graph G. Let $\lambda_1(D(G)) \geq \cdots \geq \lambda_n(D(G))$ be the distance spectrum of G. Let $m_\lambda(D)$ denote the multiplicity of the eigenvalue λ of the distance matrix D of G. As usual, let K_n, P_n and $K_{n_1,n_2,\cdots,n_k}$, where $\sum_{i=1}^{k} n_i = n$, denote the complete graph, the path and the complete k-partite graph with order n, respectively. The complete product $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2. Let $K_{s,t}^r = K_r \vee (K_s \cup K_t)$ with $r \geq 1$. A block of G is a maximal connected subgraph of G that has no cut-vertex. A graph G is a clique tree if each block of G is a clique. Let S_n^+ denote the graph obtained from $K_{1,n-1}$ by adding an edge and let $S_{a,b}$ $(a+b = n-2)$ denote the double star obtained by adding a pendent vertices to one end vertex of P_2 and b pendent vertices to the other.

The distance matrix of a connected graph has been studied extensively. Lin, Liu and Lu [4] showed that the distance matrix of a clique tree is non-singular. Moreover, they also proved that the distance matrix of a clique tree has exactly one positive D-eigenvalue. In addition, they determined the extremal graphs with maximum and minimum distance energy among all clique trees.

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Two graphs are said to be D-cospectral if they have the same distance spectrum. A graph G is said to be determined by the D-spectra if there is no other nonisomorphic graph D-cospectral to G. Lin et al.[3] proved that the complete bipartite graph and the complete split $K_a \vee K_b^c$ graph are determined by their D-spectra, and conjectured that the complete k-partite graph is determined by its D-spectra. Jin and Zhang [8] confirmed the conjecture. Lin, Zhai and Gong [6] showed that the graph $K_{s,t}^r$ is determined by its D-spectra. Lin [5] showed that connected graphs with $\lambda_n(D) \geq -1 - \sqrt{2}$ are determined by their distance spectra.

Cámara and Haemers [7] characterized graphs with $m_{-1}(A(G)) = n - i$, where $i = 2, 3$, and they showed that the graphs are determined by their adjacency spectrum, as well. So far, there are not many results on the multiplicity of eigenvalues of the distance matrix. In this paper, we characterize the graphs with $m_{-2}(D(G)) = n - i$, where $i = 1, 2, 3, 4$. Furthermore, we show that S_n^+ and $S_{a,b}$ $(a + b = n - 2)$ are determined by their D-spectrum.

2. The multiplicity of -2

In this section, we characterize the graphs with $m_{-2}(D(G)) = n - i$, where $i = 1, 2, 3, 4$. Obviously, K_1 is the only graph with $m_{-2}(D(G)) = n - 1$. Since $m_{-2}(D(K_n)) = 0$, in the following we just consider the graphs with $m_{-2}(D(G)) \geq 1$.

Lemma 2.1. *[3]. Let G be a connected graph and D be the distance matrix of G. Then $\lambda_n(D) = -2$ with multiplicity $n - k$ if and only if G is a complete k-partite graph for $2 \leq k \leq n - 1$.*

Lemma 2.2. *[8]. Let $G = K_{n_1,n_2,\cdots,n_k}$ be a complete k-partite graph with $\sum_{i=1}^{k} n_i = n$. Then G is determined by its D-spectrum.*

Theorem 2.3. *Let G be a graph on n vertices and $D(G)$ be its distance matrix. If $m_{-2}(D(G)) = n - 2$, then $G \cong K_{n_1,n_2}$, and therefore G is determined by its D-spectrum.*

Proof. Suppose that G has eigenvalue -2 with multiplicity $n-2$, then $2I + D(G)$ has rank 2. If $diam(G) \geq 3$, then $2I + D(P_4)$ is a principal submatrix of $2I + D(G)$, and $rank(2I + D(G)) \geq rank(2I + D(P_4)) = 4$, a contradiction. Since $G \ncong K_n$, $diam(G) = 2$. If G has two nonadjacent vertices with different neighbors, then G must contain H which is a triangle with one pendant edge or C_5 as an induced subgraph. However, $rank(2I + D(H)) = 4$ and $rank(2I + D(C_5)) = 5$, i.e., $rank(2I + D(G)) \geq 4$, a contradiction. Therefore, any two nonadjacent vertices of G have the same neighbors, which means that G is a complete multipartite graph. By the sufficiency of Lemma 2.1, we have that -2 is the least distance eigenvalue of G. Note that $m_{-2} = n-2$, then by the necessity of Lemma 2.1 we know that $G \cong K_{n_1,n_2}$. Clearly, G is determined by its D-spectrum by Lemma 2.2. □

Theorem 2.4. *Let G be a graph on n vertices and $D(G)$ be its distance matrix. If $m_{-2}(D(G)) = n - 3$, then $G \cong K_{n_1,n_2,n_3}$, and therefore G is determined by its D-spectrum.*

