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Numerical Semigroups in a Problem About Economic Incentives for Consumers

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Abstract. Motivated by a promotion to increase the number of musical downloads, we introduce the concept of C-incentive and show an algorithm that computes the smallest C-incentive containing a subset $X \subseteq \mathbb{N}$. On the other hand, in order to study C-incentives, we see that we can focus on numerical C-incentives. Then, we establish that the set formed by all numerical C-incentives is a Frobenius pseudo-variety and we show an algorithmic process to recurrently build such a pseudo-variety.

1. Introduction

A certain commercial music streaming service designs a new promotion for one month. Namely, depending on the demand of a song, the cost of the download is 5, 7, 9, or 11 cents. In addition, if the customer waits

- less than one hour between two downloads, then there is a discount of 3 cents for the second one;
- more than two hours between two downloads, then there is an additional charge of 2 cents for the second one.

For instance, suppose a customer buys a song for 7 cents, then thirty minutes later gets a discount of 3 cents when buying a 9 cent song. Moreover, four hours later, he has an additional charge of 2 cents when purchasing a 5 cent song; and so forth. Our purpose is to study the set $\mathcal{F}$ formed by the amounts that can appear in the customers' invoices at the end of the promotion.

It is clear that we can associate each customer with an odd finite length list $(x_1, x_2, \dots, x_n)$ such that $x_1, x_3, \dots, x_n \in \{5, 7, 9, 11\}$, $x_2, x_4, \dots, x_{n-1} \in \{-3, 0, 2\}$, and the invoice is $x_1 + x_2 + \cdots + x_n$. Thus, we have that $\mathcal{F} = \{x_1 + \cdots + x_n \mid n \text{ is an odd positive integer, } x_1, x_3, \dots, x_n \in \{5, 7, 9, 11\},\ x_2, x_4, \dots, x_{n-1} \in \{-3, 0, 2\}\} \cup \{0\}$.

In order to set out the above example in an abstract way, we give the following definition: if A, B are two non-empty subsets of $\mathbb{Z}$, then an (A, B)-*sequence* is an odd finite length list $(x_1, x_2, \dots, x_n)$, such that $x_1, x_3, \dots, x_n \in A$ and $x_2, x_4, \dots, x_{n-1} \in B$, or an empty list. (As usual, we denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of non-negative integers and the set of integers, respectively.)

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Let us take $|x| = x_1 + \cdots + x_n$, for all $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, and $|x| = 0$ if x is a empty list. We denote by $\mathrm{M}(A, B) = \{|x| \mid x \text{ is an } (A, B)\text{-sequence}\}$. Observe that, with this notation, $\mathcal{F} = \mathrm{M}(\{5, 7, 9, 11\}, \{-3, 0, 2\})$.

In the remainder of the introduction, we suppose that A is a non-empty finite subset of $\mathbb{N} \setminus \{0\}$, B is a finite subset of $\mathbb{Z}$ that contains the zero element, and $\min(A) + \min(B) \geq 0$.

We begin Section 2 showing that $\mathrm{M}(A, B)$ is a submonoid of $(\mathbb{N}, +)$. Moreover, we observe that (1) $A \subseteq \mathrm{M}(A, B)$; (2) if $b \in B$, then it could be that $b \notin \mathrm{M}(A, B)$; (3) if $x, y \in \mathrm{M}(A, B) \setminus \{0\}$, then $x + y + b \in \mathrm{M}(A, B)$ for all $b \in B$. This last fact leads us to give the concept of *C-incentive*: if C is a subset of $\mathbb{Z}$, then a C-incentive is a submonoid M of $(\mathbb{N}, +)$ such that $\{x + y\} + C \subseteq M$ for all $x, y \in M \setminus \{0\}$. Moreover, we see that $\mathrm{M}(A, B)$ is the smallest (with respect to inclusion) B-incentive, and $(B \setminus \{0\})$-incentive, containing A. In this way, following with our example, we have that $\mathcal{F}$ is the smallest $\{-3, 2\}$-incentive containing the set $\{5, 7, 9, 11\}$.

From this point to the end of the introduction, we suppose that C is a non-empty finite subset of $\mathbb{Z}$.

In Section 3 we approach the problem of computing the smallest C-incentive that contains a given set X of non-negative integers. In order to see that the above mentioned set exists, we establish the conditions that X has to satisfy with respect to C. Once that is done, we show an algorithm to compute the smallest C-incentive in the case that it exists.

Let S be a submonoid of $(\mathbb{N}, +)$. It is well known that S is a *numerical semigroup* if $\mathbb{N} \setminus S$ is a finite set or, equivalently, if $\gcd(S) = 1$. In this way, we say that M is a *numerical C-incentive* if M is a C-incentive such that $\gcd(M) = 1$. We denote by $\mathrm{I}(C) = \{M \mid M \text{ is a } C\text{-incentive}\}$ and by $\mathrm{NI}(C) = \{M \mid M \text{ is a numerical } C\text{-incentive}\}$. In Section 4 we show that $\mathrm{I}(C) \setminus \{\{0\}\} = \bigcup_{d \in D} \left\{ dS \mid S \in \mathrm{NI}\left(\frac{C}{d}\right) \right\}$, where D is the set of all positive divisors of $\gcd(C)$. Observe that this result points out that for studying C-incentives we can focus on numerical C-incentives.

In [11] the concept of Frobenius variety was introduced in order to unify several results that appeared in [1, 5, 16, 17]. Nevertheless, there exist families of numerical semigroups that are not Frobenius varieties. For instance, the family of numerical semigroups with maximal embedding dimension and fixed multiplicity (see [15]). The study, in [2], of this class of numerical semigroups led to the concept of m-variety. On the other hand, in [8] the concept of Frobenius pseudo-variety was introduced for generalizing the concepts of Frobenius variety and m-variety.

In Section 5 we prove that $\mathrm{NI}(C)$ is a Frobenius pseudo-variety. This fact, together with several results of [8], allows us to arrange the elements of $\mathrm{NI}(C)$ in a tree with root. Then, in Section 6 we give a procedure to recurrently build $\mathrm{NI}(C)$. In order to show it, we describe how the children of a vertex in the tree are computed.

In the end, in Section 7 we study the tree of numerical C-incentives containing a given set X. In particular, we determine when that tree is finite and, therefore, we can completely draw it.

To finish this introduction, we review some works that have led us to the study of C-incentives.

A (v, b, r, k)-configuration (see [3]) is a connected bipartite graph with v vertices on one side, each of them of degree r, and b vertices on the other side, each of them of degree k, and with no cycle of length 4. A (v, b, r, k)-configuration can also be seen as a combinatorial configuration (see [18]) with v points, b lines, r lines through every point and k points on every line. It is said that the 4-tuple (v, b, r, k) is configurable if a (v, b, r, k)-configuration exists. In [3] it was shown that, if (v, b, r, k) is configurable, then $vr = bk$ and, consequently, there exists d such that $v = d\frac{k}{\gcd\{r,k\}}$ and $b = d\frac{r}{\gcd\{r,k\}}$. The main result of [3] states that, if k, r are integers greater than or equal to 2, then $S_{(r,k)} = \left\{ d \in \mathbb{N} \,\middle|\, \left(d\frac{k}{\gcd\{r,k\}}, d\frac{r}{\gcd\{r,k\}}, r, k \right) \text{ is configurable} \right\}$ is a numerical semigroup. Moreover, in [18] it was proved that, if a configuration is balanced (that is, $r = k$), then $\{x + y - 1, x + y + 1\} \subseteq S_{(r,r)}$, for all $x, y \in S_{(r,r)} \setminus \{0\}$. Therefore, $S_{(r,r)}$ is a numerical $\{-1, 1\}$-incentive.

Let us observe that all the submonoids of $(\mathbb{N}, +)$ (in particular, numerical semigroups) are $\{0\}$-incentives. Thus, the results in this work can be considered as generalizations of known facts in the numerical semigroups theory. On the other hand, several particular cases of C-incentives arise as a characterization of the family of numerical semigroups that is the solution of a certain problem. For instance, C-incentives with C equal to $\{1\}$, $\{-1\}$, and $\{-1, 1\}$ are studied in [7], [12], and [9], respectively. However, in [13] the authors work directly with the definition of C-bracelet (monoids and numerical semigroups), that is just the same as C-incentive for $C \subseteq \mathbb{N}$. Finally, the families of C-incentives when $C = [-\beta, \alpha] \cap \mathbb{Z}$, for $\alpha, \beta \in \mathbb{N}$, are studied in [10] without make any reference to the concept of C-incentive. In this way, this work can be seen as an unification of the main results contained in those previous papers.

