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# Irreducible Locales

**Dharmanand Baboolal[a], Paranjothi Pillay[a]**

[a]*School of Mathematics, Statistics, and Computer Science, University of KwaZulu-Natal. P.B. X54001, Durban 4000, South Africa*

**Abstract.** Basic properties of irreducible locales which extend results contained in [4] are presented. Our main result is that every locale $L$ can be embedded as a closed nowhere dense sublocale of an irreducible locale $\mathcal{I}L$, what we call the *irreducible envelope* of $L$. The properties of *spatiality*, *subfitness*, *fitness*, *compactness*, and the *Noetherian* property are shown to be inherited and reflected by the irreducible envelope.

## 1. Introduction

The concept of an irreducible or hyperconnected topological space has been studied by several authors (see for example [1, 7–9, 12, 13]). An *irreducible* topological space is one that cannot be written as the union of two proper closed subsets. Such spaces are important in algebraic geometry: in a commutative ring $A$ the set *Spec*($A$) of all prime ideals with the Zariski topology is irreducible for a certain class of rings (see [2], pp. 12-13). In [7] such spaces are referred to as *D-spaces*, but in most of the papers in topology following this the terminology *hyperconnected* is used. To our knowledge the analogous study of irreducibility was first carried out in the pointfree setting in [4].

Our purpose is to add to the results obtained in [4] on irreducibility in the pointfree context. One of our main results is the construction of what we call the *irreducible envelope* of a locale, i.e. we show that every locale $L$ can be embedded as a closed nowhere dense sublocale of an irreducible locale $\mathcal{I}L$. This is the pointfree analog of the *hyperconnectification* of a topological space described in [1]. We show that the assignment of a locale $L$ to its irreducible envelope $\mathcal{I}L$ is functorial, and that this functor preserves open maps and closed injections. We then look at *spatiality*, *subfitness*, *fitness*, *compactness* and the *Noetherian* condition, showing that the irreducible envelope both reflects and inherits these properties. Since it will be evident from the definition that an irreducible locale is always connected and locally connected, if $L$ is non-spatial then $\mathcal{I}L$ will be a non-spatial connected and locally connected locale. We think this provides an interesting and easy negative answer to the question whether every connected, locally connected locale is spatial.

We began this project with the idea of writing out everything in the language of locales and sublocales in the spirit of the book [10], but we soon realized that certain results are better expressed in terms of frames than in terms of locales or sublocales. Thus we have no particular preference of one over the other, and we will use both forms to express our results in this paper.

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*Email addresses:* baboolald@ukzn.ac.za (Dharmanand Baboolal), pillaypi@ukzn.ac.za (Paranjothi Pillay)

**2. Preliminaries**

We recall that a *frame (locale)* $L$ is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s | s \in S\}$$

for all $x \in L$, $S \subseteq L$. The top element of $L$ is denoted by 1 and the bottom by 0. A *frame homomorphism* is a map $h : L \longrightarrow M$ between frames that preserve finitary meets (including the top 1) and arbitrary joins (including the bottom 0).

We thus have the category of frames and frame homomorphisms, which we denote by **Frm**. A frame map $h : L \longrightarrow M$ is called *dense* if $x = 0$ whenever $h(x) = 0$. For elements $a, b \in L$, we say that $a$ is *rather below* $b$, written $a \prec b$, if there exists an element $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$. This is equivalent to the condition that $a^* \vee b = 1$, where $a^*$ is the *pseudocomplement* of $a$, i.e. the largest element in $L$ whose meet with $a$ is 0. A frame $L$ is said to be *regular* if for each $a \in L$ we have $a = \bigvee x(x \prec a)$. A frame $L$ is said to be *spatial* if there is a topological space $X$ such that $L \cong \mathcal{O}X$, the frame of open subsets of $X$. It is known that $L$ is spatial if and only if whenever $a < b$ in $L$, there is a frame homomorphism $\xi : L \longrightarrow \mathbf{2}$ such that $\xi(a) = 0$ and $\xi(b) = 1$, where $\mathbf{2}$ is the two-element chain ([3]). Since the restriction of a frame homomorphism on a frame to any of its subframes is again a frame homomorphism, it follows that a subframe of a spatial frame must also be spatial.

Recall that an element $a$ in a frame $L$ is said to be *connected* if whenever $a = b \vee c$ with $b \wedge c = 0$ we have either $b = 0$ or $c = 0$. A frame is said to be connected if its top element 1 is connected, and it is said to be *locally connected* if each element in the frame can be written as a join of connected elements.

An element $a$ for which $a^* = 0$ is said to be *dense*. The meet of any two dense elements is dense. Any element above a dense one is dense.

Recall that every frame homomorphism $h : L \longrightarrow M$ has a right adjoint $h_* : M \longrightarrow L$ characterized by the condition $h(x) \leq y \Leftrightarrow x \leq h_*(y)$. If $h$ is onto then $h_*$ is given by the formula $h_*(y) = \bigvee\{x \in L | h(x) = y\}$. It follows that for such $h$ the composite map $hh_*$ is the identity map. If the map $h$ is dense and onto then it is well known that $h(a^*) = (h(a))^*$ for every $a$ and that $h_*(b^*) = (h_*(b))^*$ for every $b$. For every pair of elements $a, b$ in a frame we have the element $a \rightarrow b$ given by the Heyting operation $\rightarrow$ characterized by the condition: $x \leq a \rightarrow b$ iff $x \wedge a \leq b$.

If $L$ and $M$ are locales (frames) a localic map $f : L \longrightarrow M$ is a map that is the right adjoint of a frame map $f^* : M \longrightarrow L$. This gives us the category **Loc** of locales and localic maps. Since $f$ and $f^*$ are Galois adjoints we have $ff^*f = f$ and $f^*ff^* = f^*$. From this one gets $f$ is one to one iff $f^*$ is onto, and $f$ is onto iff $f^*$ is one to one. Apart from the fact that a localic map $f : L \longrightarrow M$ is infima preserving, it satisfies two further useful properties reflecting the fact that $f^*$ preserves the top element and $f^*$ preserves finite meet. These are:

1. $f(L \setminus \{1\}) \subseteq M \setminus \{1\}$, and
2. $f(f^*(a) \rightarrow b) = a \rightarrow f(b)$.