Proof. Suppose that G has eigenvalue -2 with multiplicity $n-3$, then $2I + D(G)$ has rank 3. If $diam(G) \geq 3$, then $2I + D(P_4)$ is a principal submatrix of $2I + D(G)$, and $rank(2I + D(G)) \geq rank(2I + D(P_4)) = 4$, a contradiction. Since $G \ncong K_n$, $diam(G) = 2$. If G has two nonadjacent vertices with different neighbors, then G must contain H which is a triangle with one pendant edge or C_5 as an induced subgraph. However, $rank(2I + D(H)) = 4$ and $rank(2I + D(C_5)) = 5$, i.e., $rank(2I + D(G)) \geq 4$, a contradiction. Therefore, any two nonadjacent vertices of G have the same neighbors, which means that G is a complete multipartite graph. By the sufficiency of Lemma 2.1, we have that -2 is the least distance eigenvalue of G. Note that $m_{-2} = n-3$, then by the necessity of Lemma 2.1 we know that $G \cong K_{n_1,n_2,n_3}$. By Lemma 2.2, G is determined by its D-spectrum. □

Let $K_{b_1,b_2,b_3}^{a_1,a_2,a_3}$ be the graph depicted in Fig 1.

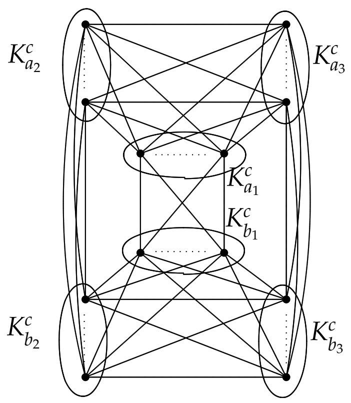

Fig. 1 The graph $K^{a_1,a_2,a_2}_{b_1,b_2,b_3}$.

The inertia of the matrix M is the triple of integers $(n_+(M), n_0(M), n_-(M))$, where $n_+(M)$, $n_0(M)$ and $n_-(M)$ denote the number of positive, 0 and negative eigenvalues of M, respectively. If $det(M) = 0$, then we call M singular; otherwise, we call M non-singular.

Lemma 2.5. *[2]. Let A be a real symmetric $n \times n$ matrix, partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

If A_{11} is square and nonsingular, then $In(A) = In(A_{11}) + In(A_{22} - A_{21}A_{11}^{-1}A_{12})$, where $In(A)$ denotes the inertia of A.

Lemma 2.6. *Let G be a connected graph with order n and diameter 2. Let $D(G)$ be its distance matrix. If $m_{-2}(D(G)) = n-4$, then $G \cong K_{n_1,n_2,n_3,n_4}$, where $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$ or $G \cong K^{a_1,a_2,a_2}_{b_1,b_2,b_3}$, where $a_1, a_2, a_3 \geq 1$, at least one of $\{b_1, b_2, b_3\}$ greater than 0 and $\sum_{i=1}^{3} a_i = n - \sum_{i=1}^{3} b_i$.*

Proof. $2I + D(G)$ is a symmetric matrix with rank 4, and we can assume that

$$2I + D(G) = \begin{pmatrix} A_1 & X \\ X^T & A_2 \end{pmatrix},$$

where A_1 is a nonsingular 4×4 matrix.

Note that $rank(2I + D(G)) = rank(A_1) = 4$. Then we get that $A_2 = X^T A_1^{-1} X$ by Lemma 2.5, where each column $\mathbf{x}$ of X satisfies $\mathbf{x}^T A_1^{-1}\mathbf{x} = 2$ and for any different columns $\mathbf{x}_i$ and $\mathbf{x}_j$ of X satisfy $\mathbf{x}_i^T A_1^{-1}\mathbf{x}_j$ =1 or 2. There are only two cases for which $rank(A_1) = 4$:

when $A_1 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$, in which case $A_1^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{pmatrix}$;

when $A_1 = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}$, in which case $A_1^{-1} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & 1 & -\frac{1}{3} & -\frac{4}{3} \end{pmatrix}$.

In the first case, the possible columns of X are $(1\ 1\ 1\ 2)^T$, $(1\ 1\ 2\ 1)^T$, $(1\ 2\ 1\ 1)^T$ and $(2\ 1\ 1\ 1)^T$. In the second case, the possible columns of X are $(2\ 1\ 1\ 2)^T$, $(1\ 2\ 1\ 2)^T$, $(1\ 1\ 2\ 1)^T$, $(2\ 2\ 1\ 2)^T$, $(2\ 1\ 2\ 1)^T$ and $(1\ 2\ 2\ 1)^T$. Combining with $A_2 = X^T A_1^{-1} X$, we can obtain the following two possibilities for $2I + D(G)$:

$$2I + D(G) = \begin{pmatrix} 2J_{n_1\times n_1} & J_{n_1\times n_2} & J_{n_1\times n_3} & J_{n_1\times n_4} \\ J_{n_2\times n_1} & 2J_{n_2\times n_2} & J_{n_2\times n_3} & J_{n_2\times n_4} \\ J_{n_3\times n_1} & J_{n_3\times n_2} & 2J_{n_3\times n_3} & J_{n_3\times n_4} \\ J_{n_4\times n_1} & J_{n_4\times n_2} & J_{n_4\times n_3} & 2J_{n_4\times n_4} \end{pmatrix}$$