2. First results

In this section, A is a non-empty finite set of positive integers, B is a finite subset of $\mathbb{Z}$ such that $0 \in B$, and, moreover, we suppose that $\min(A) + \min(B) \geq 0$.

Throughout this paper, if n is an odd positive integer, $x_1, x_3, \ldots, x_n \in A$, and $x_2, x_4, \ldots, x_{n-1} \in B$, then $(x_1, x_2, \ldots, x_n)$ is an (A, B)-*sequence*. Moreover, the empty list is an (A, B)-sequence too.

Let $\mathrm{M}(A, B) = \{|x| \mid x \text{ is an } (A, B)\text{-sequence}\}$, where $|x| = x_1 + \cdots + x_n$ if $x = (x_1, \ldots, x_n)$ and $|x| = 0$ if x is the empty list. Our first purpose is to show that $\mathrm{M}(A, B)$ is a submonoid of $(\mathbb{N}, +)$. The following result has an immediate proof.

Lemma 2.1. *If $(x_1, \ldots, x_n)$ is an (A, B)-sequence and $n \geq 3$, then $(x_3, \ldots, x_n)$ is an (A, B)-sequence.*

Lemma 2.2. *If $(x_1, \ldots, x_n)$ is an (A, B)-sequence and $n \geq 1$, then we have that $|(x_1, \ldots, x_n)| \in \mathbb{N} \setminus \{0\}$.*

Proof. Let $m = |(x_1, \ldots, x_n)|$. By induction over n, we are going to prove that $m \in \mathbb{N} \setminus \{0\}$. First, if $n = 1$, then $m = x_1 \in A \subseteq \mathbb{N} \setminus \{0\}$. Now, let us suppose that $n \geq 3$ (remember that n is odd). Since $x_1 \in A$, $x_2 \in B$, and $\min(A) + \min(B) \geq 0$, then $x_1 + x_2 \in \mathbb{N}$. Now, by Lemma 2.1, we have that $|(x_3, \ldots, x_n)| \in \mathbb{N} \setminus \{0\}$. Therefore, $m = |(x_1, \ldots, x_n)| = x_1 + x_2 + |(x_3, \ldots, x_n)| \in \mathbb{N} \setminus \{0\}$. □

Let us observe that $A \subseteq \mathrm{M}(A, B)$. However, if $b \in B$, then it is possible that $b \notin \mathrm{M}(A, B)$. Despite this situation, we have the next result.

Lemma 2.3. *If $s, t \in \mathrm{M}(A, B) \setminus \{0\}$ and $b \in B$, then $s + t + b \in \mathrm{M}(A, B)$.*

Proof. If $s, t \in \mathrm{M}(A, B) \setminus \{0\}$, then there exist two (A, B)-sequences, $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, such that $|(x_1, \ldots, x_n)| = s$ and $|(y_1, \ldots, y_n)| = t$. Obviously, $(x_1, \ldots, x_n, b, y_1, \ldots, y_n)$ is an (A, B)-sequence and, moreover, $|(x_1, \ldots, x_n, b, y_1, \ldots, y_n)| = s + t + b$. □

Let us observe that, having in mind the above lemma, then the condition $0 \in B$ allows us to assure that $\mathrm{M}(A, B)$ is closed under addition. Thus, as a direct consequence of Lemmas 2.2 and 2.3, and the definition of $\mathrm{M}(A, B)$, we can establish the announced result.

Proposition 2.4. *$\mathrm{M}(A, B)$ is a submonoid of $(\mathbb{N}, +)$.*

The previous results lead us to give the following definition.

Definition 2.5. *Let C be a subset of $\mathbb{Z}$. We say that a submonoid M of $(\mathbb{N}, +)$ is a* C-incentive *if it fulfils that $\{s + t\} + C \subseteq M$ for all $s, t \in M \setminus \{0\}$.*

We are now ready to prove the main result of this section.

Theorem 2.6. *$\mathrm{M}(A, B)$ is the smallest (with respect to inclusion) B-incentive containing A.*

Proof. By Lemma 2.3 and Proposition 2.4, $\mathrm{M}(A, B)$ is a B-incentive containing A. Let us see that $\mathrm{M}(A, B) \subseteq T$ for any T that is a B-incentive containing A. If $m \in \mathrm{M}(A, B) \setminus \{0\}$, then there exists an (A, B)-sequence $(x_1, \ldots, x_k)$ such that $|(x_1, \ldots, x_k)| = m$. By induction over k, we are going to show that $m \in T$. If $k = 1$, then $m = x_1 \in A \subseteq T$. Now, we can suppose that $k \geq 3$. By Lemma 2.1, we know that $|(x_3, \ldots, x_k)| \in T$. Moreover, since T is a B-incentive and $x_2 \in B$, we have that $x_1 + |(x_3, \ldots, x_k)| + x_2 \in T$, that is, $m = |(x_1, \ldots, x_k)| \in T$. □

The following result is easy to prove.

Proposition 2.7. *Let C be a subset of $\mathbb{Z}$ and let M be a submonoid of $(\mathbb{N}, +)$. Then M is a C-incentive if and only if M is a $(C \setminus \{0\})$-incentive.*

As a consequence of the above results, we have that $\mathrm{M}(A, B)$ is the smallest $(B \setminus \{0\})$-incentive containing A. Moreover, carrying on the example of the introduction, we get that $\mathcal{F}$ is the smallest $\{-3, 2\}$-incentive containing the set $\{5, 7, 9, 11\}$.

Remark 2.8. *The sets B and C play similar but different roles in this paper. In effect, we have talked about B-incentives and C-incentives and they are subsets of* $\mathbb{Z}$. *However, we have imposed the condition* $0 \in B$. *Why? On the one hand, because we want to ensure that* $\mathrm{M}(A, B)$ *is closed under addition. On the other hand, because we are sure that M is a submonoid of* $(\mathbb{N}, +)$ *when we say that M is a C-incentive. Therefore, we do not need additional conditions on C.*

In the next section we study how to compute the smallest C-incentive that contains a given set of positive integers. Now, to finish this section, we give a result that allows us to decide whether or not a submonoid of $(\mathbb{N}, +)$ is a C-incentive.

If Y is a non-empty subset of $\mathbb{N}$, then we denote by $\langle Y \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by Y, that is, $\langle Y \rangle = \{\lambda_1 y_1 + \cdots + \lambda_n y_n \mid n \in \mathbb{N} \setminus \{0\},\ y_1, \ldots, y_n \in Y,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$. Thus, if $M = \langle Y \rangle$, then we say that M is generated by Y or, equivalently, that Y is a *system of generators of* M. Moreover, if $M \neq \langle \tilde{Y} \rangle$ for all $\tilde{Y} \subsetneq Y$, then we say that Y is a *minimal system of generators of* M. The following result is [14, Corollary 2.8].

Lemma 2.9. *Let M be a submonoid of* $(\mathbb{N}, +)$. *Then M has a unique minimal system of generators. In addition, such a system is finite.*

If M is a submonoid of $(\mathbb{N}, +)$, then we denote by $\mathrm{msg}(M)$ the minimal system of generators of M. It is easy to show (see [14, Lemma 2.3]) that $\mathrm{msg}(M) = M^* \setminus (M^* + M^*)$ (as usual, $M^* = M \setminus \{0\}$).

Proposition 2.10. *Let C be a non-empty subset of* $\mathbb{Z}$ *and let M be a submonoid of* $(\mathbb{N}, +)$ *generated by the set of positive integers* $\{n_1, \ldots, n_p\}$. *Then M is a C-incentive if and only if* $\{n_i + n_j\} + C \subseteq M$ *for all* $i, j \in \{1, \ldots, p\}$.

