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Bernoulli Polynomials Collocation for Weakly Singular Volterra Integro-Differential Equations of Fractional Order

Haman Deilami Azodi[a], Mohammad Reza Yaghouti[a]

[a]*Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*

Abstract. This paper is concerned with a numerical procedure for fractional Volterra integro-differential equations with weakly singular kernels. The fractional derivative is in the Caputo sense. In this study, Bernoulli polynomial of first kind is used and its matrix form is given. Then, the matrix form based on the collocation points is constructed for each term of the problem. Hence, the proposed scheme simplifies the problem to a system of algebraic equations. Error analysis is also investigated. Numerical examples are announced to demonstrate the validity of the method.

1. Introduction

Fractional calculus is a fascinating topic in mathematics with diverse applications in science and technology [11, 21, 22]. In this way, many mathematicians try to introduce instrumental techniques for solving the differential and integro-differential equations of fractional order. For the existence and uniqueness of the fractional differential equations solution, we refer to [13, 35]. Also, the outcomes of local and global existence and uniqueness for the solution of fractional integro-differential equations have been taken in [30, 31], respectively.

Practically, there is a great concentration on finding the solution of fractional integro-differential equations of Volterra, Fredholm and Volterra-Fredholm types. In fact, one can see a huge number of works on the solutions of fractional integro-differential equations in the literature. For example, see [2, 4–6, 9, 14–16, 19, 20, 23, 26–29, 32, 36, 37, 39, 40, 47]. In these references, the kernel of integral parts is non-singular.

In this paper, we consider the following fractional Volterra integro-differential equation with weakly singular kernel

$$D_*^{\alpha} y(x) = p(x)y(x) + \lambda \int_0^x \frac{y(t)}{(x-t)^{\nu}} dt + g(x), \quad x \in [0\ 1], \tag{1}$$

under the initial condition

$$y(0) = c, \tag{2}$$

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Email addresses: haman.d.azodi@gmail.com (Haman Deilami Azodi), yaghouti@guilan.ac.ir (Mohammad Reza Yaghouti)

where $y(x)$ is unknown, $p(x)$ and $g(x)$ are known, λ and c are real values. Also, D_*^α denotes the Caputo fractional derivative of order α, $0 < \alpha \leq 1$ and $0 < \nu < 1$. It is notable for $\alpha = 0$ and $p(x) = 0$, (1) reduces to Abel's integral equation. Fractional integro-differential equations with weakly singular kernel have many usages in radiative equilibrium [17], heat conduction problem [41], elasticity and fracture mechanics [46]. Due to the complicated behaviour of equation (1), finding the exact solution of it is not easy. Thereby, numerical methods are required more and more. However, researchers have paid less attention to solving weakly singular Volterra integro-differential equations of fractional order. For solving (1) numerically, Zhao et al. analyzed piecewise polynomial collocation [43]. The continuation of this work, Nemati et al. applied second kind Chebyshev polynomials [33]. Furthermore, Yi et al. used CAS wavelet method [44] and Sahu et al. inspected Sinc-Galerkin (SG) method [38] for solving fractional Volterra-Fredholm integro-differential equation with a weakly singular kernel.

Recently, Bhrawy et al. derived Bernoulli polynomials successfully for the numerical solution of Fredholm integro-differential equations [8]. Bazm solved the Volterra-Fredholm-Hammerstein integral equations [7] using operational matrices of Bernoulli polynomials. Also, Tohidi et al. used Bernoulli polynomials expansion for solving fractional Volterra integro-differential equations with non-singular kernel [42]. It should be noted that Mashayekhi et al. accounted the advantages of Bernoulli polynomials over orthogonal polynomials for approximating a real function in [24, 25].

Throughout this paper, by using Bernoulli polynomials, new matrix operations, the collocation method and the Caputo fractional derivative, we intend to approximate the solution of (1) with the initial condition (2) in the form

$$y_N(x) = \sum_{n=0}^{N} a_n B_n(x). \tag{3}$$

Here, $a_n, n = 0, 1, \ldots, N$ are the unknown Bernoulli coefficients; N is selected any positive integer; $B_n(x)$ are the Bernoulli polynomials of first kind defined by [3, 10]

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i, \;\; n \in \mathbb{N}, \;\; x \geq 0,$$

in which $b_n, n = 0, 1, \ldots, N$ are bernoulli numbers. These numbers are computed using the following identity

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} b_i \frac{x^i}{i!}.$$

The first few Bernoulli numbers are

$$b_0 = 1, \; b_1 = -\frac{1}{2}, \; b_2 = \frac{1}{6}, \; b_4 = -\frac{1}{30}, \; \ldots,$$

and for $i = 1, 2, \ldots$, $b_{2i+1} = 0$. Besides this, the Bernoulli polynomials for some small values of n are

$$B_0(x) = 1, \;\; B_1(x) = x - \frac{1}{2}, \;\; B_2(x) = x^2 - x + \frac{1}{6}, \;\; B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \;\; \ldots.$$

Also, Bernoulli polynomials and Bernoulli numbers are related to each other by $b_i = B_i(0)$, $i = 0, 1, 2, \ldots$.