The regular subobjects in this category are the sublocales $S$ of $L$ which are those subsets $S$ of $L$ having the characteristic properties:

1. If $M \subseteq S$ then $\bigwedge M \in S$.
2. If $a \in L, s \in S$ then $a \rightarrow s \in S$.

The collection of sublocales $\mathcal{S}(L)$ of a locale $L$ ordered by inclusion form a co-frame, that is,

$$S \vee \bigcap_{i \in I} T_i = \bigcap_{i \in I} (S \vee T_i)$$

for sublocales $S$ and $T_i$. Here, the infimum of sublocales is just the set-theoretic intersection, and the supremum of a collection of sublocales $T_i$ is the collection $\{\bigwedge M | M \subseteq \bigcup T_i\}$. An open sublocale is one of the form $\mathfrak{o}(a) = \{a \rightarrow x | x \in L\} = \{x \in L | x = a \rightarrow x\}$, and a closed sublocale is one of the form $\mathfrak{c}(a) = \uparrow a$. The sublocales $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other in the sublocale lattice. If a sublocale $S$ is either open or closed, and $\{T_i\}$ is any collection of sublocales, then the frame law also holds, that is,

$$S \cap \bigvee T_i = \bigvee (S \cap T_i)$$

The closure of a sublocale $S$, written as $\overline{S}$, is the closed sublocale $\uparrow (\bigwedge S)$ which is the smallest closed sublocale containing $S$. $S$ is dense if $\overline{S} = L$. An open sublocale $S$ is said to be *regular open* if $S=$ int $(\overline{S})$. Here int $(T)$ is the interior of $T$ and is the largest open sublocale contained in $T$. This exists since the join of open sublocales is again open and there exists an open sublocale, namely the trivial one $\mathbb{O} = \{1\}$, contained in every sublocale. We have the following:

$$\begin{array}{llll} \mathfrak{o}(0) = \mathbb{O} & \mathfrak{o}(1) = L & \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b) & \mathfrak{o}(\bigvee_i a_i) = \bigvee_i \mathfrak{o}(a_i) \\ \overline{\mathfrak{o}(a)} = \uparrow a^* & \uparrow (a \wedge b) = \uparrow a \vee \uparrow b & \mathfrak{o}(a) = \mathfrak{o}(b) \Leftrightarrow a = b. & \end{array}$$

It can be easily seen from the definition that an open sublocale $\mathfrak{o}(a)$ is regular open iff $a = a^{**}$, that is, $a$ is regular as an element of the frame $L$. If $S$ is a sublocale then the map $\nu_S : L \longrightarrow L$ given by $\nu_S(a) = \bigwedge x(x \in S, a \leq x)$ is a *nucleus* on $L$, that is it satisfies:

1. $a \leq \nu_S(a)$ for all $a \in L$.
2. $\nu_S(a \wedge b) = \nu_S(a) \wedge \nu_S(b)$ for all $a, b \in L$.
3. $\nu_S(\nu_S(a)) = \nu_S(a)$ for all $a \in L$.

Furthermore $S = \nu_S(L)$.

If $a \in L, x \in S$ then $a \to x = \nu_S(a) \to x$, and this gives us $\mathfrak{o}(a) \cap S = \mathfrak{o}_S(\nu_S(a))$, where by $\mathfrak{o}_S(\nu_S(a))$ we mean $\{y \in S | y = \nu_S(a) \to y\}$. Thus $\mathfrak{o}(a) \cap S$ is an open sublocale of $S$, and every open sublocale of $S$ is of this form. A sublocale $S$ is said to be *connected* if whenever $S \subseteq U \vee V$ where $U$ and $V$ are open sublocales of $L$ and $S \cap U \cap V = \mathbb{O}$ then either $S \cap U = \mathbb{O}$ or $S \cap V = \mathbb{O}$. The sublocales $U$ and $V$ can be replaced equivalently by closed sublocales in the definition. It is known, or easy to show, that an open sublocale $\mathfrak{o}(a)$ is connected iff $a$ is connected as an element of the frame $L$. Just as for spaces closures of connected sublocales are again connected.

We follow closely the approach to sublocales as contained in the book [10], and much of the background material above can be found therein. For further background material on frames we refer to the book [6].

## 3. Irreducible Locales: Basic Properties

We started this study with our initial reference being [7]. At the time we were unaware of the paper [4], and therefore our definition of irreducibility was taken simply as the pointfree analog of the notion of a $D$-space ( [7]), later to be called hyperconnected space in papers in topology. In [4] the author calls a frame $L$ *irreducible* if $0 \in L$ is prime, i.e. whenever $a \wedge b = 0$ then either $a = 0$ or $b = 0$. We show the equivalence of our definition with the one in [4] in Theorem 3.4.

**Definition 3.1.** A locale $L$ is called irreducible if every non-trivial open sublocale is dense or, equivalently, $a^* = 0$ for every non-zero $a \in L$.

An irreducible locale is very far from being regular, as the following result shows.

**Proposition 3.2.** *A locale L is regular and irreducible if and only if* $L \cong \mathbf{2}$.

*Proof.* Clearly $\mathbf{2}$ is regular and irreducible. Conversely suppose $L$ is regular and irreducible. Take any $a \in L, a \neq 1$. By regularity $a = \bigvee x(x \prec a)$. For any $x \prec a$ we have $x^* \vee a = 1$ and since $a \neq 1$ we therefore have $x^* \neq 0$. By irreducibility we therefore have $x = 0$. Hence $a = 0$ and thus $L \cong \mathbf{2}$. □

**Theorem 3.3.** *The following conditions are equivalent for a locale L:*

*(a) L is irreducible.*

*(b) If the intersection of two open sublocales is trivial then one of them must be trivial.*

*(c) Every open sublocale of L is connected.*

*(d) The only regular open sublocales of L are the trivial one and L itself.*

*Proof.* $(a) \implies (b)$: Suppose $L$ is irreducible and $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathbb{O}$. Then $a \wedge b = 0$. If $a \neq 0$ then $a^* = 0$ and therefore $b = 0$ since $b \leq a^*$. Hence $\mathfrak{o}(b) = \mathbb{O}$.

$(b) \implies (c)$: We show that every element in $L$ is connected. Take any $a \in L$ and let $a = b \vee c$ with $b \wedge c = 0$. Then either $b = 0$ or $c = 0$ and so $a$ is connected.