Proof. The necessary condition is trivial. In order to see the sufficient condition, let $x, y \in M \setminus \{0\}$ and $c \in C$. By the comment above this proposition, we know that there exist $i, j \in \{1, \ldots, p\}$ and $s, t \in M$ such that $x = n_i + s$ and $y = n_j + t$. Thereby, $x + y + c = (n_i + n_j + c) + s + t \in M$. Therefore, M is a C-incentive. □

Let us see an example to illustrate the previous proposition.

Example 2.11. *We have that* $\{3, 7, 8\} + \{3, 7, 8\} + \{-3, 2\} \subseteq \langle 3, 7, 8 \rangle$. *Consequently, by Proposition 2.10, we can assert that* $\langle 3, 7, 8 \rangle$ *is a* $\{-3, 2\}$*-incentive.*

3. An algorithm for finding the smallest *C*-incentive containing a given set of positive integers

Let C be a subset of $\mathbb{Z}$. We say that $X \subseteq \mathbb{N}$ is a *C-admissible set* if there exists at least one C-incentive containing it. We begin this section by characterizing the C-admissible sets. Then, if X is a C-admissible set, we show that there exists the smallest C-incentive containing it. Finally, we give an algorithm to compute it. First of all, let us observe that sometimes there is not any C-incentive containing X, such as it is shown in the following example.

Example 3.1. *Let us see that there does not exist any* $\{-4\}$*-incentive containing the set* $\{3\}$. *In fact, by contradiction, let us suppose that M is a* $\{-4\}$*-incentive containing* $\{3\}$. *Then* $2 = 3 + 3 - 4 \in M$. *Therefore,* $1 = 2 + 3 - 4 \in M$. *Consequently,* $-2 = 1 + 1 + -4 \in M$, *which is false.*

In order to characterize the C-admissible sets, we need three lemmas. From now on, we are going to suppose that C is a non-empty finite subset of $\mathbb{Z}$ and we denote by $\theta(C) = -\min(C \cup \{0\})$.

Lemma 3.2. $S = \{0, \theta(C), \rightarrow\} = \{0\} \cup \{n \in \mathbb{N} \mid n \geq \theta(C)\}$ *is a C-incentive.*

Proof. It is clear that S is a submonoid of $(\mathbb{N}, +)$. Let $a, b \in S \setminus \{0\}$ and $c \in C$. Then $a + b + c \geq \theta(C)$ and, therefore, $a + b + c \in S$. □

Lemma 3.3. *If M is a C-incentive, then $M \subseteq \{0, \theta(C), \rightarrow\}$ or $M = \left\langle \frac{\theta(C)}{2} \right\rangle$.*

Proof. If $\theta(C) = 0$, then it is obvious that $M \subseteq \{0, \rightarrow\}$. Thereby, we can suppose that $\theta(C) = -c > 0$ for some $c \in C$. Let m be the least positive integer belonging to M. If $M \nsubseteq \{0, \theta(C), \rightarrow\}$, then $m < \theta(C)$. Since $m + m - \theta(C) < m$ and $m + m - \theta(C) = m + m + c \in M$, then $m + m - \theta(C) = 0$ and, therefore, $m = \frac{\theta(C)}{2}$. Consequently, $\left\langle \frac{\theta(C)}{2} \right\rangle \subseteq M$. Now, let us see that $M \subseteq \left\langle \frac{\theta(C)}{2} \right\rangle$. If it is not the case, then there exists $w = \min\left\{x \in M \mid x \not\equiv 0 \bmod \frac{\theta(C)}{2}\right\}$. However, $w - \frac{\theta(C)}{2} = w + \frac{\theta(C)}{2} - \theta(C) = w + m + c \in M$, contradicting the minimality of w. □

Lemma 3.4. *The monoid $T = \left\langle \frac{\theta(C)}{2} \right\rangle$ is a C-incentive if and only if $C \subseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2, -1\} \cup \mathbb{N}\right\}$.*

Proof. By Proposition 2.10, T is a C-incentive if and only if $\left\{\frac{\theta(C)}{2} + \frac{\theta(C)}{2}\right\} + C \subseteq T$. That is, T is a C-incentive if and only if, for each $c \in C$, there exists $k_c \in \mathbb{N}$ such that $\theta(C) + c = k_c \frac{\theta(C)}{2}$. From this equality, the conclusion is clear. □

Proposition 3.5. *Let X be a subset of $\mathbb{N}$. Then X is C-admissible if and only if either $X \subseteq \{0, \theta(C), \rightarrow\}$ or $X \subseteq \left\langle \frac{\theta(C)}{2} \right\rangle$ and $C \subseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2, -1\} \cup \mathbb{N}\right\}$.*

Proof. From Lemmas 3.3 and 3.4, we have the necessary condition. For the sufficient condition, we apply Lemmas 3.2 and 3.4. □

The next result has an immediate proof and, therefore, we omit it.

Lemma 3.6. *The intersection of C-incentives is a C-incentive.*

This lemma leads us to the following definition.

Definition 3.7. *Let X be a C-admissible set and let $\mathrm{L}_C(X)$ be the intersection of all C-incentives containing X. We say that $\mathrm{L}_C(X)$ is the* C-incentive generated by *X.*

As a consequence of Lemma 3.6, we have that $\mathrm{L}_C(X)$ is the smallest (with respect the inclusion) C-incentive containing X.

Let us denote by $\mathrm{I}(C) = \{M \mid M \text{ is a } C\text{-incentive}\}$.

Theorem 3.8. *With the above notation we have the following.*

1. *If $C \nsubseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2, -1\} \cup \mathbb{N}\right\}$, then $\mathrm{I}(C) = \{\mathrm{L}_C(X) \mid X \text{ is a finite subset of } \{0, \theta(C), \rightarrow\}\}$.*
2. *If $C \subseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2, -1\} \cup \mathbb{N}\right\}$, then $\mathrm{I}(C) = \left\{\mathrm{L}_C(X) \mid X \text{ is a finite subset of } \{0, \theta(C), \rightarrow\}\right\} \cup \left\langle \frac{\theta(C)}{2} \right\rangle$.*

Proof. Let us observe that, if $M \in \mathrm{I}(C)$, then M is a submonoid of $(\mathbb{N}, +)$ and, by Lemma 2.9, there exists a finite subset X of $\mathbb{N}$ such that $M = \langle X \rangle$. Thus, it is clear that $M = \mathrm{L}_C(X)$. Now, by Lemmas 3.3 and 3.4, we have that, if $X \nsubseteq \{0, \theta(C), \rightarrow\}$, then $X \subseteq \left\langle \frac{\theta(C)}{2} \right\rangle$, $C \subseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2, -1\} \cup \mathbb{N}\right\}$, and, consequently, $\mathrm{L}_C(X) = \left\langle \frac{\theta(C)}{2} \right\rangle$. □

Let us observe that $\emptyset$ is a C-admissible set and $\mathrm{L}_C(\emptyset) = \{0\}$. On the other hand, we have that X is a C-admissible set if and only if $X \setminus \{0\}$ is a C-admissible set, and that $\mathrm{L}_C(X) = \mathrm{L}_C(X \setminus \{0\})$. All these considerations allow us to focus on the computation of $\mathrm{L}_C(X)$ when X is a non-empty finite set of positive integers contained in $\{\theta(C), \rightarrow\}$.

Proposition 3.9. *Let $X = \{x_1, \ldots, x_t\}$ be a set of positive integers contained in $\{\theta(C), \rightarrow\}$ and let us suppose that $C = \{c_1, \ldots, c_q\}$. Then*

$$\mathrm{L}_C(X) = \left\{a_1x_1 + \cdots + a_tx_t + b_1c_1 + \cdots + b_qc_q \mid a_1, \ldots, a_t, b_1, \ldots, b_q \in \mathbb{N}, a_1 + \cdots + a_t > b_1 + \cdots + b_q\right\} \cup \{0\}.$$

Proof. Let $H = \{a_1x_1 + \cdots + a_tx_t + b_1c_1 + \cdots + b_qc_q \mid a_1, \ldots, a_t, b_1, \ldots, b_q \in \mathbb{N}$ and $a_1 + \cdots + a_t > b_1 + \cdots + b_q\} \cup \{0\}$. Since $a_1x_1 + \cdots + a_tx_t + b_1c_1 + \cdots + b_qc_q \geq (a_1 + \cdots + a_t)\theta(C) - (b_1 + \cdots + b_q)\theta(C) \geq \theta(C)$, then we have that $H \subseteq \mathbb{N}$.

