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Pointwise Planar Horizontal Sections Along Riemannian Submersions

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Abstract. We study Riemannian submersions with pointwise k-planar horizontal sections. We provide examples, obtain characterizations and give a geometric interpretation of such property.

1. Introduction

Let M be an $n-$dimensional submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. For any point p in M and any nonzero vector t at p tangent to M, the vector t and the normal space $(T_pM)^\perp$ determine an $(m - n + 1)$ dimensional vector space $E(p, t)$ in $\mathbb{E}^m$. The intersection of M and $E(p, t)$ gives rise a curve $\gamma(s)$ (in a neighbourhood of p), called the normal section of M at p in the direction t, where s denotes the arc length of γ. In general the normal section γ is a twisted space curve in $E(p, t)$. In particular, $\gamma^{'} \wedge \gamma^{''} \wedge \gamma^{'''} \neq 0$ at p in general. A submanifold M is said to have pointwise planar normal sections if each normal section γ at p satisfies $\gamma^{'} \wedge \gamma^{''} \wedge \gamma^{'''} = 0$ for each p in M. And a submanifold is said to have planar normal sections if its normal sections are planar curves, that is, $\gamma^{'} \wedge \gamma^{''} \wedge \gamma^{'''} \equiv 0$ for each normal section γ, [9]. The notion of submanifolds with planar normal sections was defined by Chen [9] and main results were given in [10]. Later such submanifolds have been studied by many authors and they have shown that this notion is useful for characterizing parallel submanifolds in Euclidean space, see: [1], [2], [3], [4], [5], [6], [11], [12], [13], [14], [15],[16], [19], [20].

Riemannian submersions between Riemannian manifolds were studied by O'Neill [23] and Gray [18] and they are useful for obtaining new manifolds with certain curvature. Riemannian submersions have been studied widely in differential geometry, see [17], however this subject is still an active area of differential geometry, see a recent paper [26] and references there in.

In this paper, we study the notion of planar normal sections by considering Riemannian submersion. As far as we know, there is no any study on planar normal sections along a Riemannian submersion. We provide an example, obtain certain characterizations and give a geometric meaning for a submersion to have such properties. We show that the projection of warped product submanifold onto its first factor has pointwise $2-$planar horizontal sections. Since there are many warped product submanifolds in an Euclidean space, this shows that there are many Riemannian submersions having pointwise $2-$planar horizontal sections. We also observe that Riemannian submersions with totally geodesic fibers has this property. Since there is a close relation between Riemannian submersions from spheres with totally geodesic fibers and Clifford

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algebra, this relation shows that it will be possible to use Clifford algebra and their representation to study pointwise planar horizontal sections along Riemannian submersions.

The paper is organized as follows: In section 2, we present the basic information needed for this paper. In section 3, we introduced notion of pointwise k-planar horizontal sections and give certain characterizations. We also obtain a method to obtain examples of Riemannian submersions with pointwise 2-planar horizontal sections. Then by using this method we provide a numerical example.

2. Preliminaries

In this section, we define Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let (M^m, g_M) and (N^n, g_N) Riemannian manifolds, where $dim(M) = m$, $dim(N) = n$ and $m > n$. A Riemannian submersion $F : M \longrightarrow N$ is a map of M onto N satisfying the following axioms:

(S1) F has maximal rank.

(S2) The differential F_* preserves the lenghts of horizontal vectors.

For each $q \in N$, $F^{-1}(q)$ is an $(m-n)$ dimensional submanifold of M. The submanifolds $F^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibres. A vector field on M is called horizontal if it is always orthogonal to fibres. A vector field U on M is called basic if U is horizontal and $F-$ related to a vector field U_* on N, i.e., $F_* U_p = U_{*F(p)}$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $kerF_*$ and $(kerF_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall the following lemma from O'Neill [23].

Lemma 2.1. *Let $F : M \longrightarrow N$ be a Riemannian submersion between Riemannian manifolds and U, V be basic vector fields of M. Then*

(a) $g_M(U, V) = g_N(U_*, V_*) \circ F$

(b) the horizontal part $[U, V]^{\mathcal{H}}$ of $[U, V]$ is a basic vector field and corresponds to $[U_*, V_*]$, i.e., $F_*([U, V]^{\mathcal{H}}) = [U_*, V_*]$.