The remainder of this paper proceeds as follows: In Section 2, basic definitions of fractional calculus applied further in this research are reviewed. In Section 3, the matrix relations for the Caputo fractional derivative and the weakly singular Volterra integral part are formed. Using these matrix operations and collocation method, Section 4 suggests a procedure for solving (1) under condition (2). An error analysis is investigated in Section 5. Section 6 confirms the impression of present method through several examples. Lastly, a conclusion is drawn in Section 7.

2. Preliminaries and Basic Concepts

For the convenience of the reader, we repeat the relevant materials of fractional calculus from [12, 35].

Definition 2.1. *The Riemann-Liouville's fractional order integration for the function h on $L^1[a,b]$ is defined as follows*

$$J^{\alpha}h(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}h(s)ds, & \alpha>0,\\ h(x), & \alpha=0.\end{cases}$$

The important properties of J^{α} are

- $J^{\alpha_1}J^{\alpha_2}h(x)=J^{\alpha_1+\alpha_2}h(x)$,
- $J^{\alpha_1}J^{\alpha_2}h(x)=J^{\alpha_2}J^{\alpha_1}h(x)$,
- $J^{\alpha_1}x^{\alpha_2}=\frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_1+\alpha_2+1)}x^{\alpha_1+\alpha_2}$.

Definition 2.2. *The Caputo derivative of order $\alpha>0$ is defined as*

$$D_*^{\alpha}h(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1}h^{(n)}(s)ds,\ \ n-1<\alpha<n,$$

where $x>0$ and n is an integer.

The interesting features of Caputo derivative are listed in the following

- $J^{\alpha}D_*^{\alpha}h(x)=h(x)-\sum_{i=0}^{n-1}h^{(i)}\left(0^+\right)\frac{x^i}{i!},\ \ n-1<\alpha<n,\ n\in\mathbb{N}$,
- $D_*^{\alpha}c=0$, (c is a constant),
- $D_*^{\alpha_1}x^{\alpha_2}=\begin{cases}0, & \alpha_2\in\mathbb{N}_0,\ \alpha_2<\lceil\alpha_1\rceil,\\ \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+1-\alpha_1)}x^{\alpha_2-\alpha_1}, & \alpha_2\in\mathbb{N}_0,\ \alpha_2\geq\lceil\alpha_1\rceil,\ or\ \alpha_2\notin\mathbb{N},\ \alpha_2>\lfloor\alpha_1\rfloor.\end{cases}$

3. Fundamental Matrix Relations

Firstly, let us represent $B_n(x)$ in the matrix form as follows

$$\mathbf{B}(x)=\mathbf{X}(x)\mathbf{D}^T, \tag{4}$$

where

$$\mathbf{B}(x)=[B_0(x)\ B_1(x)\ \dots\ B_N(x)],\ \ \mathbf{X}(x)=\left[1\ x\ \dots\ x^N\right],$$

and

$$\mathbf{D}=\begin{bmatrix}\binom{0}{0}b_0 & 0 & 0 & \dots & 0 & 0\\ \binom{1}{0}b_1 & \binom{1}{1}b_0 & 0 & \dots & 0 & 0\\ \binom{2}{0}b_2 & \binom{2}{1}b_1 & \binom{2}{2}b_0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \binom{N-1}{0}b_{N-1} & \binom{N-1}{1}b_{N-2} & \binom{N-1}{2}b_{N-3} & \dots & \binom{N-1}{N-1}b_0 & 0\\ \binom{N}{0}b_N & \binom{N}{1}b_{N-1} & \binom{N}{2}b_{N-2} & \dots & \binom{N}{N-1}b_1 & \binom{N}{N}b_0\end{bmatrix}.$$

For the sake of simplicity, we state (1) in the form

$$D_*^{\alpha} y(x) = p(x)y(x) + \lambda V(x) + g(x), \tag{5}$$

where

$$V(x) = \int_0^x \frac{y(t)}{(x-t)^{\nu}} dt.$$

In what follows, we exhibit matrix relations for the Caputo fractional derivative of the solution, $D_*^{\alpha} y(x)$, and the Volterra integral part $V(x)$.