$(c) \implies (d)$: If $\mathfrak{o}(a)$ is regular open then $a$ is a regular element of $L$. We cannot have both $a \neq 0$ and $a^* \neq 0$, otherwise the element $c = a \vee a^*$ would be disconnected. Thus either $a = 0$ or $a^* = 0$. If $a = 0$ then $\mathfrak{o}(a) = \mathbb{O}$. If the latter applies then $a = a^{**} = 0^* = 1$ and in this case $\mathfrak{o}(a) = L$.

$(d) \implies (a)$: Take any $a \neq 0$ in $L$. If $a^* \neq 0$ then since $a^* = a^{***}$ we have by $(d)$ that $a^* = 1$. Thus $a^{**} = 0$ and since $a \leq a^{**}$ this implies $a = 0$, a contradiction. Thus $a^* = 0$. □

In terms purely of the elements of a locale $L$ irreducibility is characterized as follows:

**Theorem 3.4.** *The following conditions are equivalent for a locale L:*
*(a) L is irreducible.*
*(b) If $u \wedge v = 0$, then either $u = 0$ or $v = 0$.*
*(c) Every element a of L is connected.*
*(d) The only regular elements of L are* 0 *and* 1.

**Corollary 3.5.** *An irreducible locale is connected and locally connected.*

*Proof.* The element 1 is connected, i.e. $L$ is connected. Each $a \neq 0$ is connected. Thus $L$ is locally connected. □

**Corollary 3.6.** ([4]) *The localic image of an irreducible locale is irreducible; equivalently every subframe of an irreducible frame is irreducible.*

*Proof.* This follows immediately from (b) of the above theorem. □

**Theorem 3.7.** ([4]) *Let S be a dense sublocale of a locale L. Then S is irreducible if and only if L is irreducible*

**Proposition 3.8.** *L is irreducible if and only if every open sublocale of L is irreducible.*

*Proof.* The necessity follows from (b) of Theorem 3.3 and the fact that an open sublocale of an open sublocale is an open sublocale of $L$. For the sufficiency, we note that the open sublocale $\mathfrak{o}(1)$ is $L$. □

Recall that a localic map $f : L \longrightarrow M$ is called *open* if it maps open sublocales to open sublocales. From [10] we have the following characterization of open localic maps.

**Theorem 3.9.** *The following conditions are equivalent for a localic map $f : L \longrightarrow M$:*
*(a) f is open.*
*(b) $f^*$ is a complete Heyting homomorphism, i.e. $f^*$ preserves arbitrary meets, and $f^*(a \to b) = f^*(a) \to f^*(b)$ for all $a, b \in M$.*
*(c) $f^*$ admits a left adjoint $f_!$ that satisfies the Frobenius identity $f_!(a \wedge f^*(b)) = f_!(a) \wedge b$ for all $a \in L$ and $b \in M$.*

**Proposition 3.10.** *Let $f : L \longrightarrow M$ be a one to one and open localic map. If M is irreducible, then so is L.*

*Proof.* Suppose $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathbb{O}$. Then $f(\mathfrak{o}(a) \cap \mathfrak{o}(b)) = f(\mathbb{O}) = \mathbb{O}$ (since $f(1) = 1$), and hence $f(\mathfrak{o}(a)) \cap f(\mathfrak{o}(b)) = \mathbb{O}$ (since $f$ is one to one). Since $M$ is irreducible, we can assume $f(\mathfrak{o}(a)) = \mathbb{O}$, say. Hence $\mathfrak{o}(a) = \mathbb{O}$ since $f(L \setminus \{1\} \subseteq M \setminus \{1\}$. Hence $L$ is irreducible. □

In [4] it is shown that every irreducible sublocale is contained in a maximal irreducible sublocale. The author uses certain key facts about prime elements in the proof. These are that:
(i) an element $a \in L$ is prime if and only if $\uparrow a$ is irreducible.
(ii) if $C \subseteq L$ is a chain of primes in $L$, then $\bigwedge C$ is prime.
(iii) any prime is above some minimal prime (by an application of Zorn's Lemma).
Here we give a proof using only properties of the sublocale lattice.

**Proposition 3.11.** *The join of a chain of irreducible sublocales of L is irreducible.*

*Proof.* Let $\{S_i\}_{i\in I}$ be a chain of irreducible sublocales of $L$ and let $S = \bigvee S_i$. Suppose $\mathfrak{o}_S(a)$ and $\mathfrak{o}_S(b)$ are two open sublocales of $S$ such that $\mathfrak{o}_S(a) \cap \mathfrak{o}_S(b) = \mathbb{O}$, where $a, b \in S$. Suppose that $\mathfrak{o}_S(a) \neq \mathbb{O}$ and $\mathfrak{o}_S(b) \neq \mathbb{O}$. Then $\mathfrak{o}(a) \cap S \neq \mathbb{O}$ and $\mathfrak{o}(b) \cap S \neq \mathbb{O}$. Since $\mathfrak{o}(a) \cap \bigvee S_i \neq \mathbb{O}$ we can use frame distributivity (since $\mathfrak{o}(a)$ is an open sublocale) to find $S_i$ such that $\mathfrak{o}(a) \cap S_i \neq \mathbb{O}$. Similarly we can find $S_j$ such that $\mathfrak{o}(b) \cap S_j \neq \mathbb{O}$. From the chainedness we can find $S_k$ such that $S_i, S_j \subseteq S_k$. Hence $\mathfrak{o}(a) \cap S_k \neq \mathbb{O}$ and $\mathfrak{o}(b) \cap S_k \neq \mathbb{O}$. But $(\mathfrak{o}(a) \cap S_k) \cap (\mathfrak{o}(b) \cap S_k) = \mathbb{O}$ and this contradicts the irreducibility of $S_k$. Hence $S$ is irreducible. □

**Theorem 3.12.** ([4]) *Let S be an irreducible sublocale of L. Then there exists a maximal irreducible sublocale $\hat{S}$ containing S. Furthermore every maximal irreducible sublocale of L is closed, so $\hat{S}$ is closed.*

*Proof.* Let $\mathcal{B} = \{T : T \text{ is a irreducible sublocale of } L \text{ and } S \subseteq T\}$ ordered by inclusion. If $\{T_i\}$ is any chain of irreducible sublocales, then $\bigvee T_i$ is irreducible so $\bigvee T_i \in \mathcal{B}$. Thus $\mathcal{B}$ has a maximal element $\hat{S}$ from Zorn's Lemma. Now the closure of a irreducible sublocale is irreducible, so $\hat{S}$ must be closed. □