It is easy to show that H is closed under addition, $0 \in H$, and that, if $x, y \in H \setminus \{0\}$, then $\{x + y\} + C \subseteq H$. Therefore, H is a C-incentive.

Now, it is obvious that $X \subseteq H$ and, consequently, $\mathrm{L}_C(X) \subseteq H$. Thus, in order to finish the proof, it is enough to show that $H \subseteq \mathrm{L}_C(X)$. Thereby, let $x = a_1x_1 + \cdots + a_tx_t + b_1c_1 + \cdots + b_qc_q \in H$ and let us apply induction over $b_1 + \cdots + b_q$ to prove that $x \in \mathrm{L}_C(X)$.

If $b_1 + \cdots + b_q = 0$, then $x = a_1x_1 + \cdots + a_tx_t \in \mathrm{L}_C(X)$. Thus, we can suppose that $b_1 + \cdots + b_q \geq 1$ and, consequently, $a_1 + \cdots + a_t \geq 2$. Thereby, there exist $j \in \{1, \ldots, q\}$ and $i \in \{1, \ldots, t\}$ such that $b_j \neq 0$ and $a_i \neq 0$. By hypothesis of induction, we have that $x - x_i - c_j \in \mathrm{L}_C(X)$. Moreover, since $a_1 + \cdots + a_t \geq 2$, we deduce that $x - x_i - c_j \neq 0$. Now, by applying that $\mathrm{L}_C(X)$ is a C-incentive, we have that $\{x - x_i - c_j\} + \{x_i\} + C \subseteq \mathrm{L}_C(X)$. Therefore, $x \in \mathrm{L}_C(X)$. $\square$

Let us illustrate the above results with several examples.

Example 3.10. *Let us compute* $\mathrm{L}_{\{-3,2\}}(\{5, 7, 9, 11\})$*. By Proposition 3.9, since* $\theta(\{-3, 2\}) = 3$ *and* $\{5, 7, 9, 11\} \subseteq \{3, \rightarrow\}$*, we have that* $\mathrm{L}_{\{-3,2\}}(\{5, 7, 9, 11\}) = \{a_15 + a_27 + a_39 + a_411 + b_1(-3) + b_22 \mid a_1, a_2, a_3, a_4, b_1, b_2 \in \mathbb{N}$ *and* $a_1 + a_2 + a_3 + a_4 > b_1 + b_2\} \cup \{0\} = \{0, 5, 7, 9, 10, 11, 12, 14, \rightarrow\} = \langle 5, 7, 9, 11, 13\rangle$.

Example 3.11. *In order to compute* $\mathrm{L}_{\{-4,6\}}(\{2, 8\})$*, observe that* $\theta(\{-4, 6\}) = 4$, $\{-4, 6\} \subseteq \{k\frac{4}{2} \mid k \in \{-2, -1\} \cup \mathbb{N}\}$*, and* $\{2, 8\} \subseteq \langle \frac{4}{2}\rangle = \langle 2\rangle$*. Now, by Proposition 3.5, we have that* $\{2, 8\}$ *is a* $\{-4, 6\}$*-admissible set and, therefore,* $\mathrm{L}_{\{-4,6\}}(\{2, 8\})$ *exists. Moreover, from Theorem 3.8, we conclude that* $\mathrm{L}_{\{-4,6\}}(\{2, 8\}) = \langle 2\rangle$.

Example 3.12. *From Proposition 3.5, we have that* $\{3\}$ *is not a* $\{-4, 6\}$*-admissible set and, consequently,* $\mathrm{L}_{\{-4,6\}}(\{3\})$ *does not exist.*

Example 3.13. *From Proposition 3.5, we have that* $\{2\}$ *is not a* $\{-4, 7\}$*-admissible set and, consequently,* $\mathrm{L}_{\{-4,7\}}(\{2\})$ *does not exist.*

Now we are ready to show the algorithm that allows us to compute $\mathrm{L}_C(X)$ if X is a non-empty finite set of positive integers contained in $\{\theta(C), \rightarrow\}$ (such as in Example 3.10). This algorithm provides us an alternative method to the one given in Proposition 3.9. Moreover, its validity and correctness is justified by Proposition 2.10.

Algorithm 3.14.

INPUT: A non-empty finite set $X \subseteq \{\theta(C), \rightarrow\}$.
OUTPUT: The minimal system of generators of $\mathrm{L}_C(X)$.
(1) $D = \emptyset$.
(2) $Y = \mathrm{msg}(\langle X\rangle)$.
(3) $E = \{s + t \mid s, t \in Y\} \setminus D$.
(4) $Z = Y \cup (\bigcup_{e \in E}\{e\} + C)$.
(5) If $\mathrm{msg}(\langle Z\rangle) = Y$*, then return* Y.
(6) Set $Y = \mathrm{msg}(\langle Z\rangle)$, $D = D \cup E$*, and go to (3).*

Let us illustrate the performance of this algorithm with an example.

Example 3.15. *Let us compute* $\mathrm{L}_{\{-3,2\}}(\{5, 7, 9, 11\})$.

- $D = \emptyset$.
- $Y = \{5, 7, 9, 11\}$.
- $E = \{10, 12, 14, 16, 18, 20, 22\}$.

- $Z = \{5, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24\}$, $\mathrm{msg}(\langle Z\rangle) = \{5, 7, 9, 11, 13\}$.
- $Y = \{5, 7, 9, 11, 13\}$, $D = \{10, 12, 14, 16, 18, 20, 22\}$.
- $E = \{24, 26\}$.
- $Z = \{5, 7, 9, 11, 13, 21, 23, 26, 28\}$, $\mathrm{msg}(\langle Z\rangle) = \{5, 7, 9, 11, 13\}$.
- $Y = \{5, 7, 9, 11, 13\}$.

Therefore, $\mathrm{L}_{\{-3,2\}}(\{5, 7, 9, 11\}) = \langle 5, 7, 9, 11, 13\rangle$.

Observe that the most complex process in Algorithm 3.14 is to compute $\mathrm{msg}(\langle Z\rangle)$, that is, the computation of the minimal system of generators of a monoid $M = \langle Z\rangle$ starting from any system of generators of it. For this purpose, we can use the `GAP` package `numericalsgps` (see [4]).

4. Numerical C-incentives

Let M be a C-incentive. We say that M is *numerical* (that is, M is a *numerical C-incentive*) if $\gcd(M) = 1$ (or, equivalently, if $\mathbb{N}\setminus M$ is a finite set). The purpose of this section is to show that, for the study of C-incentives, we can focus on numerical C-incentives.

In this section, we suppose that $C = \{c_1, \ldots, c_q\}$ is a non-empty subset of $\mathbb{Z}$ and write $\mathrm{NI(C)} = \{M \in \mathrm{I(C)} \mid M \text{ is numerical}\}$.

Proposition 4.1. *Let X be a non-empty subset of $\{\theta(C), \rightarrow\} \setminus \{0\}$. Then $\mathrm{L}_C(X)$ is a numerical semigroup if and only if* $\gcd(X \cup C) = 1$.

Proof. (Necessity.) Let us suppose that $\gcd(X \cup C) = d \neq 1$. Then it is clear that $M = \{kd \mid kd \geq \theta(C)\} \cup \{0\}$ is a C-incentive containing X and, therefore, $\mathrm{L}_C(X) \subseteq M$. Since $\mathbb{N} \setminus M$ is not a finite set, then $\mathbb{N} \setminus \mathrm{L}_C(X)$ is not finite and, consequently, $\mathrm{L}_C(X)$ is not a numerical semigroup.

(Sufficiency.) Let $H = X \cup (2X + C)$. It is clear that $H \subseteq \mathrm{L}_C(X)$. On the other hand, if $x \in X$, then $\gcd\{x, 2x + c_1, \ldots, 2x + c_q\} = \gcd\{x, c_1, \ldots, c_q\}$. Thereby, $\gcd(H) = 1$. Consequently, $\gcd(\mathrm{L}_C(X)) = 1$, that is, $\mathrm{L}_C(X)$ is a numerical semigroup. □

Corollary 4.2. *If* $\gcd(C) = 1$, *then* $\mathrm{I(C)} = \mathrm{NI(C)} \cup \{\{0\}\}$.

