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Warped Product Skew CR-Submanifolds of Kenmotsu Manifolds and their Applications

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Abstract. In this paper, we introduce the notion of warped product skew CR-submanifolds in Kenmotsu manifolds. We obtain several results on such submanifolds. A characterization for skew CR-submanifolds is obtained. Furthermore, we establish an inequality for the squared norm of the second fundamental form of a warped product skew CR-submanifold $M_1 \times_f M_\perp$ of order 1 in a Kenmotsu manifold $\tilde{M}$ in terms of the warping function such that $M_1 = M_T \times M_\theta$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively. Finally, some applications of our results are given.

1. Introduction

The notion of CR-submanifolds was introduced by Bejancu [6] as a generalization of the complex and totally real submanifolds of almost Hermitian manifolds. A more general family of submanifolds are slant submanifolds introduced and defined by B.-Y. Chen [13, 14] in 1990. A generalization of slant submanifolds was given by Papaghiuc [34] by defining semi-slant submanifolds of almost Hermitian manifolds, for which the slant and CR-submanifolds are particular cases. Later on, J.L. Cabrerizo et al. [10, 11] studied slant and semi-slant submanifolds of an almost contact metric manifold.

On the other hand, A. Carriazo defined hemi-slant submanifolds under the name of anti-slant submanifolds [12] and showed that CR-submanifols and slant submanifolds are hemi-slant submanifolds. In [37], B. Sahin studied these submanifolds under the name of hemi-slant submanifolds for their warped products.

In [35], Ronsse introduced skew CR-submanifolds of Kaehler manifolds as a generalization of slant submanifolds and CR-submanifolds. It is important to observe that semi-slant submanifolds [34] and hemi-slant submanifolds [37] are particular cases of skew CR-submanifols.

In the beginning of this century, B.-Y. Chen introduced the notion of warped product CR-submanifolds [15, 16]. On the basis of Chen's idea on warped product submanifolds many articles have been appeared (for instance see [4, 5], [9], [17], [29], [32], [31] [36]) and references therein. For a detailed survey on warped product manifolds and warped product submanifolds we referee to Chen's books [18, 20] and his survey article [19].

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Recently, Sahin [38] introduced the notion of skew CR-warped products of Kaehler manifolds which are the generalizations of CR-warped products which are introduced by B.-Y. Chen [15] and warped product hemi-slant submanifolds studied in [37].

As Kenmotsu manifolds are themselves warped product manifolds, it is interesting to study warped product submanifolds of Kenmotsu manifolds. There are many papers on warped product submanifolds of Kenmotsu manifolds (see [3], [4, 5], [2, 33]).

Motivated by the above studies, in this paper we introduce and study warped product skew CR-submanfolds of Kenmotsu manifolds. It is shown that the skew CR-warped products are the generalizations of CR-warped products studied in [3, 27] and warped product pseudo-slant submanifolds studied in [2] of Kenmotsu manifolds. The construction of warped product skew CR-submanifolds can be considered as a special case of multiply warped product submanifolds studied in [25].

The paper is organized as follows: In Section 2, we give some preliminaries (formulas and definitions) for submanifolds of Kenmotsu manifolds. Section 3 is devoted to the study of skew CR-submanifolds of Kenmotsu manifolds. Some basic lemmas are given which are useful in the next sections. In Section 4, we study warped product skew CR-submanifolds of Kenmotsu manifolds. We start with a non-trivial example of warped product skew CR-submanifolds and then we derive some useful lemmas. In Section 5, necessary and sufficient conditions for a skew CR-submanifold to be locally a warped product submanifold are obtained. In Section 6, we establish a sharp relationship for the squared norm of the second fundamental form $||h||^2$ in terms of the warping function f of a warped product skew CR-submanifold M of order 1 in Kenmotsu manifolds. The equality case is also considered. In Section 7, some applications of our results are given.

2. Preliminaries

A $(2n+1)$-dimensional Riemannian manifold $\tilde{M}$ is said to be an almost contact metric manifold [8] if it admits a $(1,1)$ tensor field φ, a vector field ξ, an 1-form η and a Riemannian metric g, which satisfy the following relations

$$\varphi^2 = -I + \eta \otimes \xi, \ \ \varphi\xi = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(\xi) = 1, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ \eta(X) = g(X, \xi), \tag{2}$$

for any vector fields X, Y on $\tilde{M}$. In addition, if

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \ \ \tilde{\nabla}_X \xi = X - \eta(X)\xi \tag{3}$$

where $\tilde{\nabla}$ is the Reimannian connection with respect to g, then $(\tilde{M}, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold [28]. The covariant derivative of φ is defined as

$$(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y \tag{4}$$

for any vector fields X, Y on $\tilde{M}$.

Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connections on the tangent and normal bundles TM and $T^\perp M$ of M, respectively, then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{5}$$

for any vector fields $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A are related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{6}$$

For any $X \in \Gamma(TM)$, we write

$$\varphi X = TX + FX, \tag{7}$$

where TX is the tangential component of φX and FX is the normal component of φX. Similarly, for any vector field N normal to M, we put

$$\varphi N = BN + CN, \tag{8}$$

where BN and CN are the tangential and normal components of φN, respectively.

The invariant and anti-invariant submanifolds are defined depending on the behaviour the tangent spaces under the action of the almost contact structure φ. A submanifold M tangent to the structure vector field ξ is said to be *invariant* (resp. *anti-invariant*) if $\varphi(T_pM) \subseteq T_pM$, $\forall\, p \in M$ (resp. $\varphi(T_pM) \subseteq T_p{}^{\perp}M$, $\forall\, p \in M$).

We denote by H, the mean curvature vector defined as $H(p) = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of the tangent space T_pM, for any $p \in M$.

Also, we set

$$\|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)) \ \text{ and } \ h_{ij}^r = g(h(e_i, e_j), e_r), \tag{9}$$

for $i, j = 1, \cdots, m$ and $r = m+1, \cdots, 2n+1$, where $\{e_{m+1}, ..., e_{2n+1}\}$ is an orthonormal basis of the normal space $T_p^{\perp}M$.

For a differentiable function f on an m-dimensional manifold M, the gradient $\vec{\nabla} f$ of f is defined as

$$g(\vec{\nabla} f, X) = X(f)$$

for any X tangent to M. As a consequence, we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^{m} (e_i(f))^2 \tag{10}$$

for an orthonormal frame $\{e_1, \cdots, e_m\}$ on M.

A submanifold M of a Riemannian manifold $\tilde{M}$ is said to be *totally umbilical* if $h(X, Y) = g(X, Y)H$ and *totally geodesic* if $h(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$. Also, M is minimal in $\tilde{M}$, if $H = 0$.

There are some other classes of submanifolds of almost contact Riemannian manifolds which are defined as follows:

A submanifold M tangent to the structure vector field ξ is said to be a *contact CR-submanifold* if there exists a pair of orthogonal distributions $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^{\perp} : p \to \mathcal{D}_p^{\perp}$, $\forall\, p \in M$, such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by ξ.
(ii) $\mathcal{D}$ is invariant by φ, i.e., $\varphi\mathcal{D} = \mathcal{D}$.
(iii) $\mathcal{D}^{\perp}$ is anti-invariant by φ, i.e., $\varphi\mathcal{D}^{\perp} \subseteq TM^{\perp}$.

Invariant and anti-invariant submanifolds are special cases of a contact CR-submanifolds. If we denote the dimensions of the distribution $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by d_1 and d_2, respectively, then M is invariant (resp. anti-invariant) if $d_2 = 0$ (resp. $d_1 = 0$).

A submanifold M is called slant [11] if for each $X \in T_pM$ linearly independent on ξ_p, the angle $\theta(X)$ between φX and T_pM is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle\xi_p\rangle$.

On a slant submanifold, if $\theta = 0$, then M is invariant and if $\theta = \frac{\pi}{2}$ then M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

A submanifold M is called semi-slant [10] if it is endowed with two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that $\mathcal{D}$ is invariant with respect to φ and $\mathcal{D}^\theta$ is a proper slant distribution.
A submanifold M is called *pseudo-slant submanifold* if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that

$$TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle$$

where $\mathcal{D}^\perp$ is an anti-invariant distribution and its orthogonal complementary distribution $\mathcal{D}^\theta$ is proper slant.

From the definition of a pseudo-slant submanifold, if we consider the dimensions $\dim \mathcal{D}^\perp = d_1$, and $\dim \mathcal{D}^\theta = d_2$, then it is clear that contact CR-submanifolds and slant submanifolds are particular classes of pseudo-slant submanifolds with $\theta = 0$ and $d_1 = 0$, respectively. Also, an invariant (resp. anti-invariant) submanifold is a pseudo-slant submanifold with $\theta = 0$ and $d_1 = 0$ (resp. $d_2 = 0$).

The normal bundle $T^\perp M$ of a pseudo-slant submanifold M is decomposed as

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \nu$$

where ν is a φ-invariant normal subbundle in the normal bundle $T^\perp M$.

A useful characterization of slant submanifolds was given in [11] as follows:

Theorem 2.1. *[11] Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi) \tag{11}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequence of the above theorem

$$g(TX, TY) = \cos^2\theta\,(g(X,Y) - \eta(X)\eta(Y)), \tag{12}$$

$$g(FX, FY) = \sin^2\theta\,(g(X,Y) - \eta(X)\eta(Y)), \tag{13}$$

for any vector fields X, Y tangent to M.

Also, for a slant submanifold of an almost contact metric manifold, we have the following useful result.

Theorem 2.2. *[41] Let M be a proper slant submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then*

$$(a)\ \ BFX = \sin^2\theta(-X + \eta(X)\xi), \quad (b)\ \ CFX = -FTX \tag{14}$$

for any $X \in \Gamma(TM)$.

3. Skew CR-submanifolds of Kenmotsu manifolds

Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$. We recall the definition of skew CR-submanifolds from [35]. Throughout the paper we consider the the structure vector field ξ is tangent to the submanifold otherwise the submanifold is C-totally real [29].