**Theorem 3.13.** *Let L be any frame and $\{L_i\}_{i\in I}$ a chain of irreducible subframes of L. Then $\bigvee_{i\in I} L_i$ is irreducible, where $\bigvee_{i\in I} L_i$ is the subframe of L generated by the $L_i$.*

*Proof.* Take $u, v \in \bigvee_{i\in I} L_i$ with $u \neq 0, v \neq 0$. Now $u$ is a join of elements $a_1 \wedge a_2 \wedge .... \wedge a_n$, where $a_k \in L_{i_k}$. Also $v$ is a join of elements $b_1 \wedge b_2 \wedge ... \wedge b_m$, where $b_t \in L_{i_t}$. Thus there exists $0 \neq a_1 \wedge a_2 \wedge ... \wedge a_n \leq u$ where $a_k \in L_{i_k}$ and $0 \neq b_1 \wedge b_2 \wedge ... \wedge b_m \leq v$ where $b_t \in L_{i_t}$. Since the $\{L_i\}_{i\in I}$ is a chain of subframes there exists $L_j$ such that $L_{i_k}, L_{i_t} \subseteq L_j$ for all $k$ and $t$. Thus $x = a_1 \wedge a_2 \wedge ... \wedge a_n \wedge b_1 \wedge b_2 \wedge ... \wedge b_m \in L_j$. Now if $x = 0$, then by the irreducibility of $Lj$ some $a_i = 0$ or some $b_i = 0$, but this is not possible. Hence $x \neq 0$ and hence $u \wedge v \neq 0$. Thus $\bigvee_{i\in I} L_i$ is irreducible. □

Using Zorn's Lemma and the above theorem we get:

**Corollary 3.14.** *Let M be a irreducible subframe of L. Then M is contained in a maximal irreducible subframe of L.*

**Theorem 3.15.** *A frame L is irreducible if and only if it contains exactly one maximal irreducible subframe.*

*Proof.* If $L$ is irreducible then it is clear that $L$ itself is the only maximal irreducible subframe. Conversely suppose $L$ contains exactly one maximal irreducible subframe. If $L$ is not irreducible then there exists $0 \neq u, 0 \neq v \in L$ such that $u \wedge v = 0$. Now $L_1 = \{0, u, 1\}$ and $L_2 = \{0, v, 1\}$ are subframes of $L$ which are easily seen to be irreducible. Then $L_1 \subseteq \hat{L}_1, L_2 \subseteq \hat{L}_2$ where $\hat{L}_1$ and $\hat{L}_2$ are maximal irreducible subframes of $L$. Thus $\hat{L}_1 = \hat{L}_2 = K$ by hypothesis. But then $u, v \in K$ and $u \wedge v = 0$ must imply $u = 0$ or $v = 0$, a contradiction. Hence $L$ is irreducible. □

Observe that if $K$ is a subframe of $L$, then any $a \in K$ which is dense as an element of $L$, is dense as an element of $K$. Thus the following result is immediate.

**Proposition 3.16.** *Let L be a frame, and let $L_d = \{a \in L : a^* = 0\} \cup \{0\}$. Then $L_d$ is an irreducible subframe of L.*

**Theorem 3.17.** *Let M be a maximal irreducible subframe of L. Then $L_d \subseteq M$.*

*Proof.* Suppose not. Then there exists $a \in L,\ a^* = 0$ and $a \notin M$. Now $K = \{0, a, 1\}$ is an irreducible subframe of $L$. We claim that the subframe $M \vee K$ is irreducible: Take $u, v \in M \vee K$ with $u \neq 0, v \neq 0$. Then there exists $0 \neq m_1 \wedge k_1 \leq u$ with $m_1 \in M, k_1 \in K$ and $0 \neq m_2 \wedge k_2 \leq v$ with $m_2 \in M, k_2 \in K$. Now $k_1, k_2 \in \{a, 1\}$. If $k_1 = 1 = k_2$, then $m_1 \wedge k_1 \wedge m_2 \wedge k_2 = m_1 \wedge m_2 \neq 0$ since $M$ is irreducible. If $k_1 = a = k_2$, then $m_1 \wedge k_1 \wedge m_2 \wedge k_2 = m_1 \wedge m_2 \wedge a \neq 0$ since otherwise $m_1 \wedge m_2 \leq a^* = 0$ would imply either $m_1 = 0$ or $m_2 = 0$ and this is not possible. If $k_1 = a$ and $k_2 = 1$, then $m_1 \wedge k_1 \wedge m_2 \wedge k_2 = m_1 \wedge m_2 \wedge a \neq 0$ as before. Thus in all cases $m_1 \wedge k_1 \wedge m_2 \wedge k_2 \neq 0$, that is, $u \wedge v \neq 0$. Hence $M \vee K$ is irreducible. But $M \not\supseteq M \vee K$ as $a \in K$ but $a \notin M$ and this contradicts the maximality of $M$. □

**Proposition 3.18.** *For any frame $L$, $L_d$ is the intersection of all maximal irreducible subframes of $L$.*

*Proof.* Let $\{M_i\}_{i\in I}$ be the collection of all maximal irreducible subframes of $L$. From above we have $L_d \subseteq \bigcap_{i\in I} M_i$. For the other direction take $x \in \bigcap_{i\in I} M_i$ and suppose $x \notin L_d$. Then $x^* \neq 0$. Now $x^* \neq 1$, otherwise $x^{**} = 0$ and hence $x = 0 \in L_d$. Then $K = \{0, x^*, 1\}$ is an irreducible subframe of $L$ and thus $K \subseteq M_k$ for some $k \in I$. But $x, x^* \in M_k,\ x \neq 0, x^* \neq 0$ and $x \wedge x^* = 0$ contradicting irreducibility of $M_k$. □

**Proposition 3.19.** *Suppose $f : L \longrightarrow M$ is an onto frame homomorphism. Then:*
*(a) $M_d \subseteq f(L_d)$.*
*(b) $M_d = f(L_d)$ if $f$ is also dense.*

*Proof.* (a) Take $a \in M_d,\ a \neq 0$. We have $f(f_*(a)) = a$ since $f$ is onto, so it suffices to show $f_*(a) \in L_d$. Now $f_*(a) \neq 0$ since $a \neq 0$. We show that $(f_*(a))^* = 0$. Take any $b$ such that $b \wedge f_*(a) = 0$. Applying $f$ gives $f(b) \wedge a = 0$ and hence $f(b) = 0$ since $a^* = 0$. Hence $b \leq f_*(0) \leq f_*(a)$ so $b = b \wedge f_*(a) = 0$. Thus $(f_*(a))^* = 0$.

