Filomat 32:9 (2018), 3397–3408
https://doi.org/10.2298/FIL1809397T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inhomogeneous Multi-Parameter Lipschitz Spaces Associated with Different Homogeneities and their Applications

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Abstract. This paper is motivated by Phong and Stein's work in [11]. The purpose of this paper is to establish the inhomogeneous multi-parameter Lipschitz spaces Lip^{α}_{com} associated with mixed homogeneities and characterize these spaces via the Littlewood-Paley theory. As applications, the boundedness of the composition of Calderón-Zygmund singular integral operators with mixed homogeneities has been considered.

1. Introduction and Statement of Main Results

This paper is motivated by Phong and Stein's work in [11]. The purpose of this paper is to introduce a new class of inhomogeneous Lipschitz spaces associated with mixed homogeneities and characterize these spaces via the Littlewood-Paley theory. We also prove that the composition of two Calderón-Zygmund singular integral operators with mixed homogeneities is bounded on these new Lipschitz spaces.

In order to explain the question we deal with let us begin with recalling the questions of composition of operators with mixed homogeneities. To be precise, let $e(\xi)$ be a function on $\mathbb{R}^n$ homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, suppose that $h(\xi)$ is a function on $\mathbb{R}^n$ homogeneous of degree 0 in the non-isotropic sense related to the heat equation, and also smooth away from the origin. Then it is well-known that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\widehat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type (1, 1). Note that any operator T_1 is bounded on the classical Hardy and homogeneous Lipschitz spaces, while T_2 is bounded on the Hardy and homogeneous Lipschitz spaces associated the non-isotropic homogeneity, which was introduced in [9]. See also the recent results of Stein and Yung [12]. Rivieré in [13] asked the question: Is the composition $T_1 \circ T_2$ still of weak-type (1,1)? Phong

2010 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B25, 46E30
Keywords. Inhomogeneous Lipschitz space, almost orthogonal estimate, composition operator, different homogeneities, Littlewood-Paley theory.
Received: 10 September 2017; Accepted: 21 October 2018
Communicated by Dragan S. Djordjević
The first author is supported by Natural Science Foundation of Jiangsu Province of China (grant no. BK20180734), Natural Science Foundation for Colleges and Universities in Jiangsu Province (grant no. 18KJB110022). The second author is supported by National Natural Science Foundation of China (Grant No. 11471338) and Guangdong Province Natural Science Foundation (Grant No. 2017A030313028).
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and Stein in [11] was the first to answer this question and they gave the necessary and sufficient conditions for which $T_1 \circ T_2$ is of weak-type (1,1). The operators Phong and Stein studied are in fact compositions with mixed type of homogeneities which arise naturally in the $\bar{\partial}$-Neumann problem. See [11] for more details. In 2013, Han etc. in [4] developed a new Hardy space theory and proved that the composition of two Calderón-Zygmund singular integral operators with different homogeneities is bounded on this new Hardy space.

Indeed, there are other questions of this type that can be asked about composition of operators associated with mixed homogeneities, which cannot be answered by using the properties of these operators separately. We mention that such a question was considered for the homogeneous Lipschitz spaces in [3]. It is well known that the classical Lipschitz spaces play an important role in harmonic analysis and partial differential equations. See [1, 2, 5–8, 10].

The purpose of this paper is to introduce a new class of inhomogeneous Lipschitz spaces associated with mixed homogeneities and characterize these spaces via the Littlewood-Paley theory. We prove that the composition of two Calderón-Zygmund singular integral operators with mixed homogeneities is bounded on these Lipschitz spaces.

In order to define the inhomogeneous Lipschitz space associated with mixed homogeneities, we recall some notations concerning mixed homogeneities. For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\delta > 0$, we consider two kinds of homogeneities

$$\delta_e : (x', x_n) \to (\delta x', \delta x_n), \quad \delta > 0$$

and

$$\delta_h : (x', x_n) \to (\delta x', \delta^2 x_n), \quad \delta > 0.$$

We denote $|x|_e = (|x'|^2 + |x_n|^2)^{1/2}$ and $|x|_h = (|x'|^2 + |x_n|)^{1/2}$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$.