For any X and Y in T_pM, we have $g(TX, Y) = -g(X, TY)$. Hence, it follows that T^2 is a symmetric operator on the tangent space T_pM, for all $p \in M$. Therefore, its eigenvalues are real and it is diagonalizable. Moreover, its eigenvalues are bounded by -1 and 0. For each $p \in M$, we may set

$$\mathcal{D}_p^\lambda = ker\{T^2 + \lambda^2(p)I\}_p,$$

where I is the identity transformation and $\lambda(p) \in [0,1]$ such that $-\lambda^2(p)$ is an eigenvalue of $T^2(p)$. We note that $\mathcal{D}_p^1 = kerF$ and $\mathcal{D}_p^0 = kerT$. $\mathcal{D}_p^1$ is the maximal φ-invariant subspace of T_pM and $\mathcal{D}_p^0$ is the maximal

φ-anti-invariant subspace of T_pM. From now on, we denote the distributions $\mathcal{D}^1$ and $\mathcal{D}^0$ by $\mathcal{D} \oplus \langle\xi\rangle$ and $\mathcal{D}^\perp$, respectively. Since T_p^2 is symmetric and diagonalizable, if $-\lambda_1^2(p), \cdots, -\lambda_k^2(p)$ are the eigenvalues of T^2 at $p \in M$, then T_pM can be decomposed as direct sum of mutually orthogonal eigenspaces, i.e.

$$T_pM = \mathcal{D}_p^{\lambda_1} \oplus \mathcal{D}_p^{\lambda_2} \cdots \oplus \mathcal{D}_p^{\lambda_k}.$$

Each $\mathcal{D}_p^{\lambda_i}$, $1 \le i \le k$, is a T-invariant subspace of T_pM. Moreover if $\lambda_i \neq 0$, then $\mathcal{D}_p^{\lambda_i}$ is even dimensional. We say that a submanifold M of a Kenmotsu manifold $\tilde{M}$ is a generic submanifold if there exists an integer k and functions λ_i, $1 \le i \le k$ defined on M with values in $(0,1)$ such that

(1) Each $-\lambda_i^2(p)$, $1 \le i \le k$ is a distinct eigenvalue of T^2 with

$$T_pM = \mathcal{D}_p \oplus \mathcal{D}_p^\perp \oplus \mathcal{D}_p^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_p^{\lambda_k} \oplus \langle\xi\rangle_p$$

for any $p \in M$.

(2) The dimensions of $\mathcal{D}_p$, $\mathcal{D}_p^\perp$ and $\mathcal{D}^{\lambda_i}$,$1 \le i \le k$ are independent on $p \in M$.

Moreover, if each λ_i is constant on M, then M is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with $k = 0$, $\mathcal{D} \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$. And slant submanifolds are also a particular class of skew CR-submanifolds with $k = 1$, $\mathcal{D} = \{0\}$, $\mathcal{D}^\perp = \{0\}$ and λ_1 is constant. Moreover, if $\mathcal{D}^\perp = \{0\}$, $\mathcal{D} \neq 0$ and $k = 1$, then M is a semi-slant submanifold. Furthermore, if $\mathcal{D} = \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $k = 1$, then M is a pseudo-slant (or hemi-slant) submanifold.

A submanifold M of a Kenmotsu manifold $\tilde{M}$ is said to be a proper skew CR-submanifold of order 1 if M is a skew CR-submanifold with $k = 1$ and λ_1 is constant. In that case, the tangent bundle of M is decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$$

The normal bundle $T^\perp M$ of a skew CR-submanifold M is decomposed as

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \nu,$$

where ν is a φ-invariant normal subbundle of $T^\perp M$.

Now, we give the following results which are useful for the further study.

Lemma 3.1. *Let M be a proper skew CR-submanifold of order* 1 *of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M. Then*

$$A_{\varphi Z}W = A_{\varphi W}Z \tag{15}$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp)$

Proof. The proof of this lemma is similar to Lemma 3.2 [2]. □

Lemma 3.2. *Let M be a proper skew CR-submanifold of order* 1 *of a Kenmotsu manifold $\tilde{M}$. Then the anti-invariant distribution $\mathcal{D}^\perp$ is always integrable.*

Proof. For any $X_1 \in \Gamma(\mathcal{D})$, $Z, W \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned} g([Z,W], X_1) &= g(\tilde{\nabla}_Z W, X_1) - g(\tilde{\nabla}_W Z, X_1) \\ &= g(\varphi\tilde{\nabla}_Z W, \varphi X_1) + \eta(\tilde{\nabla}_Z W)\eta(X_1) - g(\varphi\tilde{\nabla}_W Z, \varphi X_1) - \eta(\tilde{\nabla}_W Z)\eta(X_1). \end{aligned}$$

Using (4), we derive

$$g([Z,W], X_1) = g(\tilde{\nabla}_Z \varphi W, \varphi X_1) - g((\tilde{\nabla}_Z \varphi)W, \varphi X_1) - g(\tilde{\nabla}_W \varphi Z, \varphi X_1) + g((\tilde{\nabla}_W \varphi)Z, \varphi X_1).$$

Then from (3) and (5), we have

$$g([Z,W],X_1) = -g(A_{\varphi W}Z,\varphi X_1) + g(A_{\varphi Z}W,\varphi X_1).$$

From (15), we find

$$g([Z,W],X_1) = 0. \tag{16}$$

Similarly, for any $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z,W \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned} g([Z,W],X_2) &= g(\tilde{\nabla}_Z W, X_2) - g(\tilde{\nabla}_W Z, X_2) \\ &= g(\varphi\tilde{\nabla}_Z W,\varphi X_2) + \eta(\tilde{\nabla}_Z W)\eta(X_2) - g(\varphi\tilde{\nabla}_W Z,\varphi X_2) - \eta(\tilde{\nabla}_W Z)\eta(X_2). \end{aligned}$$

From (4), we obtain

$$g([Z,W],X_2) = g(\tilde{\nabla}_Z\varphi W,\varphi X_2) - g((\tilde{\nabla}_Z\varphi)W,\varphi X_2) - g(\tilde{\nabla}_W\varphi Z,\varphi X_2) + g((\tilde{\nabla}_W\varphi)Z,\varphi X_2).$$

Then from (3) and (7), we derive

$$g([Z,W],X_2) = g(\tilde{\nabla}_Z\varphi W, TX_2) + g(\tilde{\nabla}_Z\varphi W, FX_2) - g(\tilde{\nabla}_W\varphi Z, TX_2) - g(\tilde{\nabla}_W\varphi Z, FX_2).$$

Using (5), we get

$$g([Z,W],X_2) = g(A_{\varphi W}Z, TX_2) + g(W,\varphi\tilde{\nabla}_Z FX_2) - g(A_{\varphi Z}W, TX_2) - g(Z,\varphi\tilde{\nabla}_W FX_2).$$

Again, using (4) and (15), we obtain

$$g([Z,W],X_2) = g(\tilde{\nabla}_Z\varphi FX_2, W) - g((\tilde{\nabla}_Z\varphi)FX_2, W) - g(\tilde{\nabla}_W\varphi FX_2, Z) + g((\tilde{\nabla}_W\varphi)FX_2, Z).$$

Then from (4) and (8), we find that

$$g([Z,W],X_2) = g(\tilde{\nabla}_Z BFX_2, W) + g(\tilde{\nabla}_Z CFX_2, W) - g(\tilde{\nabla}_W BFX_2, Z) - g(\tilde{\nabla}_W CFX_2, Z).$$

Thus by Theorem 2.2, we get

$$\begin{aligned} g([Z,W],X_2) &= -\sin^2\theta g(\tilde{\nabla}_Z X_2, W) - g(\tilde{\nabla}_Z FTX_2, W) + \sin^2\theta g(\tilde{\nabla}_W X_2, Z) + g(\tilde{\nabla}_W FTX_2, Z) \\ &= \sin^2\theta g(\tilde{\nabla}_Z W, X_2) + g(A_{FTX_2}Z, W) - \sin^2\theta g(\tilde{\nabla}_W Z, X_2) - g(A_{FTX_2}W, Z). \end{aligned}$$

By the symmetric property of the shape operator, we find

$$\cos^2\theta g([Z,W],X_2) = 0.$$

Since M is a proper skew CR-submanifold, thus $\cos^2\theta \neq 0$. Then, we have

$$g([Z,W],X_2) = 0 \tag{17}$$

Also, for any $Z,W \in \Gamma(\mathcal{D}^\perp)$, we have

$$g([Z,W],\xi) = g(\tilde{\nabla}_Z W,\xi) - g(\tilde{\nabla}_W Z,\xi) = -g(\tilde{\nabla}_Z\xi, W) + g(\tilde{\nabla}_W\xi, Z).$$

By using (3), the right hand side of the above relation vanishes identically, hence we find that

$$g([Z,W],\xi) = 0. \tag{18}$$

By combining (16), (17) and (18), the result follows immediately. □

Lemma 3.3. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\theta)$. Then, we have*

$$g(\nabla_{X_1} Y_1, Z) = g(A_{\varphi Z} X_1, \varphi Y_1), \tag{19}$$

$$g(\nabla_{X_1} Y_2, Z) = \sec^2\theta \left(g(A_{\varphi Z} X_1, T Y_2) - g(A_{FTY_2} Z, X_1) \right), \tag{20}$$

$$g(\nabla_{Y_2} X_1, Z) = g(A_{\varphi Z} \varphi X_1, Y_2) \tag{21}$$

for any $X_1,\ Y_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $X_1,\ Y_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_{X_1} Y_1, Z) = g(\tilde{\nabla}_{X_1} Y_1, Z) = g(\varphi \tilde{\nabla}_{X_1} Y_1, \varphi Z) + \eta(\tilde{\nabla}_{X_1} Y_1)\eta(Z).$$

Using (4) and the fact that ξ is orthogonal to $\mathcal{D}^\perp$, we obtain

$$g(\nabla_{X_1} Y_1, Z) = g(\tilde{\nabla}_{X_1} \varphi Y_1, \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) Y_1, \varphi Z).$$

Then from (3) and (5), we get

$$g(\nabla_{X_1} Y_1, Z) = g(h(X_1, \varphi Y_1), \varphi Z).$$

Thus, (19) follows from the above relation by using (6). Also, for any $X_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} Y_2, Z) = g(\varphi \tilde{\nabla}_{X_1} Y_2, \varphi Z) + \eta(\tilde{\nabla}_{X_1} Y_2)\eta(Z).$$

Again, using (4), we get

$$g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} \varphi Y_2, \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) Y_2, \varphi Z).$$

From (3) and (7), we derive

$$\begin{aligned} g(\nabla_{X_1} Y_2, Z) &= g(\tilde{\nabla}_{X_1} T Y_2, \varphi Z) + g(\tilde{\nabla}_{X_1} F Y_2, \varphi Z) \\ &= g(h(X_1, T Y_2), \varphi Z) - g(\tilde{\nabla}_{X_1} \varphi F Y_2, Z) + g((\tilde{\nabla}_{X_1} \varphi) F Y_2, Z). \end{aligned}$$

The last term in the right hand side vanishes identically by using (3). Then from (8), the above equation takes the form

$$g(\nabla_{X_1} Y_2, Z) = g(h(X_1, T Y_2), \varphi Z) - g(\tilde{\nabla}_{X_1} B F Y_2, Z) - g(\tilde{\nabla}_{X_1} C F Y_2, Z).$$

Thus, on using Theorem 2.2, we find

$$g(\nabla_{X_1} Y_2, Z) = g(h(X_1, T Y_2), \varphi Z) + \sin^2\theta g(\tilde{\nabla}_{X_1} Y_2, Z) - \sin^2\theta \eta(Y_2) g(\tilde{\nabla}_{X_1} \xi, Z) + g(\tilde{\nabla}_{X_1} F T Y_2, Z).$$

Again, using (3) and (5), we get (20). Similarly, we have

$$g(\nabla_{Y_2} X_1, Z) = g(\tilde{\nabla}_{Y_2} X_1, Z) = g(\varphi \tilde{\nabla}_{Y_2} X_1, \varphi Z) + \eta(\tilde{\nabla}_{Y_2} X_1)\eta(Z).$$

Then from (3), we get

$$g(\nabla_{Y_2} X_1, Z) = g(\tilde{\nabla}_{Y_2} \varphi X_1, \varphi Z) = g(h(Y_2, \varphi X_1), \varphi Z) = g(A_{\varphi Z} \varphi X_1, Y_2),$$

which is (21). Hence, the lemma is proved completely. □

Lemma 3.4. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that ξ is orthogonal to $\mathcal{D}^\perp$. Then, the following hold:*

(i) If $\xi \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\theta)$, then

$$g(\nabla_{X_2} Y_2, Z) = \sec^2\theta \left(g(A_{\varphi Z} X_2, TY_2) - g(A_{FTY_2} Z, X_2) \right) \tag{22}$$

for any $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$, $Z \in \Gamma(\mathcal{D}^\perp)$.