(b) Take $a \in L_d,\ a \neq 0$. Then $a^* = 0$. Now $x \wedge f(a) = 0 \implies f_*(x) \wedge f_*f(a) = f_*(0) = 0$ ($f$ dense) $\implies f_*(x) \wedge a = 0 \implies f_*(x) = 0 \implies f(f_*(x)) = 0 \implies x = 0$ since $f$ is onto. Thus $f(a)^* = 0$ and hence $f(L_d) \subseteq M_d$ and thus we have equality. □

We call a frame homomomorphism $f : L \longrightarrow M$ *open* if its right adjoint $f_* : M \longrightarrow L$ is open as a localic map as defined earlier. Thus, according to Theorem 3.9, $f : L \longrightarrow M$ is open if $f$ has a left adjoint $f_! : M \longrightarrow L$ such that $f_!(a \wedge f(b)) = f_!(a) \wedge b$ for all $a \in M, b \in L$.

**Lemma 3.20.** *If the frame map $f : L \longrightarrow M$ is open, then $f$ maps dense elements to dense elements, i.e. if $a \in L$ and $a^* = 0$, then $f(a)^* = 0$.*

*Proof.* Let $f_!$ be the left adjoint of $f$ as described above and suppose $a^* = 0$. Thus $x \wedge f(a) = 0 \implies f_!(x \wedge f(a)) = f_!(0) = 0 \implies f_!(x) \wedge a = 0 \implies f_!(x) \leq 0 \implies x \leq f(0) \implies x = 0$. Thus $f(a)^* = 0$. □

**Proposition 3.21.** *Suppose the frame map $f : L \longrightarrow M$ is open. Then $f_d : L_d \longrightarrow M_d$ is a frame homomorphism, where $f_d$ is the restriction of $f$ to $L_d$.*

*Proof.* This follows from the fact that $L_d$ and $M_d$ are subframes of $L$ and $M$ respectively, and $f(L_d) \subseteq M_d$ from the above lemma. □

## 4. The Irreducible Envelope of a Locale

The main purpose of this section is to show that every locale $L$ can be embedded as a closed nowhere dense sublocale of a irreducible locale. We recall that a sublocale $S$ of $L$ is said to be *nowhere dense* if int $(\overline{S}) = \mathbb{O}$. We mention that in the literature (see for example Plewe [11]) a nowhere dense sublocale $S$ is defined as one having the property that $S \cap D = \mathbb{O}$, where $D$ is the smallest dense sublocale of $L$. Recall that the smallest dense sublocale is associated with the nucleus $j$ on $L$ given by $j(x) = x^{**}$ (see [6]). That the two formulations are equivalent can be seen from the following result.

**Proposition 4.1.** *If $S$ is a sublocale of $L$, then int $(\overline{S}) = \mathbb{O}$ iff $S \cap D = \mathbb{O}$.*

*Proof.* ($\Rightarrow$) Suppose int $(\overline{S}) = \mathbb{O}$. Take $x \in S \cap D$. Now $x \in D \implies x = x^{**}$. We claim that $\mathfrak{o}(x^*) \subseteq \uparrow (\bigwedge S)$: Take any $x^* \to y \in \mathfrak{o}(x^*)$, where $y \in L$. Then $\bigwedge S \wedge x^* \leq x \wedge x^* = 0$, so $\bigwedge S \leq x^* \to y$. Hence $\mathfrak{o}(x^*) \subseteq \uparrow (\bigwedge S) = \overline{S}$. Since int $(\overline{S}) = \mathbb{O}$ we must have $x^* = 0$. Thus $x^{**} = 1$, that is $x = 1$. Hence $S \cap D = \mathbb{O}$.

($\Leftarrow$) Suppose $S \cap D = \mathbb{O}$. Assume that $\mathfrak{o}(a) \subseteq \uparrow (\bigwedge S)$. Then
$a^* = a \to 0 \in \uparrow (\bigwedge S)$, that is $\bigwedge S \leq a^*$. Let $s_0 = \bigwedge S$. Now $s_0 \in S$ and $s_0 \wedge a = 0$. Thus $a^* = a \to 0 = a \to (s_0 \wedge a) = (a \to s_0) \wedge (a \to a) =$
$a \to s_0 \in S$. Since $a^* = a^{***}$ we have that $a^* \in S \cap D$. Thus $a^* = 1$ and hence $a = 0$. Thus int $(\overline{S}) = \mathbb{O}$. □

**Theorem 4.2.** *Every locale $L$ can be embedded as a closed nowhere dense sublocale of an irreducible locale.*

*Proof.* Consider the product $L \times \mathbf{2}$ in **Frm**, where **2** is the 2-element Boolean algebra. Let $\mathcal{I}L = \{(0,0),(a,1)|a \in L\}$. It is easy to see that $\mathcal{I}L$ is a subframe of $L \times \mathbf{2}$. To show $\mathcal{I}L$ is irreducible, suppose $(a,b) \wedge (c,d) = (0,0)$ where $(a,b),(c,d) \in \mathcal{I}L$. Then $(a \wedge c, b \wedge d) = (0,0)$, and hence $b \wedge d = 0$. Since $b, d$ are either 0 or 1, one of them, say $b$, must be 0. Hence $a = 0$ since $(a,b) \in \mathcal{I}L$. Thus $(a,b) = (0,0)$, so $\mathcal{I}L$ is irreducible by Theorem 3.4.

Now, the map $p : \mathcal{I}L \longrightarrow L$ defined by $p(a,b) = a$ is a frame homomorphism which is onto. Thus $L$ is a quotient frame of $\mathcal{I}L$, so the associated sublocale of $\mathcal{I}L$ is $S = p_*(L)$. We show that $S$ is closed and nowhere dense in $\mathcal{I}L$. Note that $p_*(a) = (a,1)$ for all $a \in L$. Thus $S = \{(a,1)|a \in L\}$. Further $\bigwedge S = (0,1)$, hence $\overline{S} = \uparrow (0,1) = \{(a,1)|a \in L\} = S$. Thus $S$ is closed.