(c) $[X, U]$ is vertical for any vector field X of $kerF_*$.

(d) $(\overset{M}{\nabla}_U V)^{\mathcal{H}}$ is the basic vector field corresponding to $\overset{N}{\nabla}_{U_*} V_*$.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \quad \mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{1}$$

where ∇ is the Levi-Civita connection of g_M. It is easy to see that a Riemannian submersion $F : M \longrightarrow N$ has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{vE}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{hE}$. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_X Y = \mathcal{T}_Y X, \quad \mathcal{A}_U V = -\mathcal{A}_V U = \frac{1}{2} v[U, V] \tag{2}$$

for $X, Y \in \Gamma(kerF_*)$ and $U, V \in \Gamma((kerF_*)^\perp)$.
On the other hand, from (1) we have

$$\nabla_X Y = \mathcal{T}_X Y + \hat{\nabla}_X Y \tag{3}$$

$$\nabla_X V = h\nabla_X V + \mathcal{T}_X V \tag{4}$$

for $X, Y \in \Gamma(kerF_*)$ and $U, V \in \Gamma((kerF_*)^\perp)$, where $\hat{\nabla}_X Y = v\nabla_X Y$. If U is basic, then $h\nabla_X U = \mathcal{A}_U X$.

3. Riemannian Submersions with Planar Horizontal Sections

In this section, we define Riemannian submersions with planar horizontal sections and obtain characterizations. We also relate this notion with covariant derivative of the tensor field $\mathcal{T}$. We first present the following definition.

Definition 3.1. *Let (M, g) be a $m-$dimensional Riemannian manifold and $(\mathbb{E}^n, <,>)$ be a n-dimensional Euclidean space. Consider a Riemannian submersion $F : \mathbb{E}^n \to M$ and denote its vertical distribution and horizontal distribution by $\mathcal{V}$ and $\mathcal{H}$, respectively. It is known that the vertical distribution $\mathcal{V}$ is always integrable. We denote the integral manifold of $\mathcal{V}$ by $\bar{M}$. For $p \in \bar{M}$ and a non-zero vector $X \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$, we define $(m+1)-$dimensional affine subspace $E(p, X)$ of $\mathbb{E}^n$ by*

$$E(p, X) = p + Span\{X, \mathcal{H}_p\}. \tag{5}$$

In a neighbourhood of p, the intersection $\bar{M} \cap E(p, X)$ is a regular curve $\alpha : (-\varepsilon, \varepsilon) \to \bar{M}$. We suppose that the parameter $t \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\alpha(0) = p$ and $\alpha'(0) = X$. Each choice of $X \in \mathcal{V}_p$ yields a different curve. We will call α the horizontal section of $\bar{M}$ at p in the direction of X. The Riemannian submersion F is said to have "pointwise k-planar horizontal sections (Pk-PHS)" if for each horizontal section α, the higher order derivatives

$$\{\alpha'(0), \alpha''(0), ..., \alpha^{k+1}(0)\} \tag{6}$$

are linearly dependent as vectors in $\mathbb{E}^n$.

Thus a horizontal section can be written as below

$$\alpha(t) = p + \lambda(t)X + U(t) \tag{7}$$

where $U \in \mathfrak{X}^{\mathcal{H}}(\mathbb{E}^n)$, $\lambda(t) \in \mathbb{R}$.

First of all, we have the following result.

Proposition 3.2. *Let $F : (\mathbb{E}^n, <,>) \longrightarrow (M, g_M)$ be a Riemannian submersion. Then F has P1-PHS if and only if F is a Riemannian submersion with totally geodesic fibers.*

Proof. Let $\alpha : (-\varepsilon, \varepsilon) \longrightarrow \bar{M}$, $\bar{M} \subset \mathbb{E}^n$, $\alpha(0) = p$, $\alpha'(0) = X \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$. Then if F has P1-PHS, from the definition it follows that $\alpha'(0)$ and $\alpha''(0)$ are linearly dependent. Set

$$\alpha'(0) = X \tag{8}$$

and using (3) we have

$$\alpha''(0) = \nabla_X X = \mathcal{T}_X X + \hat{\nabla}_X X. \tag{9}$$

Since $X, \hat{\nabla}_X X = v\nabla_X X \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$ and $\mathcal{T}_X X \in \mathfrak{X}^{\mathcal{H}}(\mathbb{E}^n)$; $\alpha'(0)$ and $\alpha''(0)$ are linearly dependent for only

$$\mathcal{T}_X X = 0. \tag{10}$$

Hence we have

$$\mathcal{T}_Y Z = 0$$

for $Y, Z \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$. This shows that F is a Riemannian submersion with totally geodesic fibers. The converse is clear. $\square$