(ii) If $\xi \in \Gamma(\mathcal{D})$, then

$$g(\nabla_Z V, X_2) = \sec^2\theta \left(g(A_{FTX_2} Z, V) - g(A_{\varphi V} Z, TX_2) \right), \tag{23}$$

$$g(\nabla_Z V, X_1) = -g(A_{\varphi V} Z, \varphi X_1) - \eta(X_1) g(Z, V), \tag{24}$$

for any $X_1 \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$, $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$.

(iii) If $\xi \in \Gamma(\mathcal{D}^\theta)$, then

$$g(\nabla_Z V, X_2) = \sec^2\theta \left(g(A_{FTX_2} Z, V) - g(A_{\varphi V} Z, TX_2) \right) - \eta(X_2) g(Z, V), \tag{25}$$

$$g(\nabla_Z V, X_1) = -g(A_{\varphi V} Z, \varphi X_1) \tag{26}$$

for any $X_1 \in \Gamma(\mathcal{D})$, $X_2 \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_{X_2} Y_2, Z) = g(\tilde{\nabla}_{X_2} Y_2, Z) = g(\varphi \tilde{\nabla}_{X_2} Y_2, \varphi Z) + \eta(\tilde{\nabla}_{X_2} Y_2) \eta(Z).$$

Using (4), we get

$$g(\nabla_{X_2} Y_2, Z) = g(\tilde{\nabla}_{X_2} \varphi Y_2, \varphi Z) - g((\tilde{\nabla}_{X_2} \varphi) Y_2, \varphi Z).$$

The second term in the right hand side is identically zero by using (3). Then from (7), we derive

$$g(\nabla_{X_2} Y_2, Z) = g(\tilde{\nabla}_{X_2} TY_2, \varphi Z) + g(\tilde{\nabla}_{X_2} FY_2, \varphi Z).$$

Using (4) and (7), we find

$$\begin{aligned} g(\nabla_{X_2} Y_2, Z) &= g(h(X_2, TY_2), \varphi Z) - g(\tilde{\nabla}_{X_2} \varphi FY_2, Z) + g((\tilde{\nabla}_{X_2} \varphi) FY_2, Z) \\ &= g(A_{\varphi Z} TY_2), X_2) - g(\tilde{\nabla}_{X_2} BFY_2, Z) - g(\tilde{\nabla}_{X_2} CFY_2, Z). \end{aligned}$$

Then using Theorem 2.2, we arrive at

$$g(\nabla_{X_2} Y_2, Z) = g(A_{\varphi Z} TY_2), X_2) + \sin^2\theta g(\tilde{\nabla}_{X_2} Y_2, Z) + g(\tilde{\nabla}_{X_2} FTY_2, Z).$$

Hence, the first part of the Lemma follows from the above relation by using (5) and (6). Now, for any $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z V, X_2) = g(\varphi \tilde{\nabla}_Z V, \varphi X_2) + \eta(X_2) \eta(\tilde{\nabla}_Z V).$$

Using (4), we obtain

$$g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z \varphi V, \varphi X_2) - g((\tilde{\nabla}_Z \varphi) V, \varphi X_2).$$

Then from (3) and (7), we find that

$$g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z \varphi V, TX_2) + g(\tilde{\nabla}_Z \varphi V, FX_2).$$

Again, using (4) and (5), we obtain

$$\begin{aligned} g(\nabla_Z V, X_2) &= g(\varphi\tilde{\nabla}_Z FX_2, V) - g(A_{\varphi V}Z, TX_2) \\ &= g(\tilde{\nabla}_Z \varphi FX_2, V) - g(A_{\varphi V}Z, TX_2) - g((\tilde{\nabla}_Z\varphi)FX_2, V) \\ &= g(\tilde{\nabla}_Z BFX_2, V) - g(A_{\varphi V}Z, TX_2) + g(\tilde{\nabla}_Z CFX_2, V). \end{aligned}$$

Hence by Theorem 2.2, we derive

$$\begin{aligned} g(\nabla_Z V, X_2) &= -g(A_{\varphi V}Z, TX_2) - \sin^2\theta g(\tilde{\nabla}_Z X_2, V) - g(\tilde{\nabla}_Z FTX_2, V) \\ &= -g(A_{\varphi V}Z, TX_2) + \sin^2\theta g(\nabla_Z V, X_2) + g(A_{FTX_2}Z, V) \end{aligned}$$

or,

$$\cos^2\theta g(\nabla_Z V, X_2) = g(A_{FTX_2}Z, V) - g(A_{\varphi V}Z, TX_2)$$

which gives (23). Also, for any $X_1 \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_Z V, X_1) = g(\tilde{\nabla}_Z V, X_1) = g(\varphi\tilde{\nabla}_Z V, \varphi X_1) + \eta(X_1)\eta(\tilde{\nabla}_Z V).$$

Using (3)-(5), we derive

$$\begin{aligned} g(\nabla_Z V, X_1) &= g(\tilde{\nabla}_Z \varphi V, \varphi X_1) - g((\tilde{\nabla}_Z\varphi)V, \varphi X_1) + \eta(X_1)\, g(\tilde{\nabla}_Z V, \xi) \\ &= g(\tilde{\nabla}_Z \varphi V, \varphi X_1) - \eta(X_1)\, g(\tilde{\nabla}_Z \xi, V) \\ &= -g(A_{\varphi V}Z, \varphi X_1) - \eta(X_1)\, g(Z, V), \end{aligned}$$

which is (24). Now, to prove the last part of the lemma, consider any $X_2 \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$. Then, we have

$$g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z V, X_2) = g(\varphi\tilde{\nabla}_Z V, \varphi X_2) + \eta(\tilde{\nabla}_Z V)\eta(X_2).$$

Using (4), we obtain

$$g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z \varphi V, \varphi X_2) - g((\tilde{\nabla}_Z\varphi)V, \varphi X_2) + \eta(X_2)\, g(\tilde{\nabla}_Z V, \xi).$$

Then from (3) and (7), we derive

$$g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z \varphi V, TX_2) + g(\tilde{\nabla}_Z \phi V, FX_2) - \eta(X_2)\, g(Z, V).$$

Again, using (4) and (5), we get

$$\begin{aligned} g(\nabla_Z V, X_2) &= -g(A_{\varphi V}Z, TX_2) - g(\tilde{\nabla}_Z FX_2, \varphi V) - \eta(X_2)\, g(Z, V) \\ &= -g(A_{\varphi V}Z, TX_2) + g(\varphi\tilde{\nabla}_Z FX_2, V) - \eta(X_2)\, g(Z, V) \\ &= -g(A_{\varphi V}Z, TX_2) + g(\tilde{\nabla}_Z BFX_2, V) + g(\tilde{\nabla}_Z CFX_2, V) - \eta(X_2)\, g(Z, V). \end{aligned}$$

Hence, by Theorem 2.2, we obtain

$$\begin{aligned} g(\nabla_Z V, X_2) &= -g(A_{\varphi V}Z, TX_2) - \sin^2\theta\, g(\tilde{\nabla}_Z X_2, V) + \sin^2\theta\, \eta(X_2)\, g(\tilde{\nabla}_Z \xi, V) + g(\tilde{\nabla}_Z FTX_2, V) - \eta(X_2)\, g(Z, V) \\ &= -g(A_{\varphi V}Z, TX_2) + \sin^2\theta\, g(\tilde{\nabla}_Z V, X_2) + \sin^2\theta\, \eta(X_2)\, g(Z, V) + g(A_{FTX_2}Z, V) - \eta(X_2)\, g(Z, V) \end{aligned}$$

or,

$$\cos^2\theta\, g(\nabla_Z V, X_2) = g(A_{FTX_2}Z, V) - g(A_{\varphi V}Z, TX_2) - \cos^2\theta\, \eta(X_2)\, g(Z, V)$$

which gives (25). Similarly, for any $X_1 \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_Z V, X_1) = g(\tilde{\nabla}_Z V, X_1) = g(\varphi\tilde{\nabla}_Z V, \varphi X_1) + \eta(X_1)\eta(\tilde{\nabla}_Z V).$$

Using (3) and the fact that $\xi \in \Gamma(\mathcal{D}^\theta)$, we derive

$$g(\nabla_Z V, X_1) = g(\tilde{\nabla}_Z \varphi V, \varphi X_1) - g((\tilde{\nabla}_Z \varphi)V, \varphi X_1) = -g(A_{\varphi V} Z, \varphi X_1),$$

which is (26). Hence, the proof of the lemma is complete. □

Lemma 3.5. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that ξ is orthogonal to $\mathcal{D}^\perp$. Then, we have*

$$g(\nabla_Z X_1, Y_2) = \csc^2\theta\,(g(A_{FY_2}Z, \varphi X_1) - g(A_{FTY_2}Z, X_1)) \tag{27}$$

for any $X_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $X_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_Z X_1, Y_2) = g(\tilde{\nabla}_Z X_1, Y_2) = g(\varphi\tilde{\nabla}_Z X_1, \varphi Y_2) + \eta(Y_2)\eta(\tilde{\nabla}_Z X_1).$$

Using (4), we find that

$$g(\nabla_Z X_1, Y_2) = g(\tilde{\nabla}_Z \varphi X_1, \varphi Y_2) - g((\tilde{\nabla}_Z \varphi)X_1, \varphi Y_2) - \eta(Y_2)\, g(\tilde{\nabla}_Z \xi, X_1).$$

Then from (3) and (7), we obtain

$$\begin{aligned} g(\nabla_Z X_1, Y_2) &= g(\tilde{\nabla}_Z \varphi X_1, TY_2) + g(\tilde{\nabla}_Z \varphi X_1, FY_2) \\ &= g(X_1, \varphi\tilde{\nabla}_Z TY_2) + g(h(Z, \varphi X_1), FY_2) \\ &= g(X_1, \tilde{\nabla}_Z \varphi TY_2) - g(X_1, (\tilde{\nabla}_Z \varphi)TY_2) + g(h(Z, \varphi X_1), FY_2). \end{aligned}$$

By using (3), (7) and (12), we derive

$$\begin{aligned} g(\nabla_Z X_1, Y_2) &= -\cos^2\theta\, g(\tilde{\nabla}_Z Y_2, X_1) + \cos^2\theta\, \eta(Y_2) g(X_1, \tilde{\nabla}_Z \xi) - g(A_{FTY_2}Z, X_1) + g(A_{FY_2}Z, \varphi X_1) \\ &= \cos^2\theta\, g(\nabla_Z X_1, Y_2) + g(A_{FY_2}Z, \varphi X_1) - g(A_{FTY_2}Z, X_1). \end{aligned}$$

which gives (3.13), hence the lemma is proved. □

4. Warped product skew CR-submanifolds of Kenmotsu manifolds

In [7], R.L. Bishop and B. O'Neill introduced the notion of warped product manifolds to study the manifolds of negative curvatures. These manifolds are natural generalizations of Riemannian product manifolds. The definition of a warped product is formulated as: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its canonical projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. The warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian metric g given by

$$g(X, Y) = g_1(\pi_{1*}(X), \pi_{1*}(Y)) + (f \circ \pi_1)^2\, g_2(\pi_{2*}(X), \pi_{2*}(Y))$$

for any tangent vector $X, Y \in TM$, where $*$ is the symbol for the tangent maps. If X is tangent to M_1 and V is tangent to M_2, then from lemma 7.3 of [7] we have

$$\nabla_X V = \nabla_V X = X(\ln f)V. \tag{28}$$

Recall that if $M = M_1 \times_f M_2$ is a warped product manifold, then M_1 is totally geodesic in M and M_2 is totally umbilical in M [7, 15].