To show int $S = \mathbb{O}$, let $\mathfrak{o}(z)$ be a non-trivial open sublocale of $\mathcal{I}L$. Then $z = (a,1)$ for some $a \in L$. We claim that $(a,1) \rightarrow (0,0) = (0,0)$: For suppose $(x,y) \in \mathcal{I}L$ and $(x,y) \leq (a,1) \rightarrow (0,0)$. Then $(x,y) \wedge (a,1) = (0,0)$, and since this implies $y = 0$ we must also have $x = 0$ since $(x,y) \in \mathcal{I}L$. Hence $(x,y) = (0,0)$, proving the claim. Thus $\mathfrak{o}((a,1)) \nsubseteq S$, and hence
int $S = \mathbb{O}$. □

**Remark 4.3.** The referee has pointed out that there is a quicker way of arriving at the conclusion that $p_*(L)$ is nowhere dense, albeit using facts from elsewhere. The argument goes as follows. In [5] it is shown that for an onto frame homomorphism $h : L \longrightarrow M$, the sublocale $h_*(M)$ of $L$ is nowhere dense if and only if $h_*(0)$ is a dense element. In our situation we have $p_*(a) = (a,1)$, which then implies $p_*(0) = (0,1)$, which is a dense element in $\mathcal{I}L$. Therefore $p_*(L)$ is a nowhere dense sublocale of $\mathcal{I}L$.

**Corollary 4.4.** *Every locale can be embedded as a closed sublocale of a connected, locally connected locale.*

*Proof.* This follows from the above theorem and from the fact that an irreducible locale is connected and locally connected (see Corollary 3.5). □

**Theorem 4.5.** *Given any frame map $f : L \longrightarrow M$ there exists a frame map $\mathcal{I}f : \mathcal{I}L \longrightarrow \mathcal{I}M$ such that the following diagram commutes in **Frm**:*

$$\begin{array}{ccc} \mathcal{I}L & \xrightarrow{\mathcal{I}f} & \mathcal{I}M \\ \downarrow{\scriptstyle p_L} & & \downarrow{\scriptstyle p_M} \\ L & \xrightarrow{f} & M \end{array}$$

*where $p_L$ and $p_M$ are the respective quotient frame maps described in Theorem 4.2.*

*Proof.* Define $(\mathcal{I}f)(a,b) = (f(a),b)$. Note that if $(a,b) \in \mathcal{I}L$ then $(f(a),b) \in \mathcal{I}M$. It is also easy to check that $\mathcal{I}f$ is a frame homomorphism. Furthermore $p_M((\mathcal{I}f)(a,b)) = p_M(f(a),b) = f(a) = f((p_L)(a,b))$, so the above diagram commutes. □

Let **IrrFrm** denote the full subcategory of **Frm** consisting of the irreducible frames. Then we have:

**Proposition 4.6.** *$\mathcal{I}$ : **Frm** $\longrightarrow$ **IrrFrm** is a functor.*

*Proof.* We have seen that $\mathcal{I}L$ is irreducible for every frame $L$. Furthermore, it is easy to verify that $\mathcal{I}(id) = id$ and $\mathcal{I}(f \circ g) = \mathcal{I}f \circ \mathcal{I}g$, thus making $\mathcal{I}$ a functor. □

**Remark 4.7.** The map $p_L : \mathcal{I}L \longrightarrow L$ is never injective since $p_L(0,0) = p_L(0,1)$. Therefore $\mathcal{I}$ is not a coreflector. Indeed, if it were, then $p_{\mathcal{I}L} : \mathcal{I}(\mathcal{I}L) \longrightarrow \mathcal{I}L$ would be an isomorphism. To see this, let $\tau : \mathcal{I}L \longrightarrow \mathcal{I}(\mathcal{I}L)$ be the (unique) homomorphism such that $p_{\mathcal{I}L} \cdot \tau = id_{\mathcal{I}L}$ - which exists if $\mathcal{I}L$ is a coreflector. Then the diagram

$$\begin{array}{ccccc} \mathcal{I}(\mathcal{I}L) & \xrightarrow{p_{\mathcal{I}L}} & \mathcal{I}L & \xrightarrow{\tau} & \mathcal{I}(\mathcal{I}L) \\ {\scriptstyle p_{\mathcal{I}L}}\big\downarrow & & & & \big\downarrow{\scriptstyle p_{\mathcal{I}L}} \\ \mathcal{I}L & = & \mathcal{I}L & = & \mathcal{I}L \end{array}$$

commutes, since $p_{\mathcal{I}L}\cdot\tau\cdot p_{\mathcal{I}L} = id_{\mathcal{I}L}\cdot p_{\mathcal{I}L} = p_{\mathcal{I}L}$. Since $id_{\mathcal{I}(\mathcal{I}L)} : \mathcal{I}(\mathcal{I}L) \longrightarrow \mathcal{I}(\mathcal{I}L)$ also makes the above diagram commute, we have by uniqueness that $\tau\cdot p_{\mathcal{I}L} = id_{\mathcal{I}(\mathcal{I}L)}$. Thus $p_{\mathcal{I}L}$ is an isomorphism, which is a contradiction.

The following result, however, shows that $\mathcal{I}L$ is "almost" a coreflector.

**Proposition 4.8.** *For any frame L and a homomorphism $g : M \longrightarrow L$ with M irreducible, there is a frame homomorphism $\tilde{g} : M \longrightarrow \mathcal{I}L$ such that the triangle below commutes.*

$$\begin{array}{ccc} M & \xrightarrow{\tilde{g}} & \mathcal{I}L \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle p_L} \\ L & = & L \end{array}$$

*Proof.* Define $\tilde{g} : M \longrightarrow \mathcal{I}L$ by

$$\tilde{g}(x) = \begin{cases} (0,0) & \text{if } x = 0 \\ (g(x),1) & \text{if } x \neq 0. \end{cases}$$

It is clear that $\tilde{g}$ preserves the bottom, the top, and all joins. For binary meets: if $x \wedge y = 0$, then $x = 0$ or $y = 0$ since $M$ is irreducible. Then $\tilde{g}(x \wedge y) = \tilde{g}(x) \wedge \tilde{g}(y)$. If $x \wedge y \neq 0$, then $x \neq 0$ and $y \neq 0$, and so we immediately have $\tilde{g}(x \wedge y) = \tilde{g}(x) \wedge \tilde{g}(y)$.