or

$$2I + D(G) = \begin{pmatrix} 2J_{a_1\times a_1} & J_{a_1\times a_2} & J_{a_1\times a_3} & J_{a_1\times b_1} & 2J_{a_1\times b_2} & 2J_{a_1\times b_3} \\ J_{a_2\times a_1} & 2J_{a_2\times a_2} & J_{a_2\times a_3} & 2J_{a_2\times b_1} & J_{a_2\times b_2} & 2J_{a_2\times b_3} \\ J_{a_3\times a_1} & J_{a_3\times a_2} & 2J_{a_3\times a_3} & 2J_{a_3\times b_1} & 2J_{a_3\times b_2} & J_{a_3\times b_3} \\ J_{b_1\times a_1} & 2J_{b_1\times a_2} & 2J_{b_1\times a_3} & 2J_{b_1\times b_1} & J_{b_1\times b_2} & J_{b_1\times b_3} \\ 2J_{b_2\times a_1} & J_{b_2\times a_2} & 2J_{b_2\times a_3} & J_{b_2\times b_1} & 2J_{b_2\times b_2} & J_{b_2\times b_3} \\ 2J_{b_3\times a_1} & 2J_{b_3\times a_2} & J_{b_3\times a_3} & J_{b_3\times b_1} & J_{b_3\times b_2} & 2J_{b_3\times b_3} \end{pmatrix}.$$

Therefore, $G \cong K_{n_1,n_2,n_3,n_4}$ or $G \cong K_{b_1,b_2,b_3}^{a_1,a_2,a_3}$. □

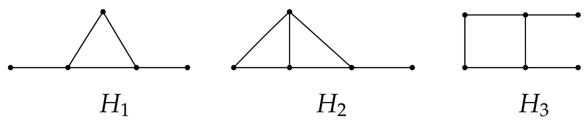

Fig. 2 The forbidden subgraphs H_1, H_2, H_3.

Let K_{n_1,n_2}^{s,t_1,t_2} denote the graph depicted in Fig 3.

Lemma 2.7. *Let G be a connected graph with order $n \geq 5$ and diameter 3. Let $D(G)$ be its distance matrix. If $m_{-2}(D(G)) = n - 4$, then $G \cong K_{n_1,n_2}^{s,t_1,t_2}$, where $s + t_1 + t_2 = n - n_1 - n_2$, $s, t_2, n_1, n_2 > 0$ and $t_1 \geq 0$.*

Proof. Suppose that G has eigenvalue -2 with multiplicity $n - 4$, then $rank(2I + D(G)) = 4$. Since $d(G) = 3$, G must contain $P_4 = v_1v_2v_3v_4$ as an induced subgraph and $N_{v_1} \cap N_{v_4} = \emptyset$. Then we can obtain the following claims.

Claim 1. For any vertex $v \in V(G)\backslash V(P_4)$, $u \in N_{v_1} \cup N_{v_2} \cup N_{v_3} \cup N_{v_4}$.

Otherwise, $2I + D(G)$ must contain a principal submatrix of the type B, where $b_i = 2$ or 3, $i = 1, 2, 3, 4$. Obviously, there are 16 possibilities of B, however, the rank of B is always greater than 4, a contradiction.

$$B = \begin{pmatrix} 2 & 1 & 2 & 3 & b_1 \\ 1 & 2 & 1 & 2 & b_2 \\ 2 & 1 & 2 & 1 & b_3 \\ 3 & 2 & 1 & 2 & b_4 \\ b_1 & b_2 & b_3 & b_4 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 2 & a \\ 1 & 1 & 2 & a & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & a_1 \\ 3 & 2 & 1 & 2 & a_2 \\ 1 & 2 & a_1 & a_2 & 2 \end{pmatrix}.$$

Claim 2. For any vertex $u \in N_{v_1}\backslash\{v_2\}$, $uv_2 \notin E(G)$.

Otherwise, $2I + D(G)$ must contain B_1, where $a = 2$ or 3, or $2I + D(H_2)$ as a principal submatrix. Note that $det(B_1) = -12a + 12 \neq 0$ and $rank(2I + D(H_2)) = 5$, a contradiction.

Similarly, we obtain that $uv_3 \notin E(G)$ for any vertex $u \in N_{v_4}\backslash\{v_3\}$.

Claim 3. $N_{v_1} \subseteq N_{v_3}$ and $N_{v_4} \subseteq N_{v_2}$.

By the symmetry, we just consider $N_{v_1} \subseteq N_{v_3}$. The result is trivial when $|N_{v_1}| = 1$. We suppose that there exists a vertex $u \in N_{v_1}$ but $u \notin N_{v_3}$, then $2I + D(G)$ must contain B_2, where $a_i = 2$ or 3. Note that $det(B_2) = 4(a_1 - 1)(2a_1 - 3a_2 + 4) \neq 0$, thus $N_{v_1} \subseteq N_{v_3}$. Similarly, $N_{v_4} \subseteq N_{v_2}$.

Claim 4. N_{v_1} and N_{v_4} are independent sets.