The above theorem shows that P1-PHS condition is a very strong condition for Riemannian submersion. Indeed, this property implies that $\mathcal{T} = 0$. But this proposition enables us to put a weaker condition for a Riemannian submersion with totally geodesic fibers. For instance, from Hermann's Theorem [17, Page:37], we can give the following result.

Corollary 3.3. *Let $F : (\mathbb{E}^n, <,>) \longrightarrow (M, g_M)$ be a Riemannian submersion. If F has P1-PHS, then F acts as the projection of a bundle associated with a principal fibre bundle with structure group the Lie group of isometries of the fibre.*

From [17, Page:41], we also write the following result by putting P1-PHS condition instead of the totally geodesicity.

Corollary 3.4. *Let $F : (\mathbb{E}^n, <,>) \longrightarrow (M, g_M)$ be a Riemannian submersion. If F has P1-PHS and $n \geq 3$, then (M, g_M) is a locally symmetric space.*

We now consider P2-PHS condition for Riemannian submersions. We first have the following result.

Theorem 3.5. *Let $F : (\mathbb{E}^n, <,>) \longrightarrow (M, g_M)$ be a Riemannian submersion. Then F has P2-PHS if and only if $(\nabla_X \mathcal{T})_X X$ and $\mathcal{T}_X X$ satisfy*

$$(\nabla_X \mathcal{T})_X X \wedge \mathcal{T}_X X = 0 \tag{11}$$

for any $X \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$.

Proof. From (3) and (4) we obtain

$$\alpha'(0) = X \tag{12}$$

$$\alpha''(0) = \nabla_X X = \mathcal{T}_X X + \hat{\nabla}_X X \tag{13}$$

$$\alpha'''(0) = h\nabla_X \mathcal{T}_X X + \mathcal{T}_X \mathcal{T}_X X + \mathcal{T}_X \hat{\nabla}_X X + \hat{\nabla}_X \hat{\nabla}_X X. \tag{14}$$

If F has P2-PHS from the definition; $\alpha'(0)$, $\alpha''(0)$ and $\alpha'''(0)$ are linearly dependent. Taking vertical parts and horizontal parts of above equations, we get

$$\mathcal{T}_X \mathcal{T}_X X + \hat{\nabla}_X \hat{\nabla}_X X = - \| \mathcal{T}_X X \|^2 X \tag{15}$$

and

$$h\nabla_X \mathcal{T}_X X = a\mathcal{T}_X X \tag{16}$$

where $a \in C^\infty(\mathbb{E}^n, \mathbb{R})$. On the other hand, for a vector field $E \in \mathfrak{X}(\mathbb{E}^n)$ and $X \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$, we have

$$(\nabla_X \mathcal{T})_X E = \nabla_X \mathcal{T}_X E - \mathcal{T}_{\nabla_X X} E - \mathcal{T}_X \nabla_X E. \tag{17}$$

Using (16) and (17), we get

$$(\nabla_X \mathcal{T})_X X = a\mathcal{T}_X X$$

which shows that $(\nabla_X \mathcal{T})_X X \wedge \mathcal{T}_X X = 0$. Conversely, if the (11) is satisfied, then it is easy to see that F has P2-PHS. □

The following result gives a geometric interpretation for the notion of the Riemannian submersion having P2-PHS.