In this section, we consider a warped product $M = M_1 \times_f M_\perp$ in a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_\theta$, where M_T, M_θ and $M_\perp$ are invariant, proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. Throughout this section we consider the structure vector field ξ is tangent to the submanifold M. Therefore, two possible cases arise:

Case 1. When ξ is tangent to $M_\perp$, then it is easy to see that the warped product is simply a Riemannian product. Thus, we will not discuss this case anymore for the non-existence of such proper warped products.

Case 2. When ξ is tangent to $M_1 = M_T \times M_\theta$. In this case either ξ is tangent to M_T or M_θ and in both subcases the warped product exists and we will discuss these kinds of warped products in our further study.

Let $M = M_1 \times_f M_\perp$ be a warped product skew CR-submanifold of order 1 of Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_\theta$ and the structure vector field ξ is tangent to M_1. Then, we call such submanifolds *skew CR-warped products* analogous to the *CR-warped products* introduced by Chen in [15, 16]. If we consider the dimensions of these submanifolds as $\dim M_T = d_1$, $\dim M_\theta = d_2$ and $\dim M_\perp = d_3$, then it is obvious that M is a CR-warped product if $d_2 = 0$ and M is a warped product pseudo-slant (or hemi-slant) submanifold if $d_1 = 0$.

Now, we provide the following non-trivial example of warped product skew CR-submanifolds of order 1 of an almost contact metric manifold.

Example 4.1. Consider a submanifold of $\mathbb{R}^{11}$ with the cartesian coordinates $(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, t)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \le i, j \le 5.$$

It is easy to show $\mathbb{R}^{11}$ is an almost contact metric manifold with respect to the Euclidean metric tensor of $\mathbb{R}^{11}$. Let us consider a submanifold M of $\mathbb{R}^{11}$ defined by the immersion χ as follows

$$\chi(u, v, w, s, r, t) = (u\cos w, u\sin w, u + v, s, 0, v\cos w, v\sin w, u - v, r, 0, t).$$

Then the tangent space of M is spanned by the following vectors

$$Z_1 = \cos w\,\frac{\partial}{\partial x_1} + \sin w\,\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3},\ Z_2 = \cos w\frac{\partial}{\partial y_1} + \sin w\,\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_3},$$
$$Z_3 = -u\sin w\,\frac{\partial}{\partial x_1} + u\cos w\,\frac{\partial}{\partial x_2} - v\sin w\,\frac{\partial}{\partial y_1} + v\cos w\,\frac{\partial}{\partial y_2},\ Z_4 = \frac{\partial}{\partial x_4},\ Z_5 = \frac{\partial}{\partial y_4},\ Z_6 = \frac{\partial}{\partial t}.$$

Then, we find

$$\varphi Z_1 = -\cos w\,\frac{\partial}{\partial y_1} - \sin w\,\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_3},\ \varphi Z_2 = \cos w\,\frac{\partial}{\partial x_1} + \sin w\,\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3},$$
$$\varphi Z_3 = u\sin w\,\frac{\partial}{\partial y_1} - u\cos w\,\frac{\partial}{\partial y_2} - v\sin w\,\frac{\partial}{\partial x_1} + v\cos w\,\frac{\partial}{\partial x_2};\ \varphi Z_4 = -\frac{\partial}{\partial y_4},\ \varphi Z_5 = \frac{\partial}{\partial x_4},\ \varphi Z_6 = 0.$$

It is easy to see that $\mathcal{D} = \mathrm{Span}\{Z_4, Z_5\}$ is an invariant distribution, $\mathcal{D}^\perp = \mathrm{Span}\{Z_3\}$ is an anti-invariant distribution and $\mathcal{D}^\theta = \mathrm{Span}\{Z_1, Z_2\}$ is a slant distribution with slant angle $\theta = \arccos(\frac{1}{3}) = 70°52'$ such that $\xi = \frac{\partial}{\partial t}$ is tangent to $\mathcal{D} \oplus \mathcal{D}^\theta$. Hence, we conclude that M is a proper skew CR-submanifold of order 1 of $\mathbb{R}^{11}$. It is easy to observe that $\mathcal{D} \oplus \mathcal{D}^\theta$ and $\mathcal{D}^\perp$ are integrable. Denoting the integral manifolds of $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by M_T, M_θ and $M_\perp$, respectively. Then the induced metric tensor g of M is given by

$$\begin{aligned} ds^2 &= 3(du^2 + dv^2) + ds^2 + dr^2 + dt^2 + (u^2 + v^2)dw^2 \\ &= g_{M_1} + (u^2 + v^2)g_{M_\perp}. \end{aligned}$$

Thus M is a warped product skew CR submanifold of $\mathbb{R}^{11}$ with the warping function $f = \sqrt{u^2 + v^2}$ such that $M_1 = M_T \times M_\theta$.

Now, we prove the following useful lemmas for a warped product skew CR-submanifold of a Kenmotsu manifold.

Lemma 4.2. *Let $M = M_1 \times_f M_\perp$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_1 and $M_1 = M_T \times M_\theta$, where M_T and M_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then, the following hold:*

(i) $\xi(\ln f) = 1$,

(ii) $g(h(X_1, Y_1), \varphi Z) = 0$,

(iii) $g(h(X_1, Z), FY_2) = h(X_1, Y_2), \varphi Z) = 0$,

(iv) $g(h(X_2, Z), FY_2) = g(h(X_2, Y_2), \varphi Z)$

for any $X_1, Y_1 \in \Gamma(TM_T)$, $X_2, Y_2 \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. For any $Z \in \Gamma(TM_\perp)$, we have $\tilde{\nabla}_Z \xi = Z$. Then from (5), we get

$$\nabla_Z \xi + h(Z, \xi) = Z.$$

Equating the tangential components and then using (28), we obtain $\xi(\ln f)Z = Z$. Taking the inner product with Z, we get (i). Now, for the other parts of the lemma we consider any $X_1, Y_1 \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$. Then, we have

$$g(h(X_1, Y_1), \varphi Z) = g(\tilde{\nabla}_{X_1} Y_1, \varphi Z) = -g(\varphi \tilde{\nabla}_{X_1} Y_1, Z).$$

Then from (4), we arrive at

$$g(h(X_1, Y_1), \varphi Z) = g((\tilde{\nabla}_{X_1} \varphi) Y_1, Z) - g(\tilde{\nabla}_{X_1} \varphi Y_1, Z) = g(\nabla_{X_1} Z, \varphi Y_1).$$

Thus, on using (28), we get $g(h(X_1, Y_1), \varphi Z) = X_1(\ln f)\, g(\varphi Y_1, Z) = 0$, which is (ii). To prove the third part of the lemma, consider any $X_1 \in \Gamma(TM_T)$, $Y_2 \in \Gamma(TM_\theta)$, and $Z \in \Gamma(TM_\perp)$. Then, we have

$$g(h(X_1, Y_2), \varphi Z) = g(\tilde{\nabla}_{X_1} Y_2, \varphi Z) = -g(\varphi \tilde{\nabla}_{X_1} Y_2, Z).$$

Using (4), we obtain

$$g(h(X_1, Y_2), \varphi Z) = g((\tilde{\nabla}_{X_1} \varphi) Y_2, Z) - g(\tilde{\nabla}_{X_1} \varphi Y_2, Z).$$

First term in the right hand side vanishes identically by using (3). Then from (7), we get

$$g(h(X_1, Y_2), \varphi Z) = -g(\tilde{\nabla}_{X_1} TY_2, Z) - g(\tilde{\nabla}_{X_1} FY_2, Z).$$

Using (5) and (28), we find that

$$g(h(X_1, Y_2), \varphi Z) = X_1(\ln f)\, g(TY_2, Z) + g(A_{FY_2} Z, X_1).$$

Hence, first equality of (iii) follows from the above relation by using (6) and the orthogonality of vector fields. For the second equality of (iii), we have

$$g(h(X_1, Y_2), \varphi Z) = g(\tilde{\nabla}_{Y_2} X_1, \varphi Z) = -g(\varphi \tilde{\nabla}_{Y_2} X_1, Z) = g((\tilde{\nabla}_{Y_2} \varphi) X_1, Z) - g(\varphi \tilde{\nabla}_{Y_2} X_1, Z).$$

From (3), (5) and (28) , we derive

$$g(h(X_1, Y_2), \varphi Z) = Y_2(\ln f)\, g(\varphi X_1, Z) = 0,$$

which is the second equality of (iii). Similarly, for any $X_2, Y_2 \in \Gamma(TM_\theta)$, and $Z \in \Gamma(TM_\perp)$, we have

$$g(h(X_2, Y_2), \varphi Z) = g(\tilde{\nabla}_{X_2} Y_2, \varphi Z) = -g(\varphi \tilde{\nabla}_{X_2} Y_2, Z).$$

From (4), we find

$$g(h(X_2, Y_2), \varphi Z) = g((\tilde{\nabla}_{X_2}\varphi)Y_2, Z) - g(\tilde{\nabla}_{X_2}\varphi Y_2, Z) = g(\tilde{\nabla}_{X_2}Z, TY_2) + g(A_{FY_2}X_2, Z).$$

Then (6) and (28), we obtain

$$g(h(X_2, Y_2), \varphi Z) = X_2(\ln f)\, g(TY_2, Z) + g(h(X_2, Z), FY_2).$$

Thus, the fourth part of the lemma follows form the above relation by using the orthogonality of vector fields. Hence, the lemma is proved completely. □

Lemma 4.3. *Let $M = M_1 \times {}_f M_\perp$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_1, where $M_1 = M_T \times M_\theta$. Then, we have*

$$g(h(X_1, Z), \varphi V) = -\varphi X_1(\ln f)\, g(Z, V) \tag{29}$$

for any $X_1 \in \Gamma(TM_T)$ and $Z, V \in \Gamma(TM_\perp)$.