By the symmetry, we just consider N_{v_1}. The result is trivial when $|N_{v_1}| = 1$ or 2. Now suppose that there are two vertices $u, v \in N_{v_1}\backslash\{v_2\}$ such that $uv \in E(G)$, then H_2 must be an induced subgraph of G since $u, v \in N_{v_3}$ by Claim 3. Note that $rank(2I + D(H_2)) = 5$, a contradiction. Thus N_{v_1} and N_{v_4} are independent sets.

$$B_3=\begin{pmatrix} 2 & 1 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 & 2 \end{pmatrix}, B_4=\begin{pmatrix} 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & a \\ 3 & 2 & 1 & 2 & 3 \\ 1 & 2 & a & 3 & 2 \end{pmatrix}, B_5=\begin{pmatrix} 2 & 1 & 2 & 3 & 2 \\ 1 & 2 & 1 & 2 & a \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 2 & 1 \\ 2 & a & 2 & 1 & 2 \end{pmatrix},$$

$$B_6=\begin{pmatrix} 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & a_1 \\ 2 & 1 & 2 & a_2 & 2 \\ 1 & 2 & a_2 & 2 & a_3 \\ 2 & a_1 & 2 & a_3 & 2 \end{pmatrix}, B_7=\begin{pmatrix} 2 & 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 & 2 & a \\ 2 & 1 & 2 & 1 & a & 2 \end{pmatrix}, B_8=\begin{pmatrix} 2 & 1 & 3 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 2 & a_1 \\ 3 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & a_2 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 2 & a_1 & 1 & a_2 & 1 & 2 \end{pmatrix}.$$

Claim 5. $G[N_{v_1}, N_{v_4}]$ is a complete bipartite graph.

Otherwise, if $|N_{v_1}| = 1$ or $|N_{v_4}| = 1$, then B_3 must be a principal submatrix of $2I + D(G)$ since $d = 3$ and Claim 4. However, $rank(2I + D(G)) \geq rank(B_3) = 5$, a contradiction. If $|N_{v_1}| > 1$ and $|N_{v_4}| > 1$, then the following fact must hold:

Fact. For any vertex $v \in N_{v_1}$ *(or* N_{v_4}*)*, $N_v \cap N_{v_4}$ *(or* N_{v_1}*)* $\neq \emptyset$.

Otherwise, without loss of generality, suppose that there exists a vertex $u \in N_{v_1}$ such that $N_u \cap N_{v_4} = \emptyset$, then $d(u, v_4) = 3$, $d(u, v_3) = 2$ *or* 3 and $d(u, v_1) = 1$, $d(u, v_2) = 2$ by Claim 4. Thus $2I + D(G)$ contains B_4 as a principal submatrix, where $a = 2$ or 3. However, $rank(2I + D(G)) \geq rank(B_4) = 5$ ($a = 2$ *or* 3), a contradiction.

If $G[N_{v_1}, N_{v_4}]$ is not a complete bipartite graph, then there exists a vertex $u \in N_{v_1}$ and a vertex $v \in N_{v_4}$ such that $uv \notin E(G)$.

Case 1. $u = v_2$ or $v = v_3$.

Without loss of generality, we just consider $u = v_2$. Then $v \neq v_3$ and $d(v, v_1) = 2$ by Fact. Thus $2I + D(G)$ contains B_5 as a principal submatrix, where $a = 2$ or 3. However, $rank(B_5) = 5$ ($a = 2$ *or* 3), a contradiction.

Case 2. $u \neq v_2$ and $v \neq v_3$.

$d(v, v_1) = 2$ by Fact and $d(u, v_2) = d(v, v_3) = 2$ by Claim 4.

Subcase 2.1 $\{uv_3, vv_2\} \not\subseteq E(G)$.

We consider the vertices v_1, v_2, v_3, u, v, then $2I + D(G)$ contains a principal submatrix of the type B_6, where $a_i = 2$ or 3, $i = 1, 2, 3$. Obviously, there are 8 possibilities of B_6, however, the rank of B_6 is always greater than 4, a contradiction.

$$B_9=\begin{pmatrix} 2 & 1 & 2 & 3 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 & a_1 & a_2 \\ 2 & 1 & 2 & a_1 & 2 & 1 \\ 2 & 1 & 2 & a_2 & 1 & 2 \end{pmatrix}, B_{10}=\begin{pmatrix} 2 & 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 & 2 & a \\ 1 & 2 & 1 & 2 & a & 2 \end{pmatrix}, B_{11}=\begin{pmatrix} 2 & 1 & 2 & 3 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

Subcase 2.2 $\{uv_3, vv_2\} \subseteq E(G)$.

Then $2I + D(G)$ contains a principal submatrix of the type B_7, where $a = 2$ or 3. Obviously, there are 2 possibilities of B_7, however, the rank of B_7 is always greater than 4, a contradiction.

Subcase 2.3 $uv_3 \in E(G)$ or $vv_2 \in E(G)$.