Proof. First of all, let us observe that, if $\gcd(C) = 1$ and we are in the case 2 of Theorem 3.8, then $\frac{\theta(C)}{2} = 1$. Therefore, $\langle \frac{\theta(C)}{2} \rangle = \mathbb{N}$, which is a numerical semigroup. In any other case, the conclusion follows from Proposition 4.1 and Theorem 3.8. □

Now we want to study the case $\gcd(C) \neq 1$. Firstly we need two lemmas.

Lemma 4.3. *Let M be a C-incentive such that $M \neq \{0\}$. Then* $\gcd(M)$ *divides* $\gcd(C)$.

Proof. Let $x \in M\setminus\{0\}$. Then $\{x, 2x+c_1, \ldots, 2x+c_q\} \subseteq M$ and, therefore, $\gcd(M) \mid \gcd\{x, 2x+c_1, \ldots, 2x+c_q\}$. Now, being that $\gcd\{x, 2x + c_1, \ldots, 2x + c_q\} = \gcd\{x, c_1, \ldots, c_q\}$ and $\gcd\{x, c_1, \ldots, c_q\} \mid \gcd\{c_1, \ldots, c_q\}$, we conclude that $\gcd(M) \mid \gcd(C)$. □

Lemma 4.4. *Let M be a submonoid of* $(\mathbb{N}, +)$ *such that $M \neq \{0\}$ and let $d = \gcd(M)$. Then M is a C-incentive if and only if $\frac{M}{d}$ is a $\frac{C}{d}$-incentive.*

Proof. (Necessity.) If $x, y \in \frac{M}{d} \setminus \{0\}$, then $dx, dy \in M \setminus \{0\}$. Since M is a C-incentive, then $\{dx + dy\} + C \subseteq M$. From Lemma 4.3, we know that $d \mid \gcd(C)$ and, consequently, $\{x + y\} + \frac{C}{d} \subseteq \frac{M}{d}$. Therefore, $\frac{M}{d}$ is a $\frac{C}{d}$-incentive.

(Sufficiency.) If $a, b \in M \setminus \{0\}$, then $\frac{a}{d}, \frac{b}{d} \in \frac{M}{d} \setminus \{0\}$. Since $\frac{M}{d}$ is a C-incentive, then $\{\frac{a}{d} + \frac{b}{d}\} + \frac{C}{d} \subseteq \frac{M}{d}$ and, therefore, $\{a + b\} + C \subseteq M$. In this way, M is a C-incentive. □

Theorem 4.5. *Let D be the set of all positive divisors of* $\gcd(C)$. *Then* $\mathrm{I}(C) \setminus \{\{0\}\} = \bigcup_{d\in D}\left\{dS \mid S \in \mathrm{NI}\left(\frac{C}{d}\right)\right\}$.

Proof. Let $M \in \mathrm{I}(C)$ such that $M \neq \{0\}$ and $\gcd(M) = d$. Then, by applying Lemmas 4.3 and 4.4, it is clear that $d \in D$ and $\frac{M}{d} \in \mathrm{NI}\left(\frac{C}{d}\right)$. For the other inclusion, by Lemma 4.4, if $d \in D$ and $S \in \mathrm{NI}\left(\frac{C}{d}\right)$, then $dS \in \mathrm{I}(C)$. □

Let us illustrate the content of the previous theorem with an example.

Example 4.6. *By Theorem 4.5, we have that* $\mathrm{I}(\{-4,6\}) = \{S \mid S \in \mathrm{NI}(\{-4,6\})\} \cup \{2S \mid S \in \mathrm{NI}(\{-2,3\})\} \cup \{\{0\}\}$. *Thus, in order to compute* $\mathrm{I}(\{-4,6\})$, *it is enough to calculate* $\mathrm{NI}(\{-4,6\})$ *and* $\mathrm{NI}(\{-2,3\})$.

We finish this section showing that, if we want to compute $\mathrm{L}_C(X)$, then we can focus on the case in which $\gcd(X \cup C) = 1$.

Lemma 4.7. *Let X be a set of positive integers such that* $\gcd(X \cup C) = d$. *Then X is C-admissible if and only if $\frac{X}{d}$ is $\frac{C}{d}$-admissible.*

Proof. It is a consequence of Proposition 3.5, having in mind both of the following facts.

1. $X \subseteq \{\theta(C), \to\}$ if and only if $\frac{X}{d} \subseteq \left\{\theta\left(\frac{C}{d}\right), \to\right\}$.
2. $X \subseteq \left\langle \frac{\theta(C)}{2} \right\rangle$ and $C \subseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2,-1\} \cup \mathbb{N}\right\}$ if and only if $\frac{X}{d} \subseteq \left\langle \frac{\theta\left(\frac{C}{d}\right)}{2} \right\rangle$ and $\frac{C}{d} \subseteq \left\{k\frac{\theta\left(\frac{C}{d}\right)}{2} \mid k \in \{-2,-1\} \cup \mathbb{N}\right\}$.

□

Proposition 4.8. *Let X be a C-admissible set such that* $\gcd(X \cup C) = d$. *Then $\frac{X}{d}$ is $\frac{C}{d}$-admissible and, moreover,* $\mathrm{L}_C(X) = d \cdot \mathrm{L}_{\frac{C}{d}}\left(\frac{X}{d}\right)$.

Proof. By Lemma 4.7 and Proposition 3.9, we have that, if $X \subseteq \{0, \theta(C), \to\}$, then $\mathrm{L}_C(X) = d \cdot \mathrm{L}_{\frac{C}{d}}\left(\frac{X}{d}\right)$. On the other hand, by Proposition 3.5, if $X \nsubseteq \{0, \theta(C), \to\}$, then $X \subseteq \left\langle \frac{\theta(C)}{2} \right\rangle$ and $C \subseteq \left\{k\frac{\theta(C)}{2} \mid k \in \{-2,-1\} \cup \mathbb{N}\right\}$. Thereby, by applying Theorem 3.8, we have that $\mathrm{L}_C(X) = \left\langle \frac{\theta(C)}{2} \right\rangle$. Moreover, by Lemma 4.7 and Theorem 3.8, we get that $\mathrm{L}_{\frac{C}{d}}\left(\frac{X}{d}\right) = \left\langle \frac{\theta\left(\frac{C}{d}\right)}{2} \right\rangle$. Therefore, $\mathrm{L}_C(X) = d \cdot \mathrm{L}_{\frac{C}{d}}\left(\frac{X}{d}\right)$. □

Let us illustrate the content of the above proposition with an example.

Example 4.9. *Let us take the sets* $C = \{-2, 2\}$ *and* $X = \{4, 6\}$. *Then* $\theta(C) = 2$ *and* $X \subseteq \{0, \theta(C), \to\}$. *By applying Proposition 3.5, we have that X is C-admissible. Since* $\gcd(X \cup C) = 2$, *by Proposition 4.8, then we have that* $\{2, 3\}$ *is* $\{-1, 1\}$*-admissible and that* $\mathrm{L}_C(X) = 2 \cdot \mathrm{L}_{\{-1,1\}}(\{2, 3\})$. *Now, from Proposition 2.10, we easily deduce that* $\langle 2, 3\rangle$ *is a* $\{-1, 1\}$*-incentive and, therefore, that* $\mathrm{L}_{\{-1,1\}}(\{2, 3\}) = \langle 2, 3\rangle$. *Consequently,* $\mathrm{L}_C(X) = 2 \cdot \langle 2, 3\rangle = \langle 4, 6\rangle$.

5. The Frobenius pseudo-variety of the numerical *C*-incentives

Let S be a numerical semigroup. The *Frobenius number* of S, denoted by $\mathrm{F}(S)$, is the greatest integer that does not belong to S (see [6]).

A *Frobenius pseudo-variety* is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfils the following conditions.

1. $\mathcal{P}$ has a maximum element $\max(\mathcal{P})$ (with respect to the inclusion order).
2. If $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$.
3. If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

Let us observe that a Frobenius pseudo-variety $\mathcal{P}$ is a Frobenius variety if and only if $\mathbb{N} \in \mathcal{P}$ (see [8, Proposition 1]).