Theorem 3.6. *Let $F : (\mathbb{E}^n, <,>) \longrightarrow (M, g_M)$ be a Riemannian submersion with non-totally geodesic fibers. Then the following three statements are equivalent*

1. $(\nabla_X \mathcal{T})_X X = 0$

2. $\sum_{XYZ}(\nabla_X\mathcal{T})_Z Y = 0$ *where* $\sum_{XYZ}$ *denotes the cyclic sum of over the vertical vector fields X, Y and Z.*
3. *Horizontal section of the fiber along F at* $p \in \mathbb{E}^n$ *is pointwise 2-planar horizontal section with p as one of its vertices.*

Proof. (1) ⇔ (2) Taking $X + Y + Z$ instead of X in $(\nabla_X\mathcal{T})_X X = 0$, we get

$$\begin{aligned}(\nabla_{X+Y+Z}\mathcal{T})_{X+Y+Z}X + Y + Z &= (\nabla_X\mathcal{T})_Y Z + (\nabla_Y\mathcal{T})_X Z + (\nabla_X\mathcal{T})_Z Y + (\nabla_Z\mathcal{T})_X Y + (\nabla_Y\mathcal{T})_Z X\\ &+ (\nabla_Z\mathcal{T})_Y X + (\nabla_X\mathcal{T})_X Y + (\nabla_Y\mathcal{T})_Y X + (\nabla_X\mathcal{T})_X Z + (\nabla_Z\mathcal{T})_Z X\\ &+ (\nabla_Y\mathcal{T})_Y Z + (\nabla_Z\mathcal{T})_Z Y + (\nabla_X\mathcal{T})_Y Y + (\nabla_X\mathcal{T})_Z Z + (\nabla_Y\mathcal{T})_X X\\ &+ (\nabla_Y\mathcal{T})_Z Z + (\nabla_Z\mathcal{T})_X X + (\nabla_Z\mathcal{T})_Y Y + (\nabla_X\mathcal{T})_Y X + (\nabla_X\mathcal{T})_Z X\\ &+ (\nabla_Y\mathcal{T})_X Y + (\nabla_Y\mathcal{T})_Z Y + (\nabla_Z\mathcal{T})_X Z + (\nabla_Z\mathcal{T})_Y Z = 0.\end{aligned} \tag{18}$$

On the other hand for $X, Y, Z \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$, taking $Y + Z$ instead of X in $(\nabla_X\mathcal{T})_X X = 0$, we obtain

$$\begin{aligned}(\nabla_{Y+Z}\mathcal{T})_{Y+Z}Y + Z &= \nabla_Y\mathcal{T}_Y Y + \nabla_Y\mathcal{T}_Y Z + \nabla_Y\mathcal{T}_Z Y + \nabla_Y\mathcal{T}_Z Z\\ &+ \nabla_Z\mathcal{T}_Y Y + \nabla_Z\mathcal{T}_Y Z + \nabla_Z\mathcal{T}_Z Y + \nabla_Z\mathcal{T}_Z Z\\ &- \mathcal{T}_{\nabla_Y Y}Y - \mathcal{T}_{\nabla_Y Y}Z - \mathcal{T}_{\nabla_Y Z}Y - \mathcal{T}_{\nabla_Y Z}Z\\ &- \mathcal{T}_{\nabla_Z Y}Y - \mathcal{T}_{\nabla_Z Y}Z - \mathcal{T}_{\nabla_Z Z}Y - \mathcal{T}_{\nabla_Z Z}Z\\ &- \mathcal{T}_Y\nabla_Y Y - \mathcal{T}_Y\nabla_Y Z - \mathcal{T}_Y\nabla_Z Y - \mathcal{T}_Y\nabla_Z Z\\ &- \mathcal{T}_Z\nabla_Y Y - \mathcal{T}_Z\nabla_Y Z - \mathcal{T}_Z\nabla_Z Y - \mathcal{T}_Z\nabla_Z Z = 0.\end{aligned} \tag{19}$$

In a similar way, for $Y - Z$ we derive

$$\begin{aligned}(\nabla_{Y-Z}\mathcal{T})_{Y-Z}Y - Z &= \nabla_Y\mathcal{T}_Y Y - \nabla_Y\mathcal{T}_Y Z - \nabla_Y\mathcal{T}_Z Y + \nabla_Y\mathcal{T}_Z Z\\ &- \nabla_Z\mathcal{T}_Y Y + \nabla_Z\mathcal{T}_Y Z + \nabla_Z\mathcal{T}_Z Y - \nabla_Z\mathcal{T}_Z Z\\ &- \mathcal{T}_{\nabla_Y Y}Y + \mathcal{T}_{\nabla_Y Y}Z + \mathcal{T}_{\nabla_Y Z}Y - \mathcal{T}_{\nabla_Y Z}Z\\ &+ \mathcal{T}_{\nabla_Z Y}Y - \mathcal{T}_{\nabla_Z Y}Z - \mathcal{T}_{\nabla_Z Z}Y + \mathcal{T}_{\nabla_Z Z}Z\\ &- \mathcal{T}_Y\nabla_Y Y + \mathcal{T}_Y\nabla_Y Z + \mathcal{T}_Y\nabla_Z Y - \mathcal{T}_Y\nabla_Z Z\\ &+ \mathcal{T}_Z\nabla_Y Y - \mathcal{T}_Z\nabla_Y Z - \mathcal{T}_Z\nabla_Z Y + \mathcal{T}_Z\nabla_Z Z = 0.\end{aligned} \tag{20}$$