Without loss of generality, suppose that $uv_3 \in E(G)$. By Fact, there must exist a vertex $u' \in N_{v_1} \backslash \{u, v_2\}$ such that $u'v \in E(G)$. We consider vertice v_1, v_2, v_4, u, u', v, then $2I + D(G)$ contains a principal submatrix of the type B_8, where $a_i = 2$ or 3 ($i = 1, 2$). Obviously, there are 4 possibilities of B_8, however, the rank of B_8 is always greater than 4, a contradiction.

Claim 6. For any vertex $w \in V(G) \backslash \{v_1, v_4, N_{v_1}, N_{v_4}\}$, w is either the vertex of N_{v_2} or the vertex of N_{v_3}.

Otherwise, H_1 must be an induced subgraph of G. Note that $rank(2I + D(H_1)) = 5$, a contradiction.

Let $N = N_{v_2} \backslash N_{v_4}$ and $N' = N_{v_3} \backslash N_{v_1}$.

Claim 7. N and N' are independent sets

Without loss of generality, we only consider N. Note that $uv_1 \notin E(G)$ for every vertex $u \in N \backslash \{v_1\}$ by Claim 4. If $|N| = 1$ or 2, then the assertion holds obviously. If $|N| \geq 3$, suppose that there exist two vertices $v, w \in N \backslash \{v_1\}$ such that $vw \in E(G)$, then $2I + D(G)$ contains B_9 as a principal submatrix, where $a_i = 2$ or 3 ($i = 1, 2$). However, $rank(B_9) = 6$, a contradiction.

Claim 8. $G[N_{v_1}, N]$ and $G[N_{v_4}, N']$ are complete bipartite graphs.

We just consider $G[N_{v_1}, N]$ by the symmetry. If $|N| = 1$ or $|N_{v_1}| = 1$, then the assertion holds by the definition of N and N_{v_1}. Now suppose that $|N_{v_1}| > 1$ and $|N| > 1$. Let $u \in N\backslash v_1$ and $v \in N_{v_1}\backslash v_2$. By the definition of N, we have $u \notin N_{v_4}$, i.e., $d(u, v_4) \neq 1$. And $d(u, v_4) \neq 2$ since H_1 is a forbidden subgraph of G. Thus $d(u, v_4) = 3$. Then $2I + D(G)$ contains a principal submatrix of the type B_{10}, where $a = 2$ or 3. However, the rank of B_{10} is always greater than 4, a contradiction.

Claim 9. $N \cup N_{v_4}$ and $N' \cup N_{v_1}$ are independent set.

Without loss of generality, we only consider $N \cup N_{v_4}$. If $|N| = 1$ or $|N_{v_4}| = 1$, then the claim holds obviously. If $|N| > 1$ and $|N_{v_4}| > 1$, let $u \in N\backslash v_1$ and $v \in N_{v_4}\backslash v_3$, then $2I + D(G)$ contains B_{11} as a principal submatrix. However, $rank(B_{11}) = 6$, a contradiction.

Let $N' = W_1 \cup W_2$, where $W_1 \subset N'$ such that $N_u \cap N \neq \emptyset$ for $u \in W_1$. Since $d = 3$, $W_2 \neq \emptyset$.

Claim 10. $G[W_1, N]$ is a complete bipartite graph.

Otherwise, H_3 must be an induced subgraph of G, however, $rank(2I + D(G)) \geq rank(2I + D(H_3)) = 6$, a contradiction.

By the above discussion, we have $G \cong K^{s,t_1,t_2}_{n_1,n_2}$, where $t_1 \geq 0$ and $t_2 \geq 1$. □

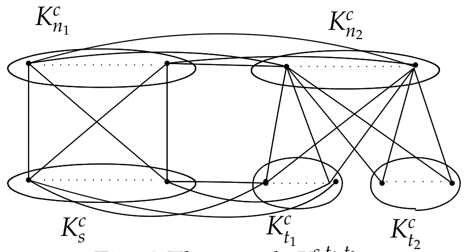

Fig. 3 The graph $K^{s,t_1,t_2}_{n_1,n_2}$.

Theorem 2.8. *Let G be a graph on n vertices and $D(G)$ be its distance matrix. If $m_{-2}(D(G)) = n - 4$, then $G \cong K_{n_1,n_2,n_3,n_4}$, $K^{a_1,a_2,a_3}_{b_1,b_2,b_3}$, or $K^{s,t_1,t_2}_{c_1,c_2}$, where $a_1, a_2, a_3 \geq 1$, at least one of $\{b_1, b_2, b_3\}$ greater than 0 and $\sum_{i=1}^{3} a_i = n - \sum_{i=1}^{3} b_i$ and $s + t_1 + t_2 = n - c_1 - c_2$, $s, t_2, c_1, c_2 > 0$ and $t_1 \geq 0$.*

Proof. $m_{-2}(D(G)) = n - 4$, then we have $rank(2I + D(G)) = 4$. If $d(G) \geq 4$, then G must contain P_5 as an induced subgraph, however, $rank(2I + D(G)) \geq rank(2I + D(P_5)) = 5$. This contradiction shows that $d(G) \leq 3$. Thus we can complete the proof by Lemmas 2.6 and 2.7. □

3. Further results on *D*-cospectral

Lemma 3.1. *[1]. For $n \times n$ matrices A and B, the following are equivalent:*

1. *A and B are cospectral;*
2. *A and B have the same characteristic polynomial;*
3. *$tr(A^i) = tr(B^i)$ for $i = 1, 2, \cdots, n$.*

Remark. Note that $tr(D^2(G)) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d^2_{ij}$. Then $tr(D^2(K^{a_1,a_2,a_3}_{b_1,b_2,b_3})) = 4n^2 - 4n - 6|E(K^{a_1,a_2,a_3}_{b_1,b_2,b_3})|$ and $tr(D^2(K^{s,t_1,t_2}_{n_1,n_2})) = 4n^2 - 4n - 6|E(K^{s,t_1,t_2}_{n_1,n_2})| + 10st_2$.