Proof. For any $X_1 \in \Gamma(TM_T)$ and $Z, V \in \Gamma(TM_\perp)$, we have

$$g(h(X_1, Z), \varphi V) = g(\tilde{\nabla}_Z X_1, \varphi V) = -g(\varphi \tilde{\nabla}_V X_1, V).$$

Then from (4), we obtain

$$g(h(X_1, Z), \varphi V) = g((\tilde{\nabla}_Z \varphi)X_1, V) - g(\tilde{\nabla}_V \varphi X_1, V).$$

First term in the right hand side is identically zero by using (3). Then from (5) and (28), we get

$$g(h(X_1, Z), \varphi V) = -\varphi X_1(\ln f)\, g(Z, V),$$

which is (29). Thus, the proof is complete. □

If we interchange X_1 by φX_1 in (29) for any $X_1 \in \Gamma(TM_T)$, then two cases arise:

(i) When $\xi \in \Gamma(TM_T)$, then

$$g(h(\varphi X_1, Z), \varphi V) = (X_1(\ln f) - \eta(X_1))\; g(Z, V), \tag{30}$$

for any $X_1 \in \Gamma(TM_T)$ and $Z, V \in \Gamma(TM_\perp)$.

(ii) When $\xi \in \Gamma(TM_\theta)$, then

$$g(h(\varphi X_1, Z), \varphi V) = X_1(\ln f)\, g(Z, V), \tag{31}$$

for any $X_1 \in \Gamma(TM_T)$ and $Z, V \in \Gamma(TM_\perp)$.

Let $M = M_1 \times_f M_\perp$ be a warped product skew CR-submanifold of a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_\theta$. We denote the tangent spaces of M_T, M_θ and $M_\perp$ by $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively. Then M is called $\mathcal{D} - \mathcal{D}^\perp$ *mixed totally geodesic* if $h(X_1, Z) = 0$, for any $X_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, respectively. Similarly, M is a $\mathcal{D}^\theta - \mathcal{D}^\perp$ *mixed totally geodesic* if $h(X_2, Z) = 0$, for any $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, respectively.

The following theorem is a consequence of Lemma 4.3.

Theorem 4.4. *Let $M = M_1 \times_f M_\perp$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_\theta$, where M_T and M_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. If M is $\mathcal{D} - \mathcal{D}^\perp$ mixed totally geodesic warped product, then f is constant on M.*

Proof. The proof follows from Lemma 4.3. □

Lemma 4.5. *Let $M = M_1 \times {}_f M_\perp$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_1, where $M_1 = M_T \times M_\theta$. Then, we have*

$$g(h(Z,V), FX_2) - g(h(Z,X_2), \varphi V) = TX_2(\ln f)\, g(Z,V) \tag{32}$$

for any $X_2 \in \Gamma(TM_\theta)$ and $Z, V \in \Gamma(TM_\perp)$.

Proof. For any $X_2 \in \Gamma(TM_\theta)$ and $Z, V \in \Gamma(TM_\perp)$, we have

$$g(h(X_2,Z), \varphi V) = g(\tilde{\nabla}_Z X_2, \varphi V) = -g(\varphi \tilde{\nabla}_Z X_2, V).$$

Then (4), we derive

$$g(h(X_2,Z), \varphi V) = g((\tilde{\nabla}_Z \varphi)X_2, V) - g(\tilde{\nabla}_Z \varphi X_2, V).$$

First term in the right hand side identically vanishes by using (3). Then from (7), we get

$$g(h(X_2,Z), \varphi V) = -g(\tilde{\nabla}_Z TX_2, V) - g(\tilde{\nabla}_Z FX_2, V).$$

Using (5) and (28), we obtain

$$g(h(X_2,Z), \varphi V) = -TX_2(\ln f)\, g(Z,V) + g(A_{FX_2}Z, V),$$

which gives (32). Hence the proof is complete. □

If we interchange X_2 by TX_2 in (32) for any $X_2 \in \Gamma(TM_\theta)$, then two cases arise:

(i) When $\xi \in \Gamma(TM_T)$, then

$$g(h(Z,V), FTX_2) - g(h(TX_2,Z), \varphi V) = -\cos^2\theta\, X_2(\ln f)\, g(Z,V), \tag{33}$$

for any $X_2 \in \Gamma(TM_\theta)$ and $Z, V \in \Gamma(TM_\perp)$.

(ii) When $\xi \in \Gamma(TM_\theta)$, then

$$g(h(Z,V), FTX_2) - g(h(TX_2,Z), \varphi V) = \cos^2\theta\, \left(\eta(X_2) - X_2(\ln f)\right)\, g(Z,V), \tag{34}$$

for any $X_2 \in \Gamma(TM_\theta)$ and $Z, V \in \Gamma(TM_\perp)$.

5. A characterization of skew CR-warped products

As we have seen that there is no proper warped product skew CR-submanifold M of order 1 of a Kenmotsu manifold $\tilde{M}$, if M is $\mathcal{D} - \mathcal{D}^\perp$ mixed totally geodesic (Theorem 4.4). Thus, for further study, we consider the warped product skew CR-submanifold of order 1 of a Kenmotsu manifold, when it is a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic. Before proving a characterization, we need the following definitions.

Definition 5.1. A *foliation* on a manifold M is an integrable subbundle $\mathcal{F}$ of the tangent bundle of M, i.e., for any sections X and Y of $\mathcal{F}$, then the Lie bracket $[X, Y]$ is a section of $\mathcal{F}$ as well.

Definition 5.2. A *foliation* L on a Riemannian manifold M is called *totally umbilical* if every leaf of L is a totally umbilical Riemannian submanifold of M. If, in addition, the mean curvature vector of every leaf is parallel in the normal bundle, then L is called a *spherical foliation*, because in this case each leaf of L is an extrinsic sphere in M. If every leaf of L is a totally geodesic submanifold of M, then L is called a *totally geodesic foliation*.

Now, we recall the following well-known result of S. Hiepko [26].

Hiepko's Theorem. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

Now, we prove the following characterization by using Hiepko's Theorem and useful lemmas of Sections 3 and Sections 4.

Theorem 5.3. *Let M be a proper skew CR-submanifold of order* 1 *of a Kenmotsu manifold $\tilde{M}$. Then M is locally a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product skew CR-submanifold if and only if*

(i) $A_{\varphi Z}X$ has no component in $\Gamma(\mathcal{D}^\theta)$ and $\Gamma(\mathcal{D})$, i.e., $A_{\varphi Z}X \in \Gamma(\mathcal{D}^\perp)$, for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

(ii) For any $X_1 \in \Gamma(\mathcal{D})$, $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$A_{\varphi Z}X_1 = -\varphi X_1(\mu)Z, \;\; A_{\varphi Z}X_2 = 0, \;\; A_{FX_2}Z = TX_2(\mu)Z, \;\; (\xi\mu) = 1 \tag{35}$$

for some smooth function μ on M satisfying $V(\mu) = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let $M = M_1 \times_f M_\perp$ be a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic proper warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_\theta$. In this theorem the tangent spaces of M_T, M_θ and $M_\perp$ are also denoted by $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively. Then, from Lemma 4.2 (ii), we have

$$A_{\varphi Z}X_1 \perp \mathcal{D}, \;\; \forall\, X_1 \in \Gamma(\mathcal{D}), \;\; Z \in \Gamma(\mathcal{D}^\perp). \tag{36}$$

Similarly, from the second equality of lemma 4.2 (iii), we have

$$A_{\varphi Z}X_1 \perp \mathcal{D}^\theta, \;\; \forall\, X_1 \in \Gamma(\mathcal{D}), \;\; Z \in \Gamma(\mathcal{D}^\perp). \tag{37}$$

Also, for any $X_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(A_{\varphi Z}X_1, \xi) = g(h(X_1, \xi), \varphi Z) = 0, \tag{38}$$

since for a submanifold of a Kenmotsu manifold $h(U, \xi) = 0$, $\forall\, U \in \Gamma(TM)$. Thus, from (36)-(38), we conclude that

$$A_{\varphi Z}X_1 \in \Gamma(\mathcal{D}^\perp), \;\; \forall\, X_1 \in \Gamma(\mathcal{D}), \;\; Z \in \Gamma(\mathcal{D}^\perp). \tag{39}$$

Similarly, from the second equality of Lemma 4.2 (iii), we have

$$A_{\varphi Z}X_2 \perp \mathcal{D}, \;\; \forall\, X_2 \in \Gamma(\mathcal{D}^\theta), \;\; Z \in \Gamma(\mathcal{D}^\perp). \tag{40}$$

Also, for a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product skew CR-submanifold, from Lemma 4.2 (iv), we have

$$A_{\varphi Z}X_2 \perp \mathcal{D}^\theta, \;\; \forall\, X_2 \in \Gamma(\mathcal{D}^\theta), \;\; Z \in \Gamma(\mathcal{D}^\perp). \tag{41}$$

On the other hand, for any $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(A_{\varphi Z}X_2, \xi) = g(h(X_2, \xi), \varphi Z) = 0. \tag{42}$$

Then, from (40)-(42), we conclude that

$$A_{\varphi Z}X_2 \in \Gamma(\mathcal{D}^\perp), \;\; \forall\, X_2 \in \Gamma(\mathcal{D}^\theta), \;\; Z \in \Gamma(\mathcal{D}^\perp). \tag{43}$$

Also, from (38) and (42), we conclude that $A_{\varphi Z}\xi$ orthogonal to both $\mathcal{D}$ and $\mathcal{D}^{\theta}$. While $g(A_{\varphi Z}\xi, \xi) = 0$, i.e., $A_{\varphi Z}\xi \perp \langle\xi\rangle$, for all $Z \in \Gamma(\mathcal{D}^{\perp})$. Thus, we find that

$$A_{\varphi Z}\xi \in \Gamma(\mathcal{D}^{\perp}), \ \ Z \in \Gamma(\mathcal{D}^{\perp}). \tag{44}$$

Thus, from (39), (43) and (44), we get $A_{\varphi Z}X \in \Gamma(\mathcal{D}^{\perp})$, for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, which is (i).

For (ii), we proceed the proof as follows: From Lemma 4.2 (ii), we have $g(A_{\varphi Z}X_1, Y_1) = 0$, for any $X_1, Y_1 \in \Gamma(\mathcal{D})$, and $Z \in \Gamma(\mathcal{D}^{\perp})$. And, from the second equality of lemma 4.2 (iii), we have $g(A_{\varphi Z}X_1, Y_2) = 0$, for any $X_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Also, for any $X_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, we have $g(A_{\varphi Z}X_1, \xi) = g(h(X_1, \xi), \varphi Z) = 0$. Thus, we conclude that $g(A_{\varphi Z}X_1, X) = 0$, for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle)$, which means that either $A_{\varphi Z}X_1 \in \Gamma(\mathcal{D}^{\perp})$ or $A_{\varphi Z}X_1 = 0$. If $A_{\varphi Z}X_1 \in \Gamma(\mathcal{D}^{\perp})$, then by taking the inner product with $V \in \Gamma(\mathcal{D}^{\perp})$ and using Lemma 4.3, we get the first relation of (ii).