Throughout this paper, we use C to denote positive constants, whose value may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. We denote by $f \sim g$ that there exists a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Now we can introduce the definition of the inhomogeneous Lipschitz space associated with different homogeneities. Denote that

$$\Delta_u f(x) = f(x-u) - f(x); \quad \Delta_u^z f(x) = f(x-u) + f(x+u) - 2f(x)$$

and

$$\Delta_v f(x) = f(x-v) - f(x); \quad \Delta_v^z f(x) = f(x-v) + f(x+v) - 2f(x).$$

Definition 1.1. *Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 0$. The inhomogeneous Lipschitz space associated with mixed homogeneities Lip^{α}_{com} is defined to be the space of all bounded continuous f defined on $\mathbb{R}^n$ such that*
(i) when $0 < \alpha_1, \alpha_2 < 1$,

$$\|f\|_{Lip^{\alpha}_{com}} := \|f\|_\infty + \sup_{u \neq 0} \frac{|\Delta_u f|}{|u|_e^{\alpha_1}} + \sup_{v \neq 0} \frac{|\Delta_v f|}{|v|_h^{\alpha_2}} + \sup_{u,v \neq 0} \frac{|\Delta_u \Delta_v f|}{|u|_e^{\alpha_1} |v|_h^{\alpha_2}} < \infty;$$

(ii) when $\alpha_1 = 1, 0 < \alpha_2 < 1$,

$$\|f\|_{Lip^{\alpha}_{com}} := \|f\|_\infty + \sup_{u \neq 0} \frac{|\Delta_u^z f|}{|u|_e} + \sup_{v \neq 0} \frac{|\Delta_v f|}{|v|_h^{\alpha_2}} + \sup_{u,v \neq 0} \frac{|\Delta_u^z \Delta_v f|}{|u|_e |v|_h^{\alpha_2}} < \infty;$$

(iii) when $0 < \alpha_1 < 1, \alpha_2 = 1$,

$$\|f\|_{Lip^{\alpha}_{com}} := \|f\|_\infty + \sup_{u \neq 0} \frac{|\Delta_u f|}{|u|_e^{\alpha_1}} + \sup_{v \neq 0} \frac{|\Delta_v^z f|}{|v|_h} + \sup_{u,v \neq 0} \frac{|\Delta_u \Delta_v^z f|}{|u|_e^{\alpha_1} |v|_h} < \infty;$$

(iv) when $\alpha_1 = \alpha_2 = 1$,

$$\|f\|_{Lip^{\alpha}_{com}} := \|f\|_{\infty} + \sup_{u\neq 0} \frac{|\Delta^z_u f|}{|u|_e} + \sup_{v\neq 0} \frac{|\Delta_v f^z|}{|v|_h} + \sup_{u,v\neq 0} \frac{|\Delta^z_u \Delta^z_v f|}{|u|_e |v|_h} < \infty.$$

When $\alpha = (\alpha_1, \alpha_2)$ *with* $\alpha_1, \alpha_2 > 1$, *we write* $\alpha_1 = m_1 + r_1$ *and* $\alpha_2 = m_2 + r_2$ *where* m_1, m_2 *are integers and* $0 < r_1, r_2 \leq 1$, $r = (r_1, r_2)$. $f \in Lip^{\alpha}_{com}$ *means that* f *is a* $C^{m_1+m_2}$ *function such that*

$$\|f\|_{Lip^{\alpha}_{com}} := \sum_{|\beta|=m_1+m_2} \|D^{\beta} f\|_{Lip^{r}_{com}} < \infty.$$

Next we will give the Littlewood-Paley characterization for Lip^{α}_{com}. For this purpose, let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ be a radial function with

$$\mathrm{supp}\widehat{\psi^{(1)}}(\xi) \subset \{\xi : 1/2 < |\xi|_e \leq 2\},$$

and let $\varphi^{(1)}$ be a radial function with

$$\widehat{\varphi^{(1)}}(0) = 1, \quad \mathrm{supp}\widehat{\varphi^{(1)}} \subset \{|\xi|_e \leq 2\}$$

satisfying

$$|\widehat{\varphi^{(1)}}(\xi', \xi_n)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_n)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