Lemma 3.2. *Let $G = K^{a_1,a_2,a_3}_{b_1,b_2,b_3}$, where $a_1, a_2, a_3 \geq 1$, at least one of $\{b_1, b_2, b_3\}$ greater than 0 and $\sum_{i=1}^{3} a_i = n - \sum_{i=1}^{3} b_i$. Then*

$$P_{D(G)}(\lambda) = (\lambda + 2)^{n-4}(\lambda^4 - (2n - 8)\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3),$$

where $\alpha_1 = 3a_1a_2 + 3a_1a_3 + 3a_1b_1 + 3a_2a_3 + 3a_2b_2 + 3a_3b_3 + 3b_1b_2 + 3b_1b_3 + 3b_2b_3 - 12n + 24$, $\alpha_2 = -4a_1a_2a_3 + 2a_1a_2b_3 + 2a_1a_3b_2 + 2a_1b_2b_3 + 2a_2a_3b_1 + 2a_2b_1b_3 + 2a_3b_1b_2 - 4b_1b_2b_3 + 12a_1a_2 + 12a_1a_3 + 12a_1b_1 + 12a_2a_3 +$

$12a_2b_2+12a_3b_3+12b_1b_2+12b_1b_3+12b_2b_3-24n+32$ *and* $\alpha_3=-3a_1a_2a_3b_1-3a_1a_2a_3b_2-3a_1a_2a_3b_3-3a_1a_2b_1b_3-3a_1a_2b_2b_3-3a_1a_3b_1b_2-3a_1a_3b_2b_3-3a_1b_1b_2b_3-3a_2a_3b_1b_2-3a_2a_3b_1b_3-3a_2b_1b_2b_3-3a_3b_1b_2b_3-8a_1a_2a_3+4a_1a_2b_3+4a_1a_3b_2+4a_1b_2b_3+4a_2a_3b_1+4a_2b_1b_3+4a_3b_1b_2-8b_1b_2b_3+12a_1a_2+12a_1a_3+12a_1b_1+12a_2a_3+12a_2b_2+12a_3b_3+12b_1b_2+12b_1b_3+12b_2b_3-16n+16$.

Proof. Note that

$$D(G)=\begin{pmatrix} 2J_{a_1}-2I_{a_1} & J_{a_1\times a_2} & J_{a_1\times a_3} & J_{a_1\times b_1} & 2J_{a_1\times b_2} & 2J_{a_1\times b_3}\\ J_{a_2\times a_1} & 2J_{a_2}-2I_{a_2} & J_{a_2\times a_3} & 2J_{a_2\times b_1} & J_{a_2\times b_2} & 2J_{a_2\times b_3}\\ J_{a_3\times a_1} & J_{a_3\times a_2} & 2J_{a_3}-2I_{a_3} & 2J_{a_3\times b_1} & 2J_{a_3\times b_2} & J_{a_3\times b_3}\\ J_{b_1\times a_1} & 2J_{b_1\times a_2} & 2J_{b_1\times a_3} & 2J_{b_1}-2I_{b_1} & J_{b_1\times b_2} & J_{b_1\times b_3}\\ 2J_{b_2\times a_1} & J_{b_2\times a_2} & 2J_{b_2\times a_3} & J_{b_2\times b_1} & 2J_{b_2}-2I_{b_2} & J_{b_2\times b_3}\\ 2J_{b_3\times a_1} & 2J_{b_3\times a_2} & J_{b_3\times a_3} & J_{b_3\times b_1} & J_{b_3\times b_2} & 2J_{b_3}-2I_{b_3} \end{pmatrix},$$

then $det(\lambda I-D(G))=(\lambda+2)^{n-6}\cdot f(\lambda)$, where

$$f(\lambda)=\begin{vmatrix} \lambda-2a_1+2 & -a_2 & -a_3 & -b_1 & -2b_2 & -2b_3\\ -a_1 & \lambda-2a_2+2 & -a_3 & -2b_1 & -b_2 & -2b_3\\ -a_1 & -a_2 & \lambda-2a_3+2 & -2b_1 & -2b_2 & -b_3\\ -a_1 & -2a_2 & -2a_3 & \lambda-2b_1+2 & -b_2 & -b_3\\ -2a_1 & -a_2 & -2a_3 & -b_1 & \lambda-2b_2+2 & -b_3\\ -2a_1 & -2a_2 & -a_3 & -b_1 & -b_2 & \lambda-2b_3+2 \end{vmatrix}.$$