3.1. Matrix relation for $D_*^{\alpha} y(x)$

We first suppose the desired solution of (1) be in the form of truncated Bernoulli series (3). Accordingly, $y(x)$ can be written in the matrix form

$$y(x) = \mathbf{B}(x)\mathbf{A}; \quad \mathbf{A} = [a_0 \ a_1 \ \dots \ a_N]^T,$$

or with the aid of (4)

$$y(x) = \mathbf{X}(x)\mathbf{D}^T\mathbf{A}. \tag{6}$$

Now, by the Caputo fractional derivative and (6), enables one to see

$$D_*^{\alpha} y(x) = \mathbf{X}^{(\alpha)}(x)\mathbf{D}^T\mathbf{A}, \tag{7}$$

in which

$$\mathbf{X}^{(\alpha)}(x) = \begin{bmatrix} D_*^{\alpha}1 & D_*^{\alpha}x & D_*^{\alpha}x^2 & \dots & D_*^{\alpha}x^N \end{bmatrix} = \begin{bmatrix} 0 & \frac{\Gamma(2)x^{1-\alpha}}{\Gamma(2-\alpha)} & \frac{\Gamma(3)x^{2-\alpha}}{\Gamma(3-\alpha)} & \cdots & \frac{\Gamma(N+1)x^{N-\alpha}}{\Gamma(N+1-\alpha)} \end{bmatrix}.$$

3.2. Matrix relation for $V(x)$

Substituting (6) into $V(x)$ results in

$$V(x) = \int_0^x \frac{\mathbf{X}(t)\mathbf{D}^T\mathbf{A}}{(x-t)^{\nu}} dt = \left(\int_0^x \frac{\mathbf{X}(t)}{(x-t)^{\nu}} dt \right) \mathbf{D}^T\mathbf{A}. \tag{8}$$

In order to construct a matrix relation for $V(x)$, we must gain an explicit formula for the integral

$$I_{i,\nu}(x) = \int_0^x \frac{t^i}{(x-t)^{\nu}} dt,$$

so that $i = 0, 1, \dots, N$. To achieve this aim, we change the variables by $t = rx$. Then $dt = xdr, \ 0 \le r \le 1$ and one can write

$$I_{i,\nu}(x) = x^{i+1-\nu} \int_0^1 (1-r)^{-\nu} r^i dr = \beta(i+1, 1-\nu)x^{i+1-\nu}, \tag{9}$$

where $\beta(.,.)$ denotes the well-known Beta function. As we know, Beta and Gamma functions are connected with each other by $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Now, employing (8) and (9) entails

$$V(x) = \mathbf{I}_{\nu}(x)\mathbf{D}^T\mathbf{A}, \tag{10}$$

where

$$\mathbf{I}_{\nu}(x) = \begin{bmatrix} \beta(1, 1-\nu)x^{1-\nu} & \beta(2, 1-\nu)x^{2-\nu} & \dots & \beta(N+1, 1-\nu)x^{N+1-\nu} \end{bmatrix}.$$

4. Method of Solution

For the implementation of numerical method, we substitute (6), (7) and (10) into (5). Consequently,

$$\mathbf{X}^{(\alpha)}(x)\mathbf{D}^T\mathbf{A} = p(x)\mathbf{X}(x)\mathbf{D}^T\mathbf{A} + \lambda\mathbf{I}_\nu(x)\mathbf{D}^T\mathbf{A} + g(x). \tag{11}$$

Now, we collocate (11) at a set of collocation points. For $x \in [0\ \ 1]$, one choice can be

$$x_i = \frac{i}{N}, \quad i = 0, 1, \dots, N.$$

This implies that

$$\mathbf{X}^{(\alpha)}(x_i)\mathbf{D}^T\mathbf{A} = p(x_i)\mathbf{X}(x_i)\mathbf{D}^T\mathbf{A} + \lambda\mathbf{I}_\nu(x_i)\mathbf{D}^T\mathbf{A} + g(x_i), \ i = 0, \dots, N.$$

Briefly, the main matrix equation is offered as

$$\left\{\mathbf{X}^{(\alpha)}\mathbf{D}^T - \mathbf{PXD}^T - \lambda\mathbf{I}_\nu\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G}, \tag{12}$$

in which

$$\mathbf{X}^{(\alpha)} = \begin{bmatrix} \mathbf{X}^{(\alpha)}(x_0) \\ \mathbf{X}^{(\alpha)}(x_1) \\ \vdots \\ \mathbf{X}^{(\alpha)}(x_N) \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} p(x_0) & 0 & 0 & \dots & 0 \\ 0 & p(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p(x_N) \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix}, \ \mathbf{I}_\nu = \begin{bmatrix} \mathbf{I}_\nu(x_0) \\ \mathbf{I}_\nu(x_1) \\ \vdots \\ \mathbf{I}_\nu(x_N) \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Except to $\mathbf{A}$ and $\mathbf{G}$ which are column vectors with $N+1$ entries, all of the matrices dimension in (12) is $(N+1) \times (N+1)$.