To see commutativity of the triangle, observe that we have $p_L(\tilde{g}(0)) = 0 = g(0)$, and if $0 \neq x \in M$, then

$$(p_L \cdot \tilde{g})(x) = p_L((g(x), 1)) = g(x),$$

from which we deduce that $p_L \cdot \tilde{g} = g$. □

In the next proposition we show that certain properties satisfied by the frame map $f : L \longrightarrow M$ are inherited by the lifted frame homomorphism $\mathcal{I}f : \mathcal{I}L \longrightarrow \mathcal{I}M$. Recall that a localic map $f : M \longrightarrow L$ is said to be closed if it maps closed sublocales to closed sublocales, and this is equivalent to saying that the corresponding frame map $f^* : L \longrightarrow M$ satisfies the condition

$$f(a \vee f^*(b)) = f(a) \vee b \ \text{ for all } a \in M, b \in L$$

(see [10]). A frame homomorphism is then said to be closed if its corresponding localic map $f_*$ is closed.

**Proposition 4.9.** *(a) If $f$ is open, then so is $\mathcal{I}f$.*
*(b) If $f$ is a closed injection, then so is $\mathcal{I}f$.*

*Proof.* (a) Suppose $f : L \longrightarrow M$ is open. We have to show that $\mathcal{I}f$ is a complete Heyting homomorphism. Firstly, we show that $\mathcal{I}L$ is closed under arbitrary meet in $L \times \mathbf{2}$: For let $X = \{(a_\alpha, b_\alpha) | \alpha \in I\} \subseteq \mathcal{I}L$, and $\bigwedge X$ the meet in $L \times \mathbf{2}$. If there exists $\beta$ such that $b_\beta = 0$, then $(a_\beta, b_\beta) = (0,0)$ since $(a_\beta, b_\beta) \in \mathcal{I}L$. Hence $\bigwedge X = (\bigwedge_{\alpha\in I} a_\alpha, \bigwedge_{\alpha\in I} b_\alpha) = (0,0) \in \mathcal{I}L$. If, on the other hand, $b_\alpha = 1$ for all $\alpha \in I$, then $\bigwedge X = \bigwedge(a_\alpha, 1) = (\bigwedge a_\alpha, 1) \in \mathcal{I}L$. Now we show $\mathcal{I}f$ preserves arbitrary meet. Taking $X$ as above, if $b_\alpha = 0$ for some $\alpha$, then $(\mathcal{I}f)(\bigwedge X) = (\mathcal{I}f)(0,0) = (0,0) = \bigwedge(\mathcal{I}f)(a_\alpha, b_\alpha)$ since $(a_\alpha, b_\alpha) = (0,0)$ for some $\alpha$. If $b_\alpha = 1$ for all $\alpha$, then

$$\begin{aligned}(\mathcal{I}f)(\bigwedge(a_\alpha,1) &= (\mathcal{I}f)(\bigwedge a_\alpha,1)\\ &= (f(\bigwedge a_\alpha),1)\\ &= (\bigwedge f(a_\alpha),1)\\ &= \bigwedge(f(a_\alpha),1)\\ &= \bigwedge(\mathcal{I}f)(a_\alpha,1);\end{aligned}$$

the third step following from the fact that $f$ is open and hence preserves arbitrary meet.

We now show that $\mathcal{I}f$ preserves the Heyting operation. For this, we observe firstly that:
(i) For $(a,1)\in\mathcal{I}L$, $(a,1)\to(0,0)=(0,0)$. (This was observed in the proof of Theorem 4.2).
(ii) For $(a,b)\in\mathcal{I}L$, $(0,0)\to(a,b)=(1,1)$ (since $\mathfrak{o}((0,0))=\mathbb{O}$).
(iii) For $a,b\in L$, $(a,1)\to(b,1)=(a\to b,1)$: To see this, for any $(x,y)\in\mathcal{I}L$ we have

$$\begin{aligned}(x,y)\le(a,1)\to(b,1) &\iff (x,y)\wedge(a,1)\le(b,1)\\ &\iff (x\wedge a, y\wedge 1)\le(b,1)\\ &\iff x\wedge a\le b,\ y\le 1\\ &\iff x\le a\to b,\ y\le 1\\ &\iff (x,y)\le(a\to b,1).\end{aligned}$$

Thus we have:
(i)

$$\begin{aligned}(\mathcal{I}f)((0,0)\to(a,b)) &= (\mathcal{I}f)((1,1))\\ &= (1,1)\\ &= (0,0)\to(f(a),b)\\ &= (\mathcal{I}f)((0,0))\to(\mathcal{I}f)((a,b)).\end{aligned}$$

(ii)

$$\begin{aligned}(\mathcal{I}f)((a,1)\to(0,0)) &= (\mathcal{I}f)((0,0))\\ &= (0,0)\\ &= (f(a),1)\to(0,0)\\ &= (\mathcal{I}f)((a,1))\to(\mathcal{I}f)((0,0)).\end{aligned}$$

(iii)

$$\begin{aligned}(\mathcal{I}f)((a,1)\to(b,1)) &= (\mathcal{I}f)((a\to b,1))\\ &= (f(a\to b),1)\\ &= (f(a)\to f(b),1) \qquad (f \text{ preserves } \to)\\ &= (f(a),1)\to(f(b),1)\\ &= (\mathcal{I}f)((a,1))\to(\mathcal{I}f)((b,1))\end{aligned}$$

Thus $\mathcal{I}f$ is open.

(b) We first observe that for any $f:L\to M$ we have:
(i) $(\mathcal{I}f)_*((0,0))=(0,0)$:

For $(x, y) \in \mathcal{I}L$, $(x, y) \leq (\mathcal{I}f)_*((0,0)) \iff (\mathcal{I}f)((x,y)) = (0,0) \iff (f(x), y) = (0,0) \iff (x,y) = (0,0)$, the latter because $y = 0$ implies $x = 0$ as $(x, y) \in \mathcal{I}L$.