$$=(\lambda+2)^{(}\lambda^4-(2n-8)\lambda^3+\alpha_1\lambda^2+\alpha_2\lambda+\alpha_3),$$

where $\alpha_1=3a_1a_2+3a_1a_3+3a_1b_1+3a_2a_3+3a_2b_2+3a_3b_3+3b_1b_2+3b_1b_3+3b_2b_3-12n+24$, $\alpha_2=-4a_1a_2a_3+2a_1a_2b_3+2a_1a_3b_2+2a_1b_2b_3+2a_2a_3b_1+2a_2b_1b_3+2a_3b_1b_2-4b_1b_2b_3+12a_1a_2+12a_1a_3+12a_1b_1+12a_2a_3+12a_2b_2+12a_3b_3+12b_1b_2+12b_1b_3+12b_2b_3-24n+32$ and $\alpha_3=-3a_1a_2a_3b_1-3a_1a_2a_3b_2-3a_1a_2a_3b_3-3a_1a_2b_1b_3-3a_1a_2b_2b_3-3a_1a_3b_1b_2-3a_1a_3b_2b_3-3a_1b_1b_2b_3-3a_2a_3b_1b_2-3a_2a_3b_1b_3-3a_2b_1b_2b_3-3a_3b_1b_2b_3-8a_1a_2a_3+4a_1a_2b_3+4a_1a_3b_2+4a_1b_2b_3+4a_2a_3b_1+4a_2b_1b_3+4a_3b_1b_2-8b_1b_2b_3+12a_1a_2+12a_1a_3+12a_1b_1+12a_2a_3+12a_2b_2+12a_3b_3+12b_1b_2+12b_1b_3+12b_2b_3-16n+16$. □

Lemma 3.3. *Let* $G=K^{s,t_1,t_2}_{n_1,n_2}$, *where* s,t_2,n_1,n_2 *are positive integers* ,$t_1\geq 0$ *and* $s+t_1+t_2=n-n_1-n_2$. *Then*

$$P_{D(G)}(\lambda)=(\lambda+2)^{n-4}[\lambda^4-(2n-8)\lambda^3+\beta_1\lambda^2+\beta_2\lambda+\beta_3],$$

where $\beta_1=3sn_1+3st_1-5st_2+3n_1n_2+3t_1n_2+3t_2n_2-12n+24$, $\beta_2=8sn_1t_2+8st_1t_2+8st_2n_2+12sn_1+12st_1-20st_2+12n_1n_2+12t_1n_2+12t_2n_2-24n+32$ *and* $\beta_3=-12sn_1t_2n_2-12st_1t_2n_2+16sn_1t_2+16st_1t_2+16st_2n_2+12sn_1+12st_1-20st_2+12n_1n_2+12t_1n_2+12t_2n_2-16n+16$.

Proof. Note that

$$D(G)=\begin{pmatrix} 2J_{n_1}-2I_{n_1} & J_{n_1\times n_2} & J_{n_1\times s} & 2J_{n_1\times t_1} & 2J_{n_1\times t_2}\\ J_{n_2\times n_1} & 2J_{n_2}-2I_{n_2} & 2J_{n_2\times s} & J_{n_2\times t_1} & J_{n_2\times t_2}\\ J_{s\times n_1} & 2J_{s\times n_2} & 2J_s-2I_s & J_{s\times t_1} & 3J_{s\times t_2}\\ 2J_{t_1\times n_1} & J_{t_1\times n_2} & J_{t_1\times s} & 2J_{t_1}-2I_{t_1} & 2J_{t_1\times t_2}\\ 2J_{t_2\times n_1} & J_{t_2\times n_2} & 3J_{t_2\times s} & 2J_{t_2\times t_1} & 2J_{t_2}-2I_{t_2} \end{pmatrix},$$

then

$$det(\lambda I-D(G))=(\lambda+2)^{n-5}g(\lambda),$$

where

$$g(\lambda)=\begin{vmatrix} \lambda-(2n_1-2) & -n_2 & -s & -2t_1 & -2t_2\\ -n_1 & \lambda-(2n_2-2) & -2s & -t_1 & -t_2\\ -n_1 & -n_2 & \lambda-(2s-2) & -t_1 & -3t_2\\ -2n_1 & -n_2 & -s & \lambda-(2t_1-2) & -2t_2\\ -2n_1 & -n_2 & -3s & -2t_1 & \lambda-(2t_2-2) \end{vmatrix}.$$

Therefore, we can get the result through a simple calculation. □

Lemma 3.4. *Let $G = S_n^+$, then G is determined by its D-spectrum.*

Proof. By Lemma 3.2 we know that

$$P_{D(S_n^+)}(\lambda) = (\lambda + 2)^{n-4}(\lambda^4 - (2n-8)\lambda^3 + (-9n+24)\lambda^2 + (-10n+22)\lambda - 3n + 5).$$

Let H be a connected graph such that $Spec(D(H)) = Spec(D(S_n^+))$, then $H \cong K_{b_1,b_2,b_3}^{a_1,a_2,a_3}$ or $H \cong K_{n_1,n_2}^{s,t_1,t_2}$.