In this section, C denotes a non-empty finite subset of $\mathbb{Z}$. Our purpose is to show that $\mathrm{NI}(C)$ is a Frobenius pseudo-variety.

Lemma 5.1. *If $r \in \mathbb{N}$, then* $\mathrm{NI}(\{r\})$ *is a Frobenius pseudo-variety and, moreover,* $\max(\mathrm{NI}(\{r\})) = \mathbb{N}$.

Proof. It is clear that $\mathbb{N} \in \mathrm{NI}(\{r\})$ and, therefore, that $\max(\mathrm{NI}(\{r\})) = \mathbb{N}$. Also, it is easy to see that, if $S, T \in \mathrm{NI}(\{r\})$, then $S \cap T \in \mathrm{NI}(\{r\})$. Finally, let us take $S \in \mathrm{NI}(\{r\}) \setminus \mathbb{N}$ and see that $S \cup \{\mathrm{F}(S)\} \in \mathrm{NI}(\{r\})$. Indeed, let $x, y \in (S \cup \{\mathrm{F}(S)\}) \setminus \{0\}$. On the one hand, if $x, y \in S$, then $x + y + r \in S \subseteq S \cup \{\mathrm{F}(S)\}$; on the other hand, if $\mathrm{F}(S) \in \{x, y\}$, then $x + y + r \geq \mathrm{F}(S)$ and, consequently, $x + y + r \in S \cup \{\mathrm{F}(S)\}$. □

As a consequence of the previous lemma, we can observe that, if $r \in \mathbb{N}$, then $\mathrm{NI}(\{r\})$ is a Frobenius variety, since $\mathbb{N} \in \mathrm{NI}(\{r\})$.

Lemma 5.2. *If r is a positive integer, then* $\mathrm{NI}(\{-r\})$ *is a Frobenius pseudo-variety. Moreover,*

$$\max(\mathrm{NI}(\{-r\})) = \begin{cases} \mathbb{N}, & \text{if } r \in \{1, 2\}, \\ \{0, r, \rightarrow\}, & \text{if } r \geq 3. \end{cases}$$

Proof. It is clear that, if $r \in \{1, 2\}$, then $\mathbb{N} \in \mathrm{NI}(\{-r\})$ and, therefore, that $\max(\mathrm{NI}(\{-r\})) = \mathbb{N}$. On the other hand, by Lemmas 3.2 and 3.3, if $r \geq 3$, then $\max(\mathrm{NI}(\{-r\})) = \{0, r, \rightarrow\}$ (observe that, if $r \geq 3$, then $\langle \frac{r}{2} \rangle$ is not a numerical semigroup). Moreover, it is not difficult to check that, if $S, T \in \mathrm{NI}(\{-r\})$, then $S \cap T \in \mathrm{NI}(\{-r\})$.

Now, let us see that, if $S \in \mathrm{NI}(\{-r\})$ and $S \neq \max(\mathrm{NI}(\{-r\}))$, then $S \cup \{\mathrm{F}(S)\} \in \mathrm{NI}(\{-r\})$. In order to do this, let us take $x, y \in (S \cup \{\mathrm{F}(S)\}) \setminus \{0\}$. Now, if $x, y \in S$, then $x + y - r \in S \subseteq S \cup \{\mathrm{F}(S)\}$. Thus, we can suppose that $\mathrm{F}(S) \in \{x, y\}$. We distinguish two cases.

1. Let us suppose that $x = \mathrm{F}(S)$ and $y \neq \mathrm{F}(S)$. Then $y \in S \setminus \{0\}$ and, since $S \subsetneq \max(\mathrm{NI}(\{-r\}))$, we deduce that $y \geq r$. Therefore, $x + y - r \geq \mathrm{F}(S)$ and, consequently, $x + y - r \in S \cup \{\mathrm{F}(S)\}$.
2. Let us suppose that $x = y = \mathrm{F}(S)$. Then $x + y - r = 2\mathrm{F}(S) - r$. We have two possibilities.
   (a) If $\mathrm{F}(S) \geq r$, then $2\mathrm{F}(S) - r \geq \mathrm{F}(S)$ and, therefore, $x + y - r \in S \cup \{\mathrm{F}(S)\}$.
   (b) If $\mathrm{F}(S) < r$, since $S \subsetneq \max(\mathrm{NI}(\{-r\}))$, then we deduce that $r = 2$ and $S = \{0, 2, \rightarrow\}$. Therefore, $S \cup \{\mathrm{F}(S)\} = \mathbb{N} \in \mathrm{NI}(\{-2\})$.

□

Let us observe that, as a consequence of the previous lemma, we have that $\mathrm{NI}(\{-2\})$ and $\mathrm{NI}(\{-1\})$ are Frobenius varieties because they contain $\mathbb{N}$. On the other hand, if $r \geq 3$, then $\mathrm{NI}(\{-r\})$ is a Frobenius pseudo-variety but not a Frobenius variety.

Lemma 5.3. *Let $\{\mathcal{P}_i\}_{i \in I}$ be a family of Frobenius pseudo-varieties. If there exists $j \in I$ such that* $\max(\mathcal{P}_j) \in \mathcal{P}_i$ *for all $i \in I$, then $\bigcap_{i \in I} \mathcal{P}_i$ is a Frobenius pseudo-variety and* $\max(\bigcap_{i \in I} \mathcal{P}_i) = \max(\mathcal{P}_j)$.

Proof. It is clear that $\max(\bigcap_{i \in I} \mathcal{P}_i) = \max(\mathcal{P}_j)$. Now, if $S, T \in \bigcap_{i \in I} \mathcal{P}_i$, then $S, T \in \mathcal{P}_i$ for all $i \in I$ and, therefore, $S \cap T \in \bigcap_{i \in I} \mathcal{P}_i$. Finally, if $S \in \bigcap_{i \in I} \mathcal{P}_i$ and $S \neq \max(\bigcap_{i \in I} \mathcal{P}_i)$, then $S \in \mathcal{P}_i$ and $S \neq \max(\mathcal{P}_i)$ for all $i \in I$. Therefore, $S \cup \mathrm{F}(S) \in \mathcal{P}_i$ for all $i \in I$. Consequently, $S \cup \mathrm{F}(S) \in \bigcap_{i \in I} \mathcal{P}_i$. □

An immediate consequence of Lemma 3.2 is the next one.

Lemma 5.4. $\{0, \theta(C), \rightarrow\} \in \mathrm{NI}(\{c\})$ *for all $c \in C$.*

We are ready to show the main result of this section.

Theorem 5.5. $\mathrm{NI}(C)$ *is a Frobenius pseudo-variety. Moreover,*

$$\max(\mathrm{NI(C)}) = \begin{cases} \mathbb{N}, & \text{if } C \subseteq \{-2, -1\} \cup \mathbb{N}, \\ \{0, \theta(C), \rightarrow\}, & \text{in other case.} \end{cases}$$

Proof. It is clear that $\mathrm{NI}(C) = \bigcap_{c\in C} \mathrm{NI}(\{c\})$. From Lemmas 5.1 and 5.2, we know that $\mathrm{NI}(\{c\})$ is a Frobenius pseudo-variety for all $c \in C$, and that, if $C \subseteq \{-2,-1\} \cup \mathbb{N}$, then $\max(\mathrm{NI}(\{c\})) = \mathbb{N}$ for all $c \in C$. Thus, from Lemma 5.3, we have that $\mathrm{NI}(C)$ is a Frobenius pseudo-variety with $\max(\mathrm{NI}(C)) = \mathbb{N}$. Now, let us suppose that $\theta(C) \geq 3$ and let $c_0 \in C$ such that $\theta(C) = -c_0$. From Lemma 5.2, we have that $\max(\mathrm{NI}(\{c_0\})) = \{0, \theta(C), \rightarrow\}$ and, from Lemma 5.4, that $\max(\mathrm{NI}(\{c_0\})) \in \mathrm{NI}(\{c\})$ for all $c \in C$. Therefore, by applying Lemma 5.3, $\mathrm{NI}(C)$ is a Frobenius pseudo-variety with $\max(\mathrm{NI}(C)) = \{0, \theta(C), \rightarrow\}$. □

Remark 5.6. *It is easy to see that, if r is a positive integer different from 2, then* $\max(\mathrm{NI}(\{-r\})) = \{0, r, \rightarrow\}$. *Moreover, as a consequence of Theorem 5.5,* $\max(\mathrm{NI}(\{-r\})) \neq \{0, 2, \rightarrow\}$ *for all $C \subseteq \mathbb{Z}$. Consequently, if* $\{0, 2, \rightarrow\} \in \mathrm{NI}(C)$, *then we conclude that* $\mathbb{N} \in \mathrm{NI}(C)$.