In the compact representation, (12) can be shown as

$$\mathbf{WA} = \mathbf{G} \quad or \quad [\mathbf{W}; \mathbf{G}], \tag{13}$$

where

$$\mathbf{W} = \mathbf{X}^{(\alpha)}\mathbf{D}^T - \mathbf{PXD}^T - \lambda\mathbf{I}_\nu\mathbf{D}^T.$$

Clearly, (13) is a linear system of algebraic equations with the unknown Bernoulli coefficients $a_0,\ a_1, \dots, a_N$.

On the other hand, the matrix form corresponding to initial condition (2) can be written as

$$\{\mathbf{B}(0)\}\,\mathbf{A} = c \quad or \quad [\mathbf{B}(0); c], \tag{14}$$

so that

$$\mathbf{B}(0) = [b_0\ b_1\ \dots\ b_N],$$

and as we noted before, $b_i;\ i = 0, 1, \dots, N$ are the Bernoulli numbers.

To determine the solution of (1) under condition (2), replacing the row vector (14) by the first row of (13), the following new augmented matrix is established

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = \begin{bmatrix} b_0 & b_1 & b_2 & \dots & b_N & ; & c \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & ; & g(x_1) \\ w_{20} & w_{21} & w_{22} & \dots & w_{2N} & ; & g(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{(N-1)0} & w_{(N-1)1} & w_{(N-1)2} & \dots & w_{(N-1)N} & ; & g(x_{N-1}) \\ w_{N0} & w_{N1} & w_{N2} & \dots & w_{NN} & ; & g(x_N) \end{bmatrix}.$$

If $rank\tilde{\mathbf{W}} = rank[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = N+1$, one can deduce

$$\mathbf{A} = \tilde{\mathbf{W}}^{-1}\tilde{\mathbf{G}}.$$

Herewith, $a_0, a_1, \dots, a_N$ are identified uniquely and (1) with the initial condition (2) has a unique solution. This solution is in the form of (3).

5. Error analysis

Let $\{B_0(x),\ B_1(x),\ \dots,\ B_N(x)\} \subset L^2[0\ 1]$ be the set of Bernoulli polynomials and

$$Y = span\,\{B_0(x),\ B_1(x),\ \dots,\ B_N(x)\}\,.$$

Assume that $h \in L^2[0\ 1]$ be an arbitrary element. Since Y is a finite dimensional vector space, h has the unique best approximation belongs to Y such as $\hat{h} \in Y$. This means for every $z \in Y$

$$\|h - \hat{h}\| \le \|h - z\|.$$

Since $\hat{h} \in Y$, there exists the unique coefficients $h_0,\ h_1,\ \dots,\ h_N$ such that

$$h \approx \hat{h} = \sum_{n=0}^{N} h_n B_n(x) = \mathbf{B}(x)\mathbf{H},$$

where

$$\mathbf{B}(x) = [B_0(x),\ B_1(x),\ \dots,\ B_N(x)]\,,\ \ \mathbf{H} = [h_0,\ h_1,\ \dots,\ h_N]^T\,.$$

Theorem 5.1. *[1] Suppose $h(x)$ be an enough smooth function on $[0\ 1]$ and $P_N[h](x)$ is the approximate polynomial of $h(x)$ in terms of Bernoulli Polynomials and $R_N[h](x)$ is the remainder term. Then, the associated formula are stated as follows*

$$h(x) = P_N[h](x) + R_N[h](x),\ \ x \in [0\ 1],$$

$$P_N[h](x) = \int_0^1 h(x)dx + \sum_{j=1}^{N} \frac{B_j(x)}{j!}\left(h^{(j-1)}(1) - h^{(j-1)}(0)\right),$$

$$R_N[h](x) = -\frac{1}{N!}\int_0^1 B_N^*(x-t)h^{(N)}(t)dt,$$

where $B_N^*(x) = B_N(x - [x])$.

Corollary 5.2. *If $h(x) \in C^\infty[0\ 1]$ and $P_N[h](x)$ is the approximate polynomial using Bernoulli polynomials, then the following error bound may be obtained*

$$\|error(h(x))\|_\infty \le \frac{2\mu}{(2\pi)^N},$$

in which μ is the maximum value of $\left|h^{(N)}(x)\right|$ on $[0\ 1]$.