Case 1. $H \cong K_{b_1,b_2,b_3}^{a_1,a_2,a_3}$.

Since $tr(D^2(H)) = tr(D^2(S_n^+))$, we know that $|V(H)| = |E(H)| = n$, i.e., H must be a unicyclic graphs. Thus $a_1 = a_2 = a_3 = 1$ and $b_1 = n-3$, $b_2 = b_3 = 0$, i.e., $H \cong S_n^+$.

Case 2. $H \cong K_{n_1,n_2}^{s,t_1,t_2}$.

Since $tr(D^2(H)) = tr(D^2(S_n^+))$, we know that $3|E(K_{n_1,n_2}^{s,t_1,t_2})| - 5st_2 = 3n$. And by $P_{D(H)}(\lambda) = P_{D(S_n^+)}(\lambda)$, we get that $\beta_2 = 8st_2(n_1+n_2+t_1) + 4(3|E(K_{n_1,n_2}^{s,t_1,t_2})| - 5st_2) - 24n + 32 = -10n + 22$, then $4st_2(n_1+n_2+t_1) = n-5$ and $n-5$ is even. In particular, $g(-2) = -12st_2n_2(n_1+t_1)$, thus we get that $4st_2n_2(n_1+t_1) = n-3$. Note that $2(n-5) = n-5+n-3-2$, then we discover that $n-5 = 2st_2[n_1+n_2+t_1+n_2(n_1+t_1)] - 1$ which means that $n-5$ is odd, a contradiction.

Therefore, S_n^+ is determined by its D-spectrum. □

The following result is proved by Xue, Liu and Jia [9] in a different way.

Theorem 3.5. *Let $G = S_{a,b}$ $(a+b = n-2)$, then G is determined by its D-spectrum.*

Proof. By Lemma 3.3 we know that

$$P_{D(S_{a,b})}(\lambda) = (\lambda+2)^{n-4}(\lambda^4 - (2n-8)\lambda^3 + (-5ab - 9n + 21)\lambda^2 + (-4ab - 12n + 20)\lambda - 4n + 4).$$

Let H be a connected graph such that $Spec(D(H)) = Spec(D(S_{a,b}))$, then $H \cong K_{b_1,b_2,b_3}^{a_1,a_2,a_3}$ or $H \cong K_{n_1,n_2}^{s,t_1,t_2}$.

Case 1. $H \cong K_{b_1,b_2,b_3}^{a_1,a_2,a_3}$.

Since $tr(D^2(H)) = tr(D^2(S_{a,b}))$, we know that $|E(H)| = |E(S_{a,b})| - \frac{5}{3}ab$. We observe that $|E(S_{a,b})| = n-1$, but it is impossible that the number of edges of H less than $n-1$.

Case 2. $H \cong K_{n_1,n_2}^{s,t_1,t_2}$.

We first know that $|E(H)| = |E(S_{a,b})| + \frac{5}{3}(st_2 - ab)$ by $tr(D^2(H)) = tr(D^2(S_{a,b}))$. Since $P_{D(H}(\lambda) = P_{D(S_{a,b})}(\lambda)$ and $g(-2) = -12st_2n_2(n_1+t_1)$, we get that $st_2n_2(n_1+t_1) = ab$, If $n_2(n_1+t_1) > 1$, then $st_2 < ab$. Thus $|E(H)| < n-1$, a contradiction. So we have $n_2(n_1+t_1) = 1$, i.e., $n_1 = n_2 = 1$ and $t_1 = 0$. Thus $H \cong S_{s,t_2}$. Combining $st_2 = ab$ the result can be obtained easily.

Therefore, $S_{a,b}$ is determined by its D-spectrum. □

Problem Is any graph G with $m_{-2}(D(G)) = n-4$ determined by its D-spectrum?

References

[1] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectra? Linear Algebra Appl. 373 (2003) 241–272.

[2] E. V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, Linear Algebra Appl. 1 (1968) 73–81.

[3] H.Q. Lin, Y. Hong, J.F. Wang, J.L. Shu, On the distance spectrum of graphs, Linear Multilinear Algebra. 439 (2013) 1662–1669.

[4] H.Q. Lin, R.F. Liu, X.W. Lu, The inertia and energy of the distance matrix of a connected graph, Linear Algebra Appl. 467 (2015) 2939.

[5] H.Q. Lin, On the least distance eigenvalue and its applications on the distance spread, Discrete Mathematics 338 (2015) 868–874.

[6] H.Q. Lin, M.Q. Zhai, S.C. Gong, On graphs with at least three distance eigenvalues less than -1, Linear Algebra Appl. 458 (2014) 548–558.

[7] M. Cámara, W.H. Haemers, Spectral characterizations of almost complete graphs, Discrete Applied Mathematics 176 (2014) 19–23.

[8] Y.L. Jin, X.D. Zhang, Complete multipartite graphs are determined by their distance spectra, Linear Algebra Appl. 448 (2014) 285–291.

[9] J. Xue, R.F. Liu, H.C. Jia, On the Distance Spectrum of Trees, Filomat 30:6 (2016) 1559–1565.