Adding and subtracting (19) and (20), we have

$$(\nabla_Y\mathcal{T})_Z Z + (\nabla_Z\mathcal{T})_Y Z + (\nabla_Z\mathcal{T})_Z Y = 0$$

and

$$(\nabla_Y\mathcal{T})_Y Z + (\nabla_Y\mathcal{T})_Z Y + (\nabla_Z\mathcal{T})_Y Y = 0$$

respectively. Hence we get

$$(\nabla_Y\mathcal{T})_Z Z + (\nabla_Z\mathcal{T})_Y Z + (\nabla_Z\mathcal{T})_Z Y + (\nabla_Y\mathcal{T})_Y Z + (\nabla_Y\mathcal{T})_Z Y + (\nabla_Z\mathcal{T})_Y Y = 0. \tag{21}$$

Repeating similar calculations for $X + Z, X - Z$ and $X + Y, X - Y$ we obtain

$$(\nabla_X\mathcal{T})_Z Z + (\nabla_Z\mathcal{T})_X Z + (\nabla_Z\mathcal{T})_Z X + (\nabla_X\mathcal{T})_X Z + (\nabla_X\mathcal{T})_Z X + (\nabla_Z\mathcal{T})_X X = 0 \tag{22}$$

and

$$(\nabla_X\mathcal{T})_Y Y + (\nabla_Y\mathcal{T})_X Y + (\nabla_Y\mathcal{T})_Y X + (\nabla_X\mathcal{T})_X Y + (\nabla_X\mathcal{T})_Y X + (\nabla_Y\mathcal{T})_X X = 0 \tag{23}$$

Using (21), (22) and (23) in (18) , we obtain

$$(\nabla_X\mathcal{T})_Y Z + (\nabla_Y\mathcal{T})_X Z + (\nabla_X\mathcal{T})_Z Y + (\nabla_Z\mathcal{T})_X Y + (\nabla_Y\mathcal{T})_Z X + (\nabla_Z\mathcal{T})_Y X = 0. \tag{24}$$

Since $\mathbb{E}^n$ is flat, using (3) and (4) we get;

$$\begin{aligned}&h\nabla_X\mathcal{T}_Y Z + \mathcal{T}_X\mathcal{T}_Y Z + \mathcal{T}_X\hat{\nabla}_Y Z + \hat{\nabla}_X\hat{\nabla}_Y Z - h\nabla_Y\mathcal{T}_X Z\\ -\ &\mathcal{T}_Y\mathcal{T}_X Z - \mathcal{T}_Y\hat{\nabla}_X Z - \hat{\nabla}_Y\hat{\nabla}_X Z - \mathcal{T}_{[X,Y]}Z - \hat{\nabla}_{[X,Y]}Z = 0\end{aligned} \tag{25}$$

where $X, Y, Z \in \mathfrak{X}^{\mathcal{V}}(\mathbb{E}^n)$. Thus taking vertical parts and horizontal parts of (25) we derive,

$$\hat{R}(X,Y)Z = -\mathcal{T}_X\mathcal{T}_Y Z + \mathcal{T}_Y\mathcal{T}_X Z \tag{26}$$

and

$$h\nabla_X\mathcal{T}_Y Z + \mathcal{T}_X\hat{\nabla}_Y Z - h\nabla_Y\mathcal{T}_X Z - \mathcal{T}_Y\hat{\nabla}_X Z - \mathcal{T}_{[X,Y]}Z = 0 \tag{27}$$

respectively, where $\hat{R}$ is the curvature tensor field of $\bar{M}$. Using (27) we find

$$(\nabla_X\mathcal{T})_Y Z - (\nabla_Y\mathcal{T})_X Z = 2(\mathcal{T}_X\mathcal{T}_Y Z - \mathcal{T}_Y\mathcal{T}_X Z). \tag{28}$$