Proof. With the aid of Theorem 5.1, it is obvious that

$$\|error(h(x))\|_\infty \le \frac{\Theta_N}{N!}\mu,$$

where Θ_N and μ are the maximum value of $|B_N(x)|$ and $\left|h^{(N)}(x)\right|$ on $[0\ 1]$, respectively.
In [18], Lehmer proved

$$-\frac{2N!}{(2\pi)^N} \le B_N(x) \le \frac{2N!}{(2\pi)^N},$$

for every $0 \le x \le 1$. Hence, $\Theta_N = \frac{2N!}{(2\pi)^N}$ and the result is satisfied.
□

Lemma 5.3. *Let $h : [0\ 1] \to \mathbb{R}$ and $J^{\alpha}(.)$ denotes the Riemann-Liouville's fractional integration operator. Then,*

$$\|J^{\alpha}(h(x))\|_{\infty} \leq \frac{1}{\Gamma(\alpha+1)}\|h(x)\|_{\infty}. \tag{15}$$

Proof. We have

$$\begin{aligned}|J^{\alpha}(h(x))| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^x (x-s)^{\alpha-1}h(s)ds\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|h(s)|\,ds \\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^x (x-s)^{\alpha-1}ds\right)\sup_{0\leq x\leq 1}|h(x)| \leq \frac{1}{\Gamma(\alpha+1)}\|h(x)\|_{\infty}.\end{aligned}$$

Ultimately,

$$\|J^{\alpha}(h(x))\|_{\infty} = \sup_{0\leq x\leq 1}|J^{\alpha}(h(x))| \leq \frac{1}{\Gamma(\alpha+1)}\|h(x)\|_{\infty}.$$

□

Theorem 5.4. *Let $y(x)$ and $y_N(x)$ be the exact and approximate solutions of (1) under condition (2). Also, assume*

- *There exist $\rho_1, \rho_2 \in \mathbb{R}^+$ such that $\left\|y(x)\right\|_{\infty} \leq \rho_1$, $\left\|p(x)\right\|_{\infty} \leq \rho_2$, $\forall x \in [0\ 1]$.*
- *$(1-\nu)\Gamma(\alpha+1) - (1-\nu)\rho_2 - (1-\nu)E(p) \neq \lambda$.*

Then,

$$\left\|y(x) - y_N(x)\right\|_{\infty} \leq \frac{(1-\nu)\Gamma(\alpha+1)E(f) + (1-\nu)\rho_1 E(p)}{(1-\nu)\Gamma(\alpha+1) - (1-\nu)\rho_2 - (1-\nu)E(p) - \lambda},$$

where

$$\begin{aligned}E(p) &= \left\|error(p(x))\right\|_{\infty} = \left\|p(x) - p_N(x)\right\|_{\infty}, \\ E(f) &= \left\|error(f(x))\right\|_{\infty} = \left\|f(x) - f_N(x)\right\|_{\infty},\ f(x) = y(0) + J^{\alpha}g(x).\end{aligned}$$

Proof. Fractional integrating from both sides of (1) and imposing the intial condition yield

$$y(x) = f(x) + J^{\alpha}(p(x)y(x)) + \lambda J^{\alpha}\left(\int_0^x \frac{y(t)}{(x-t)^{\nu}}dt\right),$$

in which $f(x) = y(0) + J^{\alpha}g(x)$.

Now, consider that $f(x)$ and $p(x)$ are expanded in terms of Bernoulli polynomials, then the obtained solution is an approximated polynomial; $y_N(x)$. Our aim is to seek an upper bound for the associated error between the exact solution $y(x)$ and the approximated solution $y_N(x)$ for (1) with the mentioned assumptions. Subsequently,

$$\begin{aligned}\|y(x) - y_N(x)\|_{\infty} &= \left\|f(x) - f_N(x) + J^{\alpha}\left(p(x)y(x) - p_N(x)y_N(x)\right) + \lambda J^{\alpha}\left(\int_0^x \frac{y(t)-y_N(t)}{(x-t)^{\nu}}dt\right)\right\|_{\infty} \\ &\leq \left\|f(x) - f_N(x)\right\|_{\infty} + \left\|J^{\alpha}\left(p(x)y(x) - p_N(x)y_N(x)\right)\right\|_{\infty} + \lambda\left\|J^{\alpha}\left(\int_0^x \frac{y(t)-y_N(t)}{(x-t)^{\nu}}dt\right)\right\|_{\infty}\end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned}\left\|J^{\alpha}\left(p(x)y(x) - p_N(x)y_N(x)\right)\right\|_{\infty} &= \left\|J^{\alpha}\left(p(x)\left(y(x) - y_N(x)\right) + \left(p(x) - p_N(x)\right)\left(y_N(x) - y(x) + y(x)\right)\right)\right\|_{\infty} \\ &\leq \left\|J^{\alpha}\left(p(x)\left(y(x) - y_N(x)\right)\right)\right\|_{\infty} + \left\|J^{\alpha}\left(\left(p(x) - p_N(x)\right)\left(y(x) - y_N(x)\right)\right)\right\|_{\infty} \\ &+ \left\|J^{\alpha}\left(\left(p(x) - p_N(x)\right)\left(y(x)\right)\right)\right\|_{\infty}.\end{aligned}$$