And let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ be a radial function with

$$\mathrm{supp}\widehat{\psi^{(2)}}(\xi) \subset \{\xi : 1/2 < |\xi|_h \leq 2\},$$

and let $\varphi^{(2)}$ be a radial function with

$$\widehat{\varphi^{(2)}}(0) = 1, \quad \mathrm{supp}\widehat{\varphi^{(1)}} \subset \{|\xi|_e \leq 2\}$$

satisfying

$$|\widehat{\varphi^{(2)}}(\xi', \xi_n)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi^{(2)}}(2^{-j}\xi', 2^{-j}\xi_n)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

Let $\psi^{(1)}_j(x) = 2^{jn}\psi^{(1)}(2^j x', 2^j x_n)$, $\psi^{(2)}_k(x) = 2^{k(n+1)}\psi^{(2)}(2^k x', 2^{2k} x_n)$ and $\psi_{j,k} = \psi^{(1)}_j * \psi^{(2)}_k(x)$. We denote $\varphi^{(i)} = \psi^{(i)}_0$, where $i = 1, 2$.

By taking the Fourier transform, we obtain the Calderón identity, that is, for $f \in L^2$,

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_{j,k} * \psi_{j,k} * f,$$

where the series converges in $L^2(\mathbb{R}^n)$ norm.

One of the main results of this paper is the following Littlewood-Paley characterization:

Theorem 1.2. $f \in Lip^{\alpha}_{com}$ *with* $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$ *if and only if* $f \in \mathcal{S}'$ *and*

$$\|\psi_{j,k} * f\|_{\infty} \leq C 2^{-j\alpha_1} 2^{-k\alpha_2},$$

where $j, k \geq 0$.

Furthermore,

$$\|f\|_{Lip^{\alpha}_{com}} \sim \sup_{j,k\geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\psi_{j,k} * f\|_{\infty}.$$

To study the boundedness of the composition of Calderón-Zygmund singular integral operators on Lip^{α}_{com}, we need the following

Definition 1.3. *A locally integrable function K_1 on $\mathbb{R}^n \setminus \{0\}$ is called a Calderón-Zygmund kernel associated with isotropic homogeneity if there exist constants $C_1, \delta > 0$ such that*
(i) $|K_1(x)| \le C_1 \frac{1}{|x|_e^n}$;
(ii) $|K_1(x)| \le C_1 \frac{1}{|x|_e^{n+\delta}}$ *for* $|x|_e \ge 1$;
(iii) $\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{K}_1(x)\right| \le A|x|_e^{-n-|\alpha|}$ *for all* $|\alpha| \ge 0$;
(iv) $\int_{r_1<|x|_e<r_2} K_1(x)dx = 0$ *for all* $0 < r_1 < r_2 < \infty$.
We say that T_1 is a Calderón-Zygmund singular integral operators associated with isotropic homogeneity if $T_1 f(x) = p.v.(K_1 * f)(x)$, where K_1 fulfills condition $(i) - (iv)$.

Definition 1.4. *Suppose that $K_2 \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ is said to be a Calderón-Zygmund kernel associated with the non-isotropic homogeneity if there exist constants $C_1, \delta > 0$ such that*
(v) $|K_2(x)| \le C_1 \frac{1}{|x|_h^{n+1}}$;
(vi) $|K_2(x)| \le C_1 \frac{1}{|x|_h^{n+\delta+1}}$ *for* $|x|_h \ge 1$;
(vii) $\left|\frac{\partial^\alpha}{\partial (x')^\alpha}\frac{\partial^\beta}{\partial (x_n)^\beta}\mathcal{K}_2(x', x_n)\right| \le B|x|_h^{-n-|\alpha|-2\beta-1}$ *for all* $|\alpha|,\ \beta \ge 0,$;
(viii) $\int_{r_1<|x|_h<r_2} K_2(x)dx = 0$ *for all* $0 < r_1 < r_2 < \infty$.
We say that T_2 is a Calderón-Zygmund singular integral