(ii) For any $(b, 1) \in \mathcal{I}M$, $(\mathcal{I}f)_*((b,1)) = (f_*(b), 1)$:

For $(x, y) \in \mathcal{I}L$,

$$\begin{aligned}
(x,y) \leq (\mathcal{I}f)_*((b,1)) &\iff (\mathcal{I}f)((x,y)) \leq (b,1) \\
&\iff (f(x), y) \leq (b,1) \\
&\iff f(x) \leq b, \quad y \leq 1 \\
&\iff x \leq f_*(b), \quad y \leq 1 \\
&\iff (x,y) \leq (f_*(b), 1)
\end{aligned}$$

Now suppose $f$ is closed and one-one. The proof that $\mathcal{I}f$ is closed will be complete if we can show the closed map condition for $\mathcal{I}f$ in each of the following four cases:

$(\mathcal{I}f)_*((0,0) \vee (\mathcal{I}f)((0,0))) = (\mathcal{I}f)_*((0,0)) \vee (0,0)$:

This follows from (i) above.

$(\mathcal{I}f)_*((0,0) \vee (\mathcal{I}f)((a,1))) = (\mathcal{I}f)_*((0,0)) \vee (a,1)$:

Using (b)(i) and (ii) from above, this amounts to showing $(f_*f(a), 1) = (a, 1)$. But since $f(f_*f(a)) = f(a)$ and $f$ is one-one, we must have $f_*f(a) = a$.

$(\mathcal{I}f)_*((b,1) \vee (\mathcal{I}f)((0,0))) = (\mathcal{I}f)_*((b,1)) \vee (0,0)$:

This is clear.

$(\mathcal{I}f)_*((b,1) \vee (\mathcal{I}f)((a,1))) = (\mathcal{I}f)_*((b,1)) \vee (a,1)$:

To see this note that

$$\begin{aligned}
(\mathcal{I}f)_*((b,1) \vee (\mathcal{I}f)((a,1))) &= (\mathcal{I}f)_*((b,1) \vee (f(a),1)) \\
&= (\mathcal{I}f)_*((b \vee f(a), 1)) \\
&= (f_*(b \vee f(a)), 1) \qquad \text{(from (b)(ii))} \\
&= (f_*(b) \vee a, 1) \qquad (f \ \text{closed}) \\
&= (f_*(b), 1) \vee (a, 1) \\
&= (\mathcal{I}f)_*((b,1)) \vee (a,1).
\end{aligned}$$

Lastly $\mathcal{I}f$ is one-one follows from $(\mathcal{I}f)((a,1)) = (\mathcal{I}f)((b,1)) \implies (f(a),1) = (f(b),1) \implies f(a) = f(b) \implies a = b$.

(c) This follows from the fact that if $f$ is one-one then so is $\mathcal{I}f$ (as seen above), and the easy observation that $f$ onto implies $\mathcal{I}f$ is onto. □

**Proposition 4.10.** *(a) $\mathcal{I}L$ cannot be regular for any $L$.*

*(b) $L$ is spatial iff $\mathcal{I}L$ is spatial.*

*Proof.* (a) We have $\mathcal{I}L = \{(0,0), (a,1) | a \in L\}$, so $|\mathcal{I}L| \geq 3$. Since $\mathcal{I}L$ is irreducible, it cannot be regular by Proposition 3.2.

(b) Suppose $L$ is spatial. Then $L \times \mathbf{2}$ is spatial, hence $\mathcal{I}L$ being a subframe of $L \times \mathbf{2}$ must be spatial. Conversely suppose $\mathcal{I}L$ is spatial. Now every complemented sublocale of a spatial locale is spatial (see [10] VI.3). Since closed sublocales are complemented, and $L$ is embedded as a closed sublocale of $\mathcal{I}L$ we must have that $L$ is spatial. □

**Remark 4.11.** Part (b) of the above proposition gives a way of producing examples of frames which are connected and locally connected but not spatial. For if $L$ is non-spatial, then $\mathcal{I}L$ is connected, locally connected (see Corollary 3.5) and non-spatial.

Recall that a locale $L$ is said to be *subfit* if whenever $a \nleq b$ there exists $c$ such that $a \vee c = 1$ and $b \vee c \neq 1$, and it is called *fit* if whenever $a \nleq b$ there exists $c$ such that $a \vee c = 1$ and $c \to b \nleq b$. A regular locale is fit, which in turn, is subfit. (See [10] V). An element $a \in L$ is called *compact* if whenever $a \leq \bigvee S$, $S \subseteq L$, there is a finite subset $T$ of $S$ such that $a \leq \bigvee T$. A locale $L$ is called *Noetherian* if and only if each of its elements is compact.

**Proposition 4.12.** *(a) L is subfit if and only if $\mathcal{I}L$ is subfit.*
*(b) L is fit if and only if $\mathcal{I}L$ is fit.*
*(c) L is compact if and only if $\mathcal{I}L$ is compact.*
*(d) L is Noetherian if and only if $\mathcal{I}L$ is Noetherian.*

*Proof.* (a) Every complemented sublocale of a subfit locale is subfit (see eg. [10] V.1). Hence if $\mathcal{I}L$ is subfit, so is $L$ since $L$ is embedded as a closed, and hence complemented, sublocale of $\mathcal{I}L$. Now suppose $L$ is subfit and $(a, 1) \nleq (b, 1)$ in $\mathcal{I}L$. Then $a \nleq b$. Hence there exists $c \in L$ such that $a \vee c = 1$ and $b \vee c \neq 1$. Then $(a, 1) \vee (c, 1) = (1, 1)$ and $(b, 1) \vee (c, 1) = (b \vee c, 1) \neq (1, 1)$. Thus $\mathcal{I}L$ is subfit.

(b) Suppose $L$ is fit and that $(a, 1) \nleq (b, 1)$ in $\mathcal{I}L$. Then $a \nleq b$ so there is a $c$ such that $a \vee c = 1$ and $c \to b \nleq b$. Then $(a, 1) \vee (c, 1) = (1, 1)$ and $(c, 1) \to (b, 1) = (c \to b, 1) \nleq (b, 1)$. Thus $\mathcal{I}L$ is subfit. The reverse follows from the fact that every sublocale of a fit locale is fit (see [10]).

(c) If $\mathcal{I}L$ is compact, then $L$ is also compact being a closed sublocale of $\mathcal{I}L$. The reverse follows routinely from the definition.

(d) This follows easily since $a \in L$ is compact if and only if $(a, 1) \in \mathcal{I}L$ is compact. □

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