By using (15), it follows

$$\left\|J^{\alpha}\left(p(x)y(x)-p_N(x)y_N(x)\right)\right\|_{\infty}\leq\frac{1}{\Gamma(\alpha+1)}\left\|p(x)\right\|_{\infty}\left\|y(x)-y_N(x)\right\|_{\infty}+\frac{1}{\Gamma(\alpha+1)}\left\|p(x)-p_N(x)\right\|_{\infty}\left\|y(x)\right\|_{\infty}$$
$$+\frac{1}{\Gamma(\alpha+1)}\left\|p(x)-p_N(x)\right\|_{\infty}\left\|y(x)-y_N(x)\right\|_{\infty}.$$

Since $\left\|y(x)\right\|_{\infty}\leq\rho_1$ and $\left\|p(x)\right\|_{\infty}\leq\rho_2$, we write

$$\begin{aligned}\left\|J^{\alpha}\left(p(x)y(x)-p_N(x)y_N(x)\right)\right\|_{\infty}&\leq\frac{1}{\Gamma(\alpha+1)}\rho_2\left\|y(x)-y_N(x)\right\|_{\infty}+\frac{1}{\Gamma(\alpha+1)}\rho_1E(p)\\&+\frac{1}{\Gamma(\alpha+1)}E(p)\left\|y(x)-y_N(x)\right\|_{\infty}.\end{aligned}\tag{17}$$

Moreover, for $x\in[0\ 1]$, we imply

$$\left|\int_0^x\frac{y(t)-y_N(t)}{(x-t)^{\nu}}dt\right|\leq\left(\int_0^x\frac{dt}{(x-t)^{\nu}}\right)\left\|y(x)-y_N(x)\right\|_{\infty}\leq\frac{1}{1-\nu}\left\|y(x)-y_N(x)\right\|_{\infty}.$$

Equivalently,

$$\left\|\int_0^x\frac{y(t)-y_N(t)}{(x-t)^{\nu}}dt\right\|_{\infty}\leq\frac{1}{1-\nu}\left\|y(x)-y_N(x)\right\|_{\infty}.\tag{18}$$

By applying (15) and (18), we conclude

$$\left\|J^{\alpha}\left(\int_0^x\frac{y(t)-y_N(t)}{(x-t)^{\nu}}dt\right)\right\|_{\infty}\leq\frac{1}{\Gamma(\alpha+1)}\left\|\int_0^x\frac{y(t)-y_N(t)}{(x-t)^{\nu}}dt\right\|_{\infty}\leq\frac{1}{(1-\nu)\Gamma(\alpha+1)}\left\|y(x)-y_N(x)\right\|_{\infty}.\tag{19}$$

Eventually, combination of (16), (17) and (19) yields

$$\left\|y(x)-y_N(x)\right\|_{\infty}\leq\frac{(1-\nu)\Gamma(\alpha+1)E(f)+(1-\nu)\rho_1E(p)}{(1-\nu)\Gamma(\alpha+1)-(1-\nu)\rho_2-(1-\nu)E(p)-\lambda}.$$

□

6. Numerical examples

In this section, three examples are dedicated to evaluate the efficiency of the proposed method. All of them are performed by MATLAB R2015a software on a 64-bit PC with 2.20 GHz processor and 8 GB memory. We report the results of applying our method through several tables and figures. In these examples, we utilize the following notations

$$|e_N(x)|=|y(x)-y_N(x)|,\quad \|e_N\|_{\infty}=\max_{x\in[0\ 1]}|e_N(x)|,$$

in which $y(x)$ and $y_N(x)$ allude to the exact and approximate solutions, respectively.

Example 6.1. *Let (1) be as follows*

$$D_*^{\alpha}y(x)=p(x)y(x)+\int_0^x\frac{y(t)}{(x-t)^{\frac{1}{2}}}dt+g(x),\tag{20}$$

where

$$p(x)=-\frac{16}{15}x^{\frac{1}{2}},\ g(x)=2x,$$

with the initial condition $y(0)=0$. The exact solution of (16) when $\alpha=1$ is $y(x)=x^2$.

Suppose $N = 2$ and $\alpha = 1$. After computation of relevant matrices and considering the initial value, the final augmented matrix is acquired as follows

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = \begin{bmatrix} 1 & -1/2 & 1/6 & ; & 0 \\ -394/597 & 713/577 & -111/7064 & ; & 1 \\ -14/15 & 6/5 & 10/9 & ; & 2 \end{bmatrix}.$$

Accordingly, the Bernoulli coefficients a_0, a_1 and a_2 are

$$a_0 = 1/3,\ \ a_1 = 1,\ \ a_2 = 1.$$

Therefore, the solution of (20) for $N = 2$, $y(0) = 0$ and $\alpha = 1$ is calculated as

$$\begin{aligned} y_2(x) &= a_0 B_0(x) + a_1 B_1(x) + a_2 B_2(x) \\ &= \left(\frac{1}{3}\right)(1) + (1)\left(x - \frac{1}{2}\right) + (1)\left(x^2 - x + \frac{1}{6}\right) \\ &= x^2. \end{aligned}$$

This problem has been solved in [33] with $\alpha = 1$, approximately. The important point to mention is that the present method concludes the exact solution. Also, Figure 1 portrays the treatment of solution for $N = 2$ and various amounts of α. We realize that when α tends to 1, approximate solutions are close to the exact solution for $\alpha = 1$.