Thus from (26) and (28) we get

$$(\nabla_X\mathcal{T})_Y Z - (\nabla_Y\mathcal{T})_X Z = -2\hat{R}(X,Y,Z). \tag{29}$$

Appliying cyclic permutation to (29), we have

$$2(\nabla_Y\mathcal{T})_X Z - 2\hat{R}(X,Y,Z) + 2(\nabla_X\mathcal{T})_Z Y - 2\hat{R}(Z,X,Y) + 2(\nabla_Z\mathcal{T})_Y X - 2\hat{R}(Y,Z,X) = 0. \tag{30}$$

Using the first Bianchi identity, we arrived at

$$(\nabla_Y\mathcal{T})_X Z + (\nabla_X\mathcal{T})_Z Y + (\nabla_Z\mathcal{T})_Y X = 0. \tag{31}$$

This means $\sum_{XYZ}(\nabla_X\mathcal{T})_Z Y = 0$. The converse is clear.

(1) $\Leftrightarrow$ (3) If (1) is satisfied, we have $(\nabla_X\mathcal{T})_X X = 0$ so $(\nabla_X\mathcal{T})_X X$ and $\mathcal{T}_X X$ are linearly dependent. Then from (3) we have

$$k^2 \;=\; g(\nabla_X X, \nabla_X X) = g(\mathcal{T}_X X, \mathcal{T}_X X)$$

where k is the curvature of horizontal section. Taking covariant derivative of k^2 with respect to s, and using (4) we obtain

$$\begin{aligned}\frac{dk^2}{ds} &= 2g(\nabla_X\mathcal{T}_X X, \mathcal{T}_X X)\\ &= 2g((\nabla_X\mathcal{T})_X X, \mathcal{T}_X X)\end{aligned} \tag{32}$$

which implies $\frac{dk^2}{ds} = 0$. Conversely, if F has planar horizontal section at $p \in \mathbb{E}^n$ with p as one of its vertices, then Theorem 3.5 and (32) tell us both $(\nabla_X\mathcal{T})_X X$ and $\mathcal{T}_X X$ are linearly dependent and they are orthogonal to each other which implies that $(\nabla_X\mathcal{T})_X X = 0$.

□

In 1969, Bishop and O'Neill [8] introduced a new concept of product manifolds, called warped product manifolds, as follows. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, $f : M_1 \to (0,\infty)$ and $\pi : M_1 \times M_2 \to M_1$, $\sigma : M_1 \times M_2 \to M_2$ the projection maps given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for every $(x, y) \in M_1 \times M_2$. Denote the warped product manifold $M = (M_1 \times_f M_2, g)$, where

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + f(\pi(x,y))^2 g_2(\sigma_* X, \sigma_* Y)$$

for every X and Y of M and $*$ is symbol for the tangent map. The manifolds M_1 and M_2 are called the base and the fiber of M. Bishop and O'Neill constructed a large variety of complete Riemannian manifolds of everywhere negative sectional curvature by using warped product. For each $y \in M_2$, the map $\pi|_{(M_1\times y)}$ is an isometry onto M_1. For each $x \in M_1$, the map $\sigma|_{(x\times M_2)}$ is a positive homothety onto M_2, with scalar factor $1/f(x)$. For each $(x, y) \in M$, the leaf $M_1 \times y$ and the fiber $x \times M_2$ are orthogonal at (x, y). The fibers $x \times M_2 = \pi^{-1}(x)$ and the leaves $M_1 \times y = \sigma^{-1}(y)$ of the warped product are totally geodesic and totally umbilical, respectively.