Now, for the second relation of (ii), form Lemma 4.2 (iii), we have $g(A_{\varphi Z}X_2, X_1) = 0$, for any $X_1 \in \Gamma(\mathcal{D})$, $X_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. And, for a $\mathcal{D}^{\theta} - \mathcal{D}^{\perp}$ mixed totally geodesic warped product skew CR-submanifold, from Lemma 4.2 (iv), we have $g(A_{\varphi Z}X_2, Y_2) = 0$, for any $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. On the other hand, for any $X_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, we have $g(A_{\varphi Z}X_2, \xi) = g(h(X_2, \xi), \varphi Z) = 0$. Hence, we conclude that $g(A_{\varphi Z}X_2, X) = 0$, for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle)$, which means that either $A_{\varphi Z}X_2 \in \Gamma(\mathcal{D}^{\perp})$ or $A_{\varphi Z}X_2 = 0$. If $A_{\varphi Z}X_2 \in \Gamma(\mathcal{D}^{\perp})$, then taking the inner product with $V \in \Gamma(\mathcal{D}^{\perp})$, we have $g(A_{\varphi Z}X_2, V) = g(h(X_2, V), \varphi Z) = 0$, by using the $\mathcal{D}^{\theta} - \mathcal{D}^{\perp}$ mixed totally geodesic condition. Hence, in both cases $A_{\varphi Z}X_2 = 0$, which is the second relation of (ii).

Similarly, from Lemma 4.2 (iii), we have $g(A_{FX_2}Z, X_1) = 0$, for any $X_1 \in \Gamma(\mathcal{D})$, $X_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. And, for a $\mathcal{D}^{\theta} - \mathcal{D}^{\perp}$ mixed totally geodesic warped product skew CR-submanifold, we have $g(A_{FX_2}Z, Y_2) = 0$, for any $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Also, for any $X_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, we have $g(A_{FX_2}Z, \xi) = g(h(Z, \xi), FX_2) = 0$. Thus, we conclude that $g(A_{FX_2}Z, X) = 0$, for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle)$, which means that either $A_{FX_2}Z \in \Gamma(\mathcal{D}^{\perp})$ or $A_{FX_2}Z = 0$. If $A_{FX_2}Z \in \Gamma(\mathcal{D}^{\perp})$, then from Lemma 4.5, for a $\mathcal{D}^{\theta} - \mathcal{D}^{\perp}$ mixed totally geodesic warped product submanifold, we find the third relation of (ii). The last relation of (ii) follows from Lemma 4.3 (i).

Conversely, suppose that M is a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that (i) and (ii) hold. Then, from Lemma 3.3 and the given conditions of (ii), we have

$$g(\nabla_{X_1}Y_1, Z) = 0, \quad g(\nabla_{X_1}Y_2, Z) = 0, \quad g(\nabla_{Y_2}X_1, Z) = 0 \tag{45}$$

for any $X_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Similarly, from Lemma 3.4 (i) and the given conditions of (ii), we find that

$$g(\nabla_{X_2}Y_2, Z) = 0, \tag{46}$$

for any $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Thus, the relations (45) and (46) imply that the leaves of $\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle$ are totally geodesic in M. Consider M_1 be a leaf of $\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle$, thus M_1 is totally geodesic in M. On the other hand, from Lemma 3.2, $\mathcal{D}^{\perp}$ is always integrable. If we consider the integral manifold $M_{\perp}$ of $\mathcal{D}^{\perp}$ and $h^{\perp}$ be the second fundamental form of $M_{\perp}$ in M, then for any $X_1 \in \Gamma(\mathcal{D})$ and $Z, V \in \Gamma(\mathcal{D}^{\perp})$, we have

$$g(h^{\perp}(Z, V), X_1) = g(\nabla_Z V, X_1) = g(\tilde{\nabla}_Z V, X_1) = -g(\tilde{\nabla}_Z X_1, V).$$

Using (2), (4) and the fact that ξ is orthogonal to $\mathcal{D}^{\perp}$, we obtain

$$g(h^{\perp}(Z, V), X_1) = g((\nabla_Z \varphi)X_1, \varphi V) - g(\nabla_Z \varphi X_1, \varphi V).$$

Then from (3) and (5), we arrive at

$$g(h^{\perp}(Z, V), X_1) = -\eta(X_1)g((Z, V) - g(h(\varphi X_1, Z), \varphi V) = -\eta(X_1)g((Z, V) - g(A_{\varphi V}\varphi X_1, Z).$$

Using the given hypothesis of the theorem i.e., the first relation of (ii) by interchanging X_1 by φX_1, we derive

$$g(h^{\perp}(Z, V), X_1) = -X_1(\mu)\, g(Z, V).$$

Thus, from the gradient definition, we find

$$g(h^{\perp}(Z,V),X_1) = -g(\vec{\nabla}\mu, X_1)\, g(Z,V). \tag{47}$$

Similarly, for any $X_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z, V \in \Gamma(\mathcal{D}^{\perp})$, we have

$$g(h^{\perp}(Z,V),TX_2) = g(\tilde{\nabla}_Z V, TX_2) = g(\tilde{\nabla}_Z V, \varphi X_2) - g(\tilde{\nabla}_Z V, FX_2).$$

Using the covariant derivative property of the connection and (2), we obtain

$$g(h^{\perp}(Z,V),TX_2) = g(\tilde{\nabla}_Z FX_2, V) - g(\varphi\tilde{\nabla}_Z V, X_2) = -g(A_{FX_2}Z,V) + g((\tilde{\nabla}_Z\varphi)V, X_2) - g(\tilde{\nabla}_Z\varphi V, X_2)$$

Then from (3), (5) and the hypothesis of the theorem, i.e., the third relation of (ii), we derive

$$g(h^{\perp}(Z,V),TX_2) = -TX_2(\mu)\, g(Z,V) + g(A_{\varphi V}Z, X_2).$$

From the gradient definition and the symmetric property of shape operator, we find that

$$g(h^{\perp}(Z,V),TX_2) = -g(\vec{\nabla}\mu, TX_2)\, g(Z,V) + g(A_{\varphi V}X_2, Z).$$

Second term in the right hand side of the above equation vanishes identically by using the second relation of (ii), thus, we obtain

$$g(h^{\perp}(Z,V),TX_2) = -g(\vec{\nabla}\mu, TX_2)\, g(Z,V). \tag{48}$$

Also, for any $Z, V \in \Gamma(\mathcal{D}^{\perp})$, we have

$$g(h^{\perp}(Z,V),\xi) = g(\tilde{\nabla}_Z V, \xi) = -g(\tilde{\nabla}_Z \xi, V) = -g(Z,V).$$

Then, from the hypothesis of the theorem, i.e., the last relation of (ii), we find that

$$g(h^{\perp}(Z,V),\xi) = -(\xi\mu)\, g(Z,V) = -g(\vec{\nabla}\mu, \xi)\, g(Z,V). \tag{49}$$

Thus, from (47)-(49), we conclude that

$$g(h^{\perp}(Z,V),X) = -g(\vec{\nabla}\mu, X)\, g(Z,V), \tag{50}$$

for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle)$, which means that

$$h^{\perp}(Z,V) = -\vec{\nabla}\mu\, g(Z,V). \tag{51}$$

The relation (51) implies that $M_{\perp}$ is totally umbilical in M with mean curvature vector $H^{\perp} = -\vec{\nabla}\mu$. Now, we have to show that $H^{\perp}$ is parallel with respect to the normal connection D^N of $M_{\perp}$ in M. For this, consider any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, thus we have

$$g(D^N_Z\vec{\nabla}\mu, X) = g(\nabla_Z\vec{\nabla}\mu, X) = g(\nabla_Z\vec{\nabla}^T\mu, X_1) + g(\nabla_Z\vec{\nabla}^{\theta}\mu, X_2) + g(\nabla_Z\vec{\nabla}^{\xi}\mu, \xi),$$

where $\vec{\nabla}^T\mu$, $\vec{\nabla}^{\theta}\mu$ and $\vec{\nabla}^{\xi}\mu$ are the gradient components of μ on M along $\mathcal{D}$, $\mathcal{D}^{\theta}$ and $\langle\xi\rangle$, respectively. Using the Riemannian metric property, we derive

$$\begin{aligned} g(D^N_Z\vec{\nabla}\mu, X) &= Zg(\vec{\nabla}^T\mu, X_1) - g(\vec{\nabla}^T\mu, \nabla_Z X_1) + Zg(\vec{\nabla}^{\theta}\mu, X_2) - g(\vec{\nabla}^{\theta}\mu, \nabla_Z X_2) + Zg(\vec{\nabla}^{\xi}\mu, \xi) - Zg(\vec{\nabla}^{\xi}\mu, \nabla_Z\xi) \\ &= Z(X_1\mu) - g(\vec{\nabla}^T\mu, [Z,X_1]) - g(\vec{\nabla}^T\mu, \nabla_{X_1}Z) + Z(X_2\mu) - g(\vec{\nabla}^{\theta}\mu, [Z,X_2]) - g(\vec{\nabla}^{\theta}\mu, \nabla_{X_2}Z) \\ &\quad + Z(\xi\mu) - g(\vec{\nabla}^{\xi}\mu, [Z,\xi]) - g(\vec{\nabla}^{\xi}\mu, \nabla_{\xi}Z). \end{aligned}$$

Now, using the definition of Lie bracket and a property of Riemannian connection, the above relation will be

$$g(D^N_Z\vec{\nabla}\mu, X) = X_1(Z\mu) + g(\nabla_{X_1}\vec{\nabla}^T\mu, Z) + X_2(Z\mu) + g(\nabla_{X_2}\vec{\nabla}^\theta\mu, Z) + \xi(Z\mu) + g(\nabla_\xi\vec{\nabla}^\xi\mu, Z) = 0, \tag{52}$$

since $(Z\mu) = 0$, for any $Z \in \Gamma(\mathcal{D}^\perp)$ and $\nabla_{X_1}\vec{\nabla}^T\mu + \nabla_{X_2}\vec{\nabla}^\theta\mu + \nabla_\xi\vec{\nabla}^\xi\mu = \nabla_X\vec{\nabla}\mu$ is orthogonal to $\mathcal{D}^\perp$, for any $X \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle)$ as we know that $\vec{\nabla}\mu$ is the gradient along M_1 and M_1 is totally geodesic in M. This means that the mean curvature vector $H^\perp$ of $M_\perp$ is parallel. Thus, the leaves of $\mathcal{D}^\perp$ are totally umbilical with non vanishing parallel mean curvature vector $-\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient of the function μ, i.e., $M_\perp$ is an extrinsic sphere in M. Hence, by Hiepko's Theorem, M is a warped product submanifold, which completes the proof. □

6. Inequalities for skew CR-warped products

In this section, we establish two estimates for the squared norm of the second fundamental form of a warped product skew CR submanifold $M = M_1 \times_f M_\perp$ in a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_\theta$, where M_T and M_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. First, we construct the following frame fields for a warped product skew CR-submanifold.