Remark 5.7. *From Theorem 5.5,* $\mathrm{NI}(C)$ *is a Frobenius variety if and only if $C \subseteq \{-2,-1\} \cup \mathbb{N}$. Several of these families have been studied in some previous works. For instance,* $\mathrm{NI}(\{1\})$, $\mathrm{NI}(\{-1\})$, $\mathrm{NI}(\{-1, 1\})$, $\mathrm{NI}(C)$ *(for $C \subseteq \mathbb{N}$), and* $\mathrm{NI}([-\beta, \alpha] \cap \mathbb{Z})$ *(for $\alpha, \beta \in \mathbb{N}$) are analysed in [7], [12], [9], [13], and [10], respectively.*

6. The tree of the numerical *C*-incentives

Our purpose in this section is to arrange the elements of $\mathrm{NI}(C)$ in a tree with root and characterize the children in such a tree. Thus, as the main result of this paper, we obtain an algorithmic process that allows us to recurrently build the elements of $\mathrm{NI}(C)$.

Recall that a *graph* G is a pair (V, E), where V is a non-empty set, whose elements are the *vertices* of G, and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$, whose elements are the *edges* of G. Moreover, a *path (of length n)* connecting the vertices x and y of G is a sequence of different edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

We say that a graph G is a *tree* if there exists a vertex v^* (the *root* of G) such that, for every other vertex x of G, there exists a unique path connecting x and v^*. If (x, y) is an edge of the tree, then we say that x is a *child* of y.

In this section, we suppose that C is a non-empty finite subset of $\mathbb{Z}$. We define the graph $\mathrm{G}(C)$ in the following way: $\mathrm{NI}(C)$ is the set of vertices of $\mathrm{G}(C)$, and $(S, S') \in \mathrm{NI}(C) \times \mathrm{NI}(C)$ is an edge of $\mathrm{G}(C)$ if $S' = S \cup \{\mathrm{F}(S)\}$.

It is well known (see [14]) that, if M is a submonoid of $(\mathbb{N}, +)$ and $x \in M$, then $M \setminus \{x\}$ is a monoid if and only if $x \in \mathrm{msg}(M)$. As a consequence of [8, Lemma 12, Theorem 3], we have the next result.

Theorem 6.1. *The graph* $\mathrm{G}(C)$ *is a tree whose root is* $\max(\mathrm{NI}(C))$. *Moreover, the children of a vertex* $S \in \mathrm{NI}(C)$ *are the elements of the set* $\{S \setminus \{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S),\ \text{and } S \setminus \{x\} \in \mathrm{NI}(C)\}$.

In the following proposition we characterize the minimal generators x of a C-incentive M such that $M \setminus \{x\}$ is also a C-incentive.

Proposition 6.2. *Let M be a C-incentive and $x \in \mathrm{msg}(M)$. Then $M \setminus \{x\}$ is a C-incentive if and only if* $\{x\} - C \subseteq (\mathbb{Z} \setminus M) \cup \mathrm{msg}(M \setminus \{x\}) \cup \{x, 0\}$.

Proof. (Necessity.) If $x - c \notin (\mathbb{Z} \setminus M) \cup \mathrm{msg}(M \setminus \{x\}) \cup \{x, 0\}$ for some $c \in C$, then we can assert that $x - c \in M \setminus \{x, 0\}$ and $x - c \notin \mathrm{msg}(M \setminus \{x\})$. Therefore, $x - c = m + n$ for some $m, n \in M \setminus \{x, 0\}$ and, consequently, $m + n + c = x \notin M \setminus \{x\}$. Thereby, $M \setminus \{x\}$ is not a C-incentive.

(Sufficiency.) If we take $m, n \in M \setminus \{x, 0\}$, then $\{m + n\} + C \subseteq M$. Let us suppose that $m + n + c = x$ for some $c \in C$. In such a case, $x - c \notin (\mathbb{Z} \setminus M) \cup \mathrm{msg}(M \setminus \{x\}) \cup \{x, 0\}$ that is a contradiction. Thus, $m + n + c \neq x$ for all $c \in C$ and, consequently, $\{m + n\} + C \subseteq M \setminus \{x\}$. □

In order to facilitate the construction of the tree $\mathrm{G}(C)$, we study the relation between the minimal generators of a numerical semigroup S and the minimal generators of $S \setminus \{x\}$, where x is a minimal generator of S that is greater than $\mathrm{F}(S)$. First of all, let us observe that, if S is minimally generated by $\{m, m+1, \ldots, 2m-1\}$ (that is, $S = \{0, m, \rightarrow\}$), then $S \setminus \{m\}$ is minimally generated by $\{m + 1, m + 2, \ldots, 2m + 1\}$. In other case we use the next result, which is a reformulation of [7, Corollary 18].

Proposition 6.3. *Let S be a numerical semigroup with minimal system of generators $\left\{n_1 < \ldots < n_p\right\}$. If $j \in \{2, \ldots, p\}$ and $n_j > \mathrm{F}(S)$, then*

$$\mathrm{msg}(S \setminus \{n_j\}) = \begin{cases} \{n_1, \ldots, n_p\} \setminus \{n_j\}, & \textit{if there exists } i \in \{2, \ldots, p\} \setminus \{j\} \textit{ such that } n_j + n_1 - n_i \in S, \\ \Big(\{n_1, \ldots, n_p\} \setminus \{n_j\}\Big) \cup \{n_j + n_1\}, & \textit{in other case.} \end{cases}$$

Let S be a numerical C-incentive, $m = \min(\mathrm{msg}(S))$, and $x \in \mathrm{msg}(S)$. From Proposition 6.3, if $(\{x\} - C) \cap (\mathrm{msg}(M \setminus \{x\}) \setminus \mathrm{msg}(M)) \neq \emptyset$, then $-m \in C$. On the other hand, $x \in \{x\} - C$ if and only if $0 \in C$. These two facts allow us to give the following improvement of Proposition 6.2 (see Remark 6.5).

Proposition 6.4. *Let S be a numerical C-incentive, $m = \min(\mathrm{msg}(S))$, and $x \in \mathrm{msg}(S)$. Let us suppose that $-m \notin C$. Then $S \setminus \{x\}$ is a numerical C-incentive if and only if $\{x\} - C \subseteq (\mathbb{Z} \setminus S) \cup \mathrm{msg}(S) \cup \{x\}$.*

Remark 6.5. *Observe that, by applying Proposition 6.2, we have to compute $\mathrm{msg}(S \setminus \{x\})$ in order to assert that $S \setminus \{x\}$ is a numerical C-incentive. That is, we assert after computing. However, by Proposition 6.4, we have only to use $\mathrm{msg}(S)$. Of course, if we want to build the tree, we have to compute $\mathrm{msg}(S \setminus \{x\})$. Therefore, we now compute after asserting.*

Let us see an example that illustrates the contents of this section.