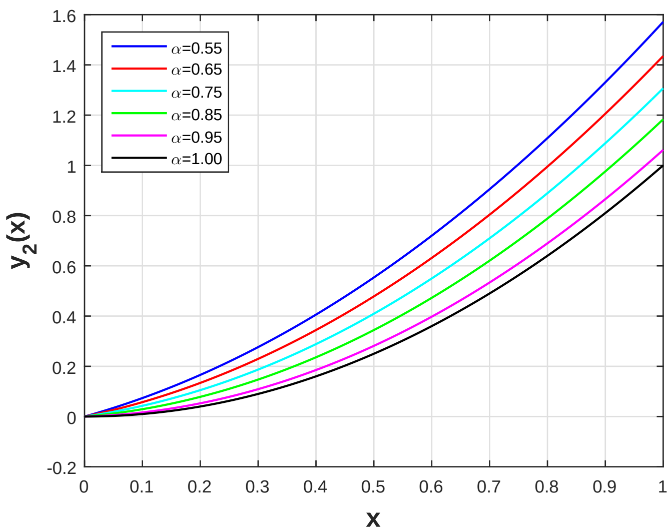

Figure 1: Solution of Example 6.1 for $N = 2$ and different α

Example 6.2. *We consider the following Abel's integral equation*

$$y(x) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\arcsin\left(\frac{1-x}{1+x}\right) - \frac{1}{4}\int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}}dt,$$

with the exact solution $y(x) = \frac{1}{\sqrt{x+1}}$.

Notice that $\alpha = 0$. Applying the proposed scheme for $N = 4$ and $N = 8$, we specify

$$y_4(x) = 0.05055244x^4 - 0.1859123x^3 + 0.3386473x^2 - 0.4961732x + 1.0,$$

and

$$y_8(x) = 0.007695764x^8 - 0.04292183x^7 + 0.1122915x^6 - 0.1915496x^5 + 0.256009x^4 - 0.3090655x^3 + 0.3746316x^2 - 0.4999841x + 1.0\,.$$

Table 1 compares the numerical solutions of our method with those of Block-Pulse functions method [34] and Legendre wavelets method [45]. Obviously, the present method is in better agreement with exact solution. Figure 2 is devoted to L^∞ error of this problem for $1 \le N \le 10$. It is clear that when N is increased sufficiently, the error decreases.

Table 1: Comparison of present method with BPFs [34] and LWs [45] methods for Example 6.2

x	BPFs method [34]			LWs [45]	Present method		Exact solution
	$k = 16$	$k = 32$	$k = 64$	$k = 1, M = 5$	$N = 4$	$N = 8$	
0.0	0.997340	0.999123	0.999993	0.999432	1.000000	1.000000	1.000000
0.2	0.911748	0.912305	0.912873	0.912320	0.912905	0.912871	0.912871
0.4	0.848041	0.845156	0.845154	0.845321	0.845110	0.845154	0.845154
0.6	0.788293	0.790527	0.790562	0.790539	0.790604	0.790569	0.790569
0.8	0.746027	0.745361	0.745316	0.745342	0.745315	0.745356	0.745356
1.0	0.704230	0.707120	0.707103	0.707163	0.707114	0.707107	0.707107

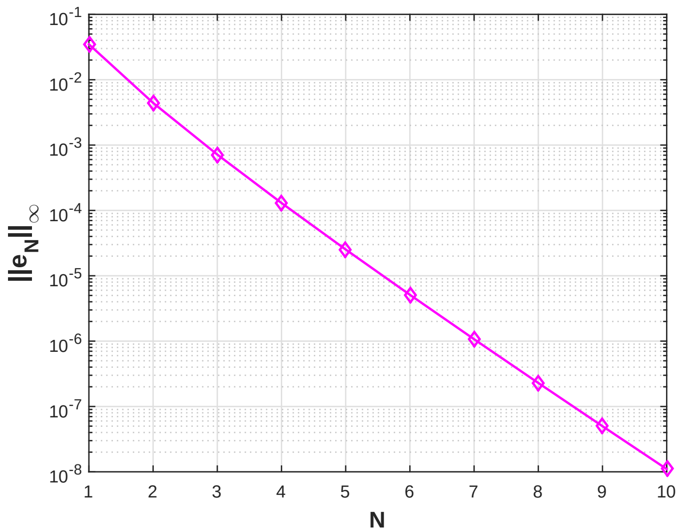

Figure 2: L^∞ error of Example 6.2 for $N = 1, \ldots, 10$

Example 6.3. *Consider (1) in the following*

$$D_*^{\frac{1}{3}} y(x) = p(x)y(x) + \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt + g(x), \tag{21}$$