Let $M = M_1 \times_f M_\perp$ be a m-dimensional warped product skew CR-submanifold of order 1 of a $(2n + 1)$-dimensional Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ tangent to M_T, where $M_1 = M_T \times M_\theta$. Let us consider the dimensions $\dim M_T = 2p + 1$, $\dim M_\theta = 2q$ and $\dim M_\perp = s$ and their corresponding tangent spaces are denoted by $\mathcal{D} \oplus \langle\xi\rangle$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively. We set the orthonormal frame fields of $\mathcal{D} \oplus \langle\xi\rangle$ as follows

$$\{e_1, e_2, \cdots, e_p, e_{p+1} = \varphi e_1, \cdots, e_{2p} = \varphi e_p, e_{2p+1} = \xi\}$$

and the orthonormal frame fields of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively are

$$\{e_{2p+2} = e^*_1, \cdots, e_{2p+q+1} = e^*_q, e_{2p+q+2} = e^*_{q+1} = \sec\theta\, Te^*_1, \cdots, e_{2p+2q+1} = e^*_{2q} = \sec\theta\, Te_q{}^*\}$$

and

$$\{e_{2p+1+2q+1} = \hat{e}_1, \cdots, e_m = e_{2p+1+2q+s} = \hat{e}_s\}.$$

Then the orthonormal frames of the normal subbundles $F\mathcal{D}^\theta$, $\varphi\mathcal{D}^\perp$ and ν, respectively are

$$\{e_{m+1} = \tilde{e}_1 = \csc\theta\, Fe^*_1, \cdots e_{m+q} = \tilde{e}_q = \csc\theta\, Fe_q, e_{m+q+1} = \tilde{e}_{q+1} = \csc\theta \sec\theta FTe^*_1,$$

$$\cdots, e_{m+2q} = \tilde{e}_{2q} = \csc\theta \sec\theta FTe^*_q\},$$

$$\{e_{m+2q+1} = \tilde{e}_{2q+1} = \varphi\hat{e}_1, \cdots, e_{m+2q+s} = \tilde{e}_{2q+s} = \varphi\hat{e}_s\}$$

and

$$\{e_{m+2q+s+1}, \cdots, e_{2n+1}\}.$$

It is clear that $\dim\nu = (2n + 1 - m - 2q - s)$.

Now, we establish the following relationship for the squared norm of the second fundament form of the warped product skew CR-submanifold in Kenmotsu manifolds.

Theorem 6.1. *Let $M = M_1 \times {}_fM_\perp$ be a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_T, where $M_1 = M_T \times M_\theta$. Then* *(i) The squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq s\left(\cot^2\theta\,\|\vec{\nabla}^{\theta}\ln f\|^2\right) + 2s\left(\|\vec{\nabla}^{T}\ln f\|^2 - 1\right) \tag{53}$$

where $\vec{\nabla}^{T}\ln f$ *and* $\vec{\nabla}^{\theta}\ln f$ *are the gradient components of the function* $\ln f$ *along* M_T *and* M_θ*, respectively and* $s = \dim M_\perp$.

(ii) If equality sign in (i) holds, then M_1 *is a totally geodesic submanifold and* $M_\perp$ *is a totally umbilical submanifold of* $\tilde{M}$.

Proof. From the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i,e_j),h(e_i,e_j)) = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m} g(h(e_i,e_j),e_r)^2.$$

Using the constructed frame fields, we find

$$\begin{aligned}\|h\|^2 =& \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),e_r)^2 + 2\sum_{r=m+1}^{2n+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{2q} g(h(e_i,e_j^*),e_r)^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),e_r)^2\\ &+ 2\sum_{r=m+1}^{2n+1}\sum_{i=1}^{2q}\sum_{j=1}^{s} g(h(e_i^*,\hat{e}_j),e_r)^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),e_r)^2 + 2\sum_{r=m+1}^{2n+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),e_r)^2\end{aligned} \tag{54}$$

Fourth term in the right hand side vanishes identically by using the $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic condition, thus we derive

$$\begin{aligned}\|h\|^2 =& \sum_{r=m+1}^{m+2q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),e_r)^2 + \sum_{r=m+2q+1}^{m+2q+s}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),e_r)^2 + \sum_{r=m+2q+s+1}^{2n+1}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),e_r)^2\\ &+ 2\sum_{r=m+1}^{m+2q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2q} g(h(e_i,e_j^*),e_r)^2 + 2\sum_{r=m+2q+1}^{m+2q+s}\sum_{i=1}^{2p+1}\sum_{j=1}^{2q} g(h(e_i,e_j^*),e_r)^2 + 2\sum_{r=m+2q+s+1}^{2n+1}\sum_{i}^{2p+1}\sum_{j=1}^{2q} g(h(e_i,e_j^*),e_r)^2\\ &+ \sum_{r=m+1}^{m+2q}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),e_r)^2 + \sum_{r=m+2q+1}^{m+2q+s}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),e_r)^2 + \sum_{r=m+2q+s+1}^{2n+1}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),e_r)^2\\ &+ \sum_{r=m+1}^{m+2q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),e_r)^2 + \sum_{r=m+2q+1}^{m+2q+s}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),e_r)^2 + \sum_{r=m+2q+s+1}^{2n+1}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),e_r)^2\\ &+ 2\sum_{r=m+1}^{m+2q}\sum_{i=1}^{2p+1}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),e_r)^2 + 2\sum_{r=m+2q+1}^{m+2q+s}\sum_{i=1}^{2p+1}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),e_r)^2 + 2\sum_{r=m+2q+s+1}^{2n+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),e_r)^2.\end{aligned} \tag{55}$$

Since we could not find the relations for a warped product in the form $g(h(U,W),\nu)$, for any U,W either in $\mathcal{D}\oplus\langle\xi\rangle$ or $\mathcal{D}^\theta$ or $\mathcal{D}^\perp$, therefore we will leave the positive third, sixth, ninth, twelfth and fifteenth terms in the right hand side of (55). Then, we find

$$\begin{aligned}\|h\|^2 \geq & \sum_{r=1}^{2q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 + \sum_{r=1}^{s}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\varphi\tilde{e}_r)^2 + 2\sum_{r=1}^{2q}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,e_j^*),\tilde{e}_r)^2 \\ & + 2\sum_{r=1}^{s}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,e_j^*),\varphi\hat{e}_r)^2 + \sum_{r=1}^{2q}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2 + \sum_{r=1}^{s}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\varphi\hat{e}_r)^2 \\ & + \sum_{r=1}^{2q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=1}^{s}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),\varphi\hat{e}_r)^2 + 2\sum_{r=1}^{2q}\sum_{i=1}^{2p+1}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 \\ & + 2\sum_{r=1}^{s}\sum_{i=1}^{2p+1}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),\varphi\hat{e}_r)^2. \end{aligned} \tag{56}$$

The second and fourth terms vanish identically by using Lemma 4.2 (ii) and Lemma 4.2 (iii), respectively and for a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product, the sixth term vanishes identically by using Lemma 4.2 (iv). Also, we could not find the relations for a warped product in the forms $g(h(X_1,Y_1),F\mathcal{D}^\theta)$, $g(h(X_2,Y_2),F\mathcal{D}^\theta)$, $g(h(X_1,X_2),F\mathcal{D}^\theta)$ and $g(h(Z,V),\varphi\mathcal{D}^\perp)$, for any $X_1,Y_1\in\Gamma(\mathcal{D}\oplus\langle\xi\rangle)$, $X_2,Y_2\in\Gamma(\mathcal{D}^\theta)$ and $Z,V\in\Gamma(\mathcal{D}^\perp)$. Hence, by leaving these positive terms in the right hand side of (56) and using the constructed frame fields, we obtain

$$\begin{aligned}\|h\|^2 \geq & \sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),\csc\theta\, Fe_r^*)^2 + \sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),\csc\theta\ \sec\theta\, FTe_r^*)^2 \\ & + 2\sum_{j,r=1}^{s}\sum_{i=1}^{2p} g(h(e_i,\hat{e}_j),\varphi\hat{e}_r)^2 + 2\sum_{j,r=1}^{s} g(h(e_{2p+1},\hat{e}_j),\varphi\hat{e}_r). \end{aligned} \tag{57}$$

Since $e_{2p+1}=\xi$ and for a submanifold of a Kenmotsu manifold, we have $h(\xi,U)=0$, for any $U\in\Gamma(TM)$, thus the last term in the right hand side of (57) vanishes identically. Then, we derive

$$\begin{aligned}\|h\|^2 \geq & \csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),Fe_r^*)^2 + \csc^2\theta\sec^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),FTe_r^*)^2 \\ & + 2\sum_{j,r=1}^{s}\sum_{i=1}^{p} g(h(e_i,\hat{e}_j),\varphi\hat{e}_r)^2 + 2\sum_{j,r=1}^{s}\sum_{i=1}^{p} g(h(\varphi e_i,\hat{e}_j),\varphi\hat{e}_r)^2. \end{aligned}$$

Then, from (29), (30), (32) and (33), we arrive at

$$\begin{aligned}\|h\|^2 \geq & \csc^2\theta\sum_{i,j=1}^{s}\sum_{r=1}^{q}\Big(Te_r^*(\ln f)\, g(\hat{e}_i,\hat{e}_j)\Big)^2 + \cot^2\theta\sum_{i,j=1}^{s}\sum_{r=1}^{q}\Big(e_r^*(\ln f)\, g(\hat{e}_i,\hat{e}_j)\Big)^2 \\ & + 2\sum_{j,r=1}^{s}\sum_{i=1}^{p}\Big(\varphi e_i(\ln f)\, g(\hat{e}_j,\hat{e}_r)\Big)^2 + 2\sum_{j,r=1}^{s}\sum_{i=1}^{p}(e_i(\ln f)-\eta(e_i))^2\, g(\hat{e}_j,\hat{e}_r)^2. \end{aligned}$$

Since $\eta(e_i)=0$, $\forall\ i=1,\cdots,2p$ and $\eta(e_{2p+1})=1$, thus we obtain

$$\begin{aligned}\|h\|^2 \geq & s\csc^2\theta\sum_{r=1}^{2q}(Te_r^*(\ln f))^2 - s\csc^2\theta\sum_{r=q+1}^{2q}(Te_r^*(\ln f))^2 \\ & + s\cot^2\theta\sum_{r=1}^{q}(e_r^*(\ln f))^2 + 2s\sum_{i=1}^{2p+1}(e_i(\ln f))^2 - 2s(e_{2p+1}(\ln f))^2. \end{aligned}$$