It is easy to prove that the first projection $\pi : M_1 \times_f M_2 \longrightarrow M_1$ is a Riemannian submersion whose vertical and horizontal spaces at any point $p = (p_1, p_2)$ are respectively identified with $T_{p_2}M_2, T_{p_1}M_1$. Since the horizontal distribution is integrable, the invariant $\mathcal{A}$ associated with π vanishes. To compute the invariant $\mathcal{T}$, for any $X, Y \in \mathfrak{X}^{\mathcal{V}}(M)$, $U \in \mathfrak{X}^{\mathcal{H}}(M)$ applying (1), one obtains:

$$\mathcal{T}_X Y = -\frac{1}{f} g(X, Y) grad f, \tag{33}$$

and any fibre of π (which is identified with M_2) turns out to be a totally umbilical submanifold of $M_1 \times_f M_2$ with mean curvature vector field $H = -\frac{1}{f} grad f$. Thus, if f is a non-constant function, the fibres of π are not minimal submanifolds of $M_1 \times_f M_2$, for details, see: [17].

Proposition 3.7. *Let $M = M_1 \times_f M_2$ be a warped product submanifold of an Euclidean space $\mathbb{E}^n$. Then the projection $\pi : M_1 \times_f M_2 \longrightarrow M_1$, as a Riemannian submersion, has $P2 - PHS$.*

Proof. To show this, we compute $(\nabla_X \mathcal{T})_X X$. First of all, by straightforward calculation we have

$$(\nabla_X \mathcal{T})_X X = \nabla_X \mathcal{T}_X X - \mathcal{T}_{\nabla_X X} X - \mathcal{T}_X \nabla_X X. \tag{34}$$

Then using (33) we get

$$(\nabla_X \mathcal{T})_X X = \nabla_X(-g(X, X) grad(ln f)) + g(\nabla_X X, X) grad(ln f) + g(X, \nabla_X X) grad(ln f).$$

Hence we obtain $(\nabla_X \mathcal{T})_X X = 0$ which shows that π is a P2-PHS. □

Above proposition gives a method to construct certain submersion to have $P2 - PHS$ property. As an application of above method we give the following example.

Example 3.8. *Let $M(r, t, s) = (rcoss, rsins, tcoss, tsins)$ be a submanifold in $\mathbb{E}^4$. It is easy to see that this is a warped product submanifold $M = M_1 \times_f M_2$ with the warping function $\sqrt{r^2 + t^2}$ in the form*

$$g = dt^2 + dr^2 + (r^2 + t^2) ds^2.$$

Now consider the Riemannian submersion $\pi : M_1 \times_{\sqrt{r^2+t^2}} M_2 \longrightarrow M_1$. Then by direct computations the vertical distribution is

$$kerF_* = span\{-rsins\frac{\delta}{\delta x_1} + rcoss\frac{\delta}{\delta x_2} - tsins\frac{\delta}{\delta x_3} + tcoss\frac{\delta}{\delta x_4}\} = span\{\gamma'\}$$

and the horizontal distribution is spanned by Z_1, Z_2 where

$$Z_1 = coss\frac{\delta}{\delta x_1} + sins\frac{\delta}{\delta x_2}, \qquad Z_2 = coss\frac{\delta}{\delta x_3} + sins\frac{\delta}{\delta x_4}.$$

Then it is easy to see that $\gamma', \gamma'', \gamma'''$ are linearly dependent.

Corollary 3.9. *Let $F : M_1 \longrightarrow M_2$ be a Riemannian submersion with totally umbilical fibers. Then F has pointwise 2-planar horizontal section if the mean curvature vector field of the fibers are parallel along fibers.*

Remark 3.10. *From [10], we know that if a submanifold has planar normal section at $p \in \mathbb{E}^n$ with p as one of its vertices, then implies that this submanifold is a parallel submanifold. But our Theorem 3.6 shows that if F has planar horizontal section $p \in \mathbb{E}^n$ with p as one of its vertices, then the tensor field $\mathcal{T}$ may not be zero. Thus our results are different from immersion case.*

Remark 3.11. *One can see that a Riemannian submersion with totally geodesic fibers has $P2 - PHS$ property, although the converse is not true. Ranjan classified the base manifold of the Riemannian submersion with totally geodesic fibers from certain spheres of Euclidean spaces onto Riemannian manifolds by using Clifford algebras and their representations via O'Neill's tensor field $\mathcal{A}$ in [24], see also [17, Section 4 of Chapter 2]. Similar results have been obtained by other authors in [7] and [25]. This suggests that it may be possible to obtain certain characterizations of Riemannian submersions with pointwise k-planar horizontal sections by using Clifford algebras and their representations. This problem will be our next research problem.*

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