where

$$p(x) = -\frac{32}{35}x^{\frac{1}{2}}, \; g(x) = \frac{6x^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \left(\frac{32}{35} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)}\right)x^{\frac{11}{6}} + \Gamma\left(\frac{7}{3}\right)x,$$

with initial value $y(0) = 0$. The exact solution of (21) is $y(x) = x^3 + x^{\frac{4}{3}}$.

We apply the present method for solving (21) with the aforesaid initial condition. Figure 3 indicates L^∞ error for $1 \le N \le 15$. It yields that for large enough N, the infinity norm of error decreases. Table 2 summarizes the results of present method for $N = 3, 7, 10, 15$. It also exposes that in comparison with Sinc-Galerkin (SG) method [38] for $N = 30$, our method provides more accurate solutions for (21) by smaller number of basis functions.

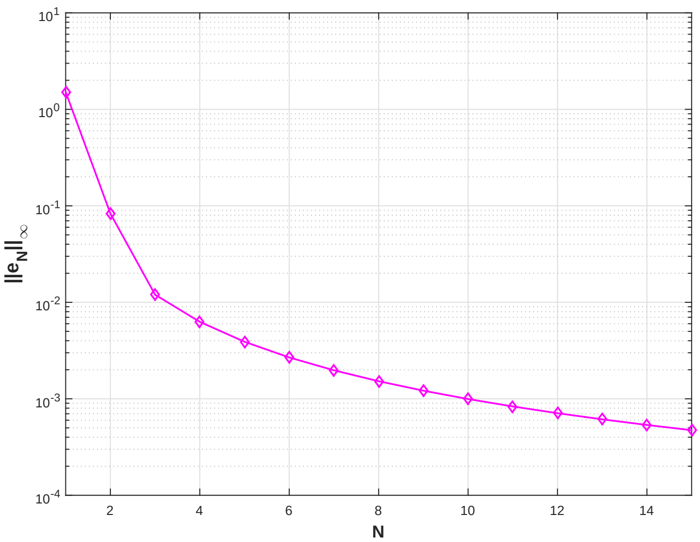

Figure 3: L^∞ error of Example 6.3 for $N = 1, \ldots, 15$

Table 2: $|e_N(x)|$ of SG method [38] and our method for Example 6.3

x	SG, $N = 30$ [38]	Present method			
		$N = 3$	$N = 7$	$N = 10$	$N = 15$
0.1	1.76957e-3	1.15359e-2	1.39847e-3	4.50863e-4	1.40173e-4
0.2	1.60604e-4	1.14446e-2	8.10077e-4	3.29955e-4	1.18347e-4
0.3	5.18220e-3	9.43810e-3	8.00536e-4	3.23756e-4	1.15312e-4
0.4	2.79194e-3	7.72486e-3	8.30684e-4	3.28951e-4	1.18204e-4
0.5	3.69227e-4	7.00071e-3	8.46847e-4	3.42813e-4	1.23729e-4
0.6	5.05652e-3	7.32665e-3	8.97221e-4	3.59210e-4	1.30666e-4
0.7	5.53609e-4	8.42791e-3	9.52796e-4	3.77599e-4	1.38411e-4
0.8	3.04882e-3	9.82763e-3	9.89910e-4	3.94238e-4	1.46628e-4
0.9	2.94377e-3	1.09167e-2	1.07960e-3	4.10377e-4	1.54998e-4

7. Conclusion

The fractional Volterra integro-differential equation involving weakly singular kernel is an applied equation and solving it exactly is usually difficult. Unlike the equations with non-singular kernels, there are a few articles in the literature related to the solution of this special type of integro-differential equations. This paper proposed a convenient scheme by means of Bernoulli polynomials, matrix operations and collocation method for solving the mentioned problem.

One of the profitable characteristics of the suggested method was that all of the calculations were displayed in the matrix form. This manner causes simplicity in the computer programming. Furthermore, if the problem has an exact solution in the polynomial form, one can find it by using small number of collocation nodes. To get the best approximate solution of the equation, the truncation limit N must be chosen large enough. The comparison between the numerical results of our method with exact solution and other existing methods revealed that our method generates noticeable approximations.

We also think that the discussed approach can be developed to a system of fractional singular Volterra integro-differential equations, which will be suitable matter for future study.

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