Using (10) and Lemma 4.2 (i), we find

$$\begin{aligned}\|h\|^2 &\geq s\csc^2\theta\|T\vec{\nabla}^\theta \ln f\|^2 - s\csc^2\theta\sum_{r=1}^{q} g(e^*_{q+r}, T\vec{\nabla}^\theta \ln f)^2\\ &+ s\cot^2\theta\sum_{r=1}^{q}(e^*_r(\ln f))^2 + 2s\|\vec{\nabla}^T \ln f\|^2 - 2s\\ &= s\cot^2\theta\|\vec{\nabla}^\theta \ln f\|^2 - s\csc^2\theta\sum_{r=1}^{q} g(\sec\theta Te^*_r, T\vec{\nabla}^\theta \ln f)^2\\ &+ s\cot^2\theta\sum_{r=1}^{q}(e^*_r(\ln f))^2 + 2s\left(\|\vec{\nabla}^T \ln f\|^2 - 1\right).\end{aligned}$$

Then, from the gradient definition, we obtain

$$\begin{aligned}\|h\|^2 &\geq s\cot^2\theta\|\vec{\nabla}^\theta \ln f\|^2 - s\cot^2\theta\sum_{r=1}^{q}(e^*_r(\ln f))^2\\ &+ s\cot^2\theta\sum_{r=1}^{q}(e^*_r(\ln f))^2 + 2s\left(\|\vec{\nabla}^T \ln f\|^2 - 1\right)\end{aligned}$$

which is inequality (i). To prove the equality case of (53), we proceed as follows: From the given mixed totally geodesic condition, we have

$$h(\mathcal{D}^\theta, \mathcal{D}^\perp) = 0. \tag{58}$$

On the other hand, leaving the third term in (55) and the first term in (56), we respectively have

$$h(\mathcal{D}, \mathcal{D}) \perp \nu \quad \text{and} \quad h(\mathcal{D}, \mathcal{D}) \perp F\mathcal{D}^\theta, \quad \Rightarrow \quad h(\mathcal{D}, \mathcal{D}) \subseteq \varphi\mathcal{D}^\perp. \tag{59}$$

Also, from Lemma 4.2 (ii), we have

$$h(\mathcal{D}, \mathcal{D}) \perp \varphi\mathcal{D}^\perp. \tag{60}$$

Then, from (59) and (60), we conclude that

$$h(\mathcal{D}, \mathcal{D}) = 0. \tag{61}$$

Similarly, from the leaving ninth term in the right hand side of (55) and leaving fifth term in the right hand side of (56), we find

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp \nu \quad \text{and} \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp F\mathcal{D}^\theta, \quad \Rightarrow \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subseteq \varphi\mathcal{D}^\perp. \tag{62}$$

And for a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product, from Lemma 4.2 (iv), we have

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp \varphi\mathcal{D}^\perp. \tag{63}$$

Thus, from (62) and (63), we arrive at

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0. \tag{64}$$

From the leaving sixth term in the right hand side of (55) and leaving third term in (56), we respectively find that

$$h(\mathcal{D}, \mathcal{D}^\theta) \perp \nu \quad \text{and} \quad h(\mathcal{D}, \mathcal{D}^\theta) \perp F\mathcal{D}^\theta, \quad \Rightarrow \quad h(\mathcal{D}, \mathcal{D}^\theta) \subseteq \varphi\mathcal{D}^\perp. \tag{65}$$

Also, from Lemma 4.2 (iii), we obtain

$$h(\mathcal{D},\mathcal{D}^\theta)\perp\varphi\mathcal{D}^\perp. \tag{66}$$

Then, from (65) and (66), we conclude that

$$h(\mathcal{D},\mathcal{D}^\theta)=0. \tag{67}$$

Since M_1 is totally geodesic in M [7, 15], using this fact with (58), (61), (64) and (67), we get M_1 is totally geodesic in $\tilde{M}$. On the other hand, leaving the fifteenth term in the right hand side of (55), we find $h(\mathcal{D},\mathcal{D}^\perp)\perp\nu$. Also, from Lemma 4.2 (iii), we obtain $h(\mathcal{D},\mathcal{D}^\perp)\perp F\mathcal{D}^\theta$. Thus, we conclude that

$$h(\mathcal{D},\mathcal{D}^\perp)\subseteq\varphi\mathcal{D}^\perp. \tag{68}$$

And, the leaving twelfth term in the right hand side of (55) and the leaving sixth term in the right hand side of (56), we respectively have

$$h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\nu \quad\text{and}\quad h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\varphi\mathcal{D}^\perp, \quad\Rightarrow\quad h(\mathcal{D}^\perp,\mathcal{D}^\perp)\subseteq F\mathcal{D}^\theta. \tag{69}$$

Also, from Lemma 4.3 and Lemma 4.5, we respectively have

$$g(h(X_1,Z),\varphi V)=-\varphi X_1(\ln f)\,g(Z,V) \tag{70}$$

and

$$g(h(Z,V),FX_2)=TX_2(\ln f)\,g(Z,V), \tag{71}$$

for any $X_1\in\Gamma(\mathcal{D}\oplus\langle\xi\rangle)$, $X_2\in\Gamma(\mathcal{D}^\theta)$ and $Z,V\in\Gamma(\mathcal{D}^\perp)$. Since $M_\perp$ is totally umbilical in M [7, 15], using this fact with (58) and (68)-(71), we observe that $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$. Hence, the theorem is proved completely. □

If the structure vector field ξ is tangent to M_θ, then we have the following result.

Theorem 6.2. *Let $M=M_1\times{}_fM_\perp$ be a $\mathcal{D}^\theta-\mathcal{D}^\perp$ mixed totally geodesic warped product skew CR-submanifold of order* 1 *of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_θ, where $M_1=M_T\times M_\theta$. Then*

(i) The squared norm of the second fundamental form satisfies

$$\|h\|^2\geq s\,\cot^2\theta\left(\|\vec{\nabla}^\theta\ln f\|^2-1\right)+2s\|\vec{\nabla}^T\ln f\|^2 \tag{72}$$

where $\vec{\nabla}^T\ln f$ and $\vec{\nabla}^\theta\ln f$ are the gradient components of the function $\ln f$ along M_T and M_θ, respectively.

(ii) If the equality sign in (i) holds, then M_1 is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

We can prove this theorem like Theorem 5.3, just we have to handle the structure vector field ξ. In this case the dimensions of M_T and M_θ respectively are $2p$ and $2q+1$ and the orthonormal frames of their tangent spaces $\mathcal{D}$ and $\mathcal{D}^\theta\oplus\langle\xi\rangle$, respectively are $\{e_1,e_2,\cdots,e_p,e_{p+1}=\varphi e_1,\cdots,e_{2p}=\varphi e_p\}$ and $\{e_{2p+1}=e_1^*,\cdots,e_{2p+q}=e_q^*,e_{2p+q+1}=e_{q+1}^*=\sec\theta\,Te_1^*,\cdots,e_{2p+2q}=e_{2q}^*=\sec\theta\,Te_q{}^*,e_{2p+2q+1}=e_{2q+1}^*=\xi\}$.

7. Some Applications

In this section, we give some applications of our derived results.

For the warped product skew CR-submanifolds of the form $M=M_1\times_f M_\perp$ of a Kenmotsu manifold $\tilde{M}$ such that $M_1=M_T\times M_\theta$, if $\dim M_\theta=0$, then the warped product skew CR-submanifolds turn into CR-warped products $M=M_T\times_f M_\perp$ which have been studied in [3, 27]. Hence, Theorem 5.3 generalise a result of [27] as follows:
If we put $\dim M_\theta=0$ in Theorem 5.3, then the warped product is of the form $M=M_T\times_f M_\perp$, a contact CR-warped product in a Kenmotsu manifold $\tilde{M}$. Thus, we have the following special case of Theorem 5.3.

Corollary 7.1. *(Theorem 3.4 [27]) A proper contact CR-submanifold of a Kenmotsu manifold $\tilde{M}$ is locally a contact CR-warped product if and only if*

$$A_{\varphi Z}X_1 = -(\varphi X_1 \mu)Z, \quad \forall \ X_1 \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle), \quad Z \in \Gamma(\mathcal{D}^\perp) \tag{73}$$

for some function μ on M satisfying $V\mu = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$.

On the other hand, in a warped product skew CR-submanifold $M = M_1 \times_f M_\perp$ such that $M_1 = M_T \times M_\theta$, if $\dim M_T = 0$, then the warped product skew CR-submanifold turns into a warped product pseudo-slant submanifold $M = M_\theta \times_f M_\perp$ and the case has been considered in [2]. In this case, Theorem 4.1 of [2] is a special case of Theorem 5.3, by interchanging X_2 by TX_2 in the third relation of Theorem 5.3 as follows:

Corollary 7.2. *(Theorem 4.1 [2]) Let M be a proper pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then M is locally a mixed totally geodesic warped product submanifold if and only if*

$$A_{\varphi Z}X_2 = 0 \ \ \text{and} \ \ A_{FTX_2}Z = \cos^2\theta \left(\eta(X_2) - (X_2\mu)\right) Z \tag{74}$$

for any $Z \in \Gamma(\mathcal{D}^\perp)$ and $X_2 \in \Gamma(\mathcal{D}^\theta \oplus \langle \xi \rangle)$ for some smooth function μ on M such that $V(\mu) = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$.

Similarly, Theorem 3.1 of [3] is a special case of Theorem 6.1 as follows:
If we consider $\dim M_\theta = 0$ in Theorem 6.1, then the inequality (53) is true for contact CR-warped products which have been considered in [3].

Corollary 7.3. *(Theorem 3.1 [3]) Let $\tilde{M}$ be a $(2n+1)$-dimensional Kenmotsu manifold and $M = M_T \times_f M_\perp$ an m-dimensional contact CR-warped product submanifold, such that M_T is a $(2p+1)$-dimensional invariant submanifold tangent to ξ and $M_\perp$ a s-dimensional anti-invariant submanifold of $\tilde{M}$. Then*

(i) The squared norm of the second fundamental form of M satisfies

$$\|h\|^2 \geq 2s\left(\|\vec{\nabla}^T \ln f\|^2 - 1\right) \tag{75}$$

where $\vec{\nabla}^T \ln f$ is the gradient of $\ln f$.

(ii) If the equality sign of (75) holds identically, then M_T is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$. Moreover, M is a minimal submanifold of $\tilde{M}$.

On the other hand, if we consider $\dim M_T = 0$ in Theorem 6.2, then the warped product skew CR-submanifold M turns to the warped product pseudo-slant submanifold $M = M_\theta \times_f M_\perp$ and the inequality (72) generalise Theorem 5.1 of [2] as follows.

Corollary 7.4. *(Theorem 5.1 [2]) Let $M = M_\theta \times_f M_\perp$ be a mixed totally geodesic warped product pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$ with their real dimensions $(2q+1)$ and s, respectively. Then*

(i) The squared norm of the second fundamental form h of M satisfies

$$\|h\|^2 \geq s\cot^2\theta \left(\|\vec{\nabla}^\theta \ln f\|^2 - 1\right) \tag{76}$$

where $\vec{\nabla}^\theta \ln f$ is gradient of the function $\ln f$ along M_θ.

(ii) If equality sign of (76) holds identically, then M_θ is totally geodesic and $M_\perp$ is totally umbilical in $\tilde{M}$.

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