Example 6.6. *We are going to build the tree associated to the numerical $\{-3, 2\}$-incentives.*

$$\begin{tikzcd}
 & & \langle 3,4,5\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
 & \langle 4,5,6,7\rangle \arrow[d, dash] & & \langle 3,5,7\rangle \arrow[d, dash] \\
 & \langle 5,6,7,8,9\rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 3,7,8\rangle \arrow[d, dash] \\
\langle 6,7,8,9,10,11\rangle \arrow[d, dash] & & \langle 5,7,8,9,11\rangle \arrow[d, dash] & \langle 3,8,10\rangle \arrow[d, dash] \\
\ldots\ldots\ldots\ldots & & \langle 5,7,9,11,13\rangle & \langle 3,8,13\rangle
\end{tikzcd}$$

By Theorem 5.5, we know that $\max(\mathrm{NI}(\{-3,2\})) = \{0, 3, \rightarrow\} = \langle 3,4,5\rangle$. By applying Theorem 6.1 and Propositions 6.2, 6.3, and 6.4 (in fact, we apply Proposition 6.2 only when $\min(\mathrm{msg}(S)) = 3$), we have that

- $\langle 4,5,6,7\rangle = \langle 3,4,5\rangle \setminus \{3\}$ *and* $\langle 3,5,7\rangle = \langle 3,4,5\rangle \setminus \{4\}$ *are the two children of* $\langle 3,4,5\rangle$.
- $\langle 5,6,7,8,9\rangle = \langle 4,5,6,7\rangle \setminus \{4\}$ *is the unique child of* $\langle 4,5,6,7\rangle$.
- $\langle 3,7,8\rangle = \langle 3,5,7\rangle \setminus \{5\}$ *is the unique child of* $\langle 3,5,7\rangle$.
- $\langle 6,7,8,9,10,11\rangle = \langle 5,6,7,8,9\rangle \setminus \{5\}$ *and* $\langle 5,7,8,9,11\rangle = \langle 5,6,7,8,9\rangle \setminus \{6\}$ *are the two children of* $\langle 5,6,7,8,9\rangle$.
- $\langle 3,8,10\rangle = \langle 3,7,8\rangle \setminus \{7\}$ *is the unique child of* $\langle 3,7,8\rangle$.
- $\langle 6,7,8,9,10,11\rangle$ *has three children.*
- $\langle 5,7,9,11,13\rangle = \langle 5,7,8,9,11\rangle \setminus \{8\}$ *is the unique child of* $\langle 5,7,8,9,11\rangle$.
- $\langle 3,8,13\rangle = \langle 3,8,10\rangle \setminus \{10\}$ *is the unique child of* $\langle 3,8,10\rangle$.
- $\langle 5,7,9,11,13\rangle$ *has not got any child.*
- $\langle 3,8,13\rangle$ *has not got any child.*
- *And so on.*

Remark 6.7. *In Example 6.6 we have an infinite tree, that is, a tree with infinitely many elements. For instance, the branch* $\langle 3,4,5\rangle$, $\langle 4,5,6,7\rangle$, $\langle 5,6,7,8,9,\rangle$, $\langle 6,7,8,9,10,11\rangle$, ... *has no end. However, if we only take into account numerical C-incentives with Frobenius number (or genus) less than or equal to a fixed number, then we are going to obtain trees with finitely many elements. (Recall that, if S is a numerical semigroup, then the genus of S is equal to the cardinality of* $\mathbb{N} \setminus S$.)

7. *C*-incentives containing a given *C*-admissible set

Let X be a subset of $\mathbb{N} \setminus \{0\}$ such that it is C-admissible. We denote by $\mathrm{I}(C, X) = \{M \in \mathrm{I}(C) \mid X \subseteq M\}$ and by $\mathrm{NI}(C, X) = \{S \in \mathrm{NI}(C) \mid X \subseteq S\}$. In this section, our main purpose is to show an algorithmic process that allows us to compute $\mathrm{I}(C, X)$. In order to do that, and first of all, we are going to see that we can focus on the computation of $\mathrm{NI}(C, X)$.

Lemma 7.1. *If* $M \in \mathrm{I}(C, X)$, *then* $\gcd(M)$ *divides* $\gcd(C \cup X)$.

Proof. By applying Lemma 4.3, we know that $\gcd(M)$ divides $\gcd(C)$. Moreover, since $X \subseteq M$, we have that $\gcd(M)$ divides $\gcd(X)$. Thus, $\gcd(M)$ divides $\gcd(C \cup X)$. □

As a consequence of the previous lemma, we have the following one.

Lemma 7.2. *If* $\gcd(C \cup X) = 1$, *then* $\mathrm{I}(C, X) = \mathrm{NI}(C, X)$.

Let M be a submonoid of $(\mathbb{N}, +)$ such that $M \neq \{0\}$ and let $d = \gcd(M)$. In such a situation, from Lemma 4.4, we have that $M \in \mathrm{I}(C)$ if and only if $\frac{M}{d} \in \mathrm{I}\left(\frac{C}{d}\right)$. Moreover, it is clear that $X \subseteq M$ if and only if $\frac{X}{d} \subseteq \frac{M}{d}$. In this way, we can establish the next result.

Lemma 7.3. *Let M be a submonoid of* $(\mathbb{N}, +)$ *such that* $M \neq \{0\}$ *and let* $d = \gcd(M)$. *Then* $M \in \mathrm{I}(C, X)$ *if and only if* $\frac{M}{d} \in \mathrm{I}\left(\frac{C}{d}, \frac{X}{d}\right)$.

Now, from Lemmas 7.1, 7.2, and 7.3, we have the following result.

Theorem 7.4. *Let D the set formed by all positive divisors of* $\gcd(C \cup X)$. *Then* $\mathrm{I}(C, X) = \bigcup_{d \in D} \left\{dS \mid S \in \mathrm{NI}\left(\frac{C}{d}, \frac{X}{d}\right)\right\}$.

On the other hand, from Theorem 5.5, we easily conclude the next one.

Theorem 7.5. $\mathrm{NI}(C, X)$ *is a Frobenius pseudo-variety. Moreover,*

$$\max(\mathrm{NI}(C, X)) = \begin{cases} \mathbb{N}, & \text{if } C \subseteq \{-2, -1\} \cup \mathbb{N}, \\ \{0, \theta(C), \rightarrow\}, & \text{in other case.} \end{cases}$$

Now, as expected, we define the graph $G(C, X)$ as follows: $\mathrm{NI}(C, X)$ is the set of vertices of $\mathrm{G}(C, X)$, and $(S, S') \in \mathrm{NI}(C, X) \times \mathrm{NI}(C, X)$ is an edge of $\mathrm{G}(C, X)$ if $S' = S \cup \{\mathrm{F}(S)\}$. The next result is a direct consequence of Theorem 6.1.

Theorem 7.6. *The graph* $\mathrm{G}(C, X)$ *is a tree whose root is* $\max(\mathrm{NI}(C, X))$. *Moreover, the children of a vertex* $S \in \mathrm{NI}(C, X)$ *are the elements of the set* $\{S \setminus \{a\} \mid a \in \mathrm{msg}(S),\ a > \mathrm{F}(S), S \setminus \{a\} \in \mathrm{NI}(C), \text{ and } a \notin X\}$.

By combining Theorems 7.5, 7.6, and Propositions 6.2, 6.3, and 6.4, we can recurrently build the tree $G(C, X)$ as shown in the next example.

Example 7.7. *The tree associated to the numerical* $\{-3, 2\}$-*incentives containing the set* $\{5\}$ *is the following one.*

$$
\begin{tikzcd}[column sep=small]
 & \langle 3,4,5\rangle \arrow[dl, no head] \arrow[dr, no head] & \\
\langle 4,5,6,7\rangle \arrow[d, no head] & & \langle 3,5,7\rangle \\
\langle 5,6,7,8,9\rangle \arrow[d, no head] & & \\
\langle 5,7,8,9,11\rangle \arrow[d, no head] & & \\
\langle 5,7,9,11,13\rangle & &
\end{tikzcd}
$$

In order to justify it, we can review the computations of Example 6.6.

In the previous example, we have obtained a finite tree. Indeed, this fact can be characterized in terms of C and X.

Theorem 7.8. $\mathrm{NI}(C, X)$ *is finite if and only if* $\gcd(C \cup X) = 1$.

Proof. (Necessity.) By applying Proposition 4.8, if $\gcd(C \cup X) = d \neq 1$, then we have that $\gcd(\mathrm{L}_C(X)) \neq 1$. Now, since $\mathrm{L}_C(X) \cup \{k, \rightarrow\} \in \mathrm{NI}(C, X)$ for all $k \in \mathbb{N}$ such that $k \geq \theta(C)$, then we conclude that $\mathrm{NI}(C, X)$ is infinite.

(Sufficiency.) From Lemma 7.2, we know that, if $\gcd(C \cup X) = 1$, then $\mathrm{L}_C(X) \in \mathrm{NI}(C, X)$. Since $\mathrm{L}_C(X)$ is contained in all elements of $\mathrm{NI}(C, X)$ and $\mathbb{N} \setminus \mathrm{L}_C(X)$ is finite, we easily deduce that $\mathrm{NI}(C, X)$ is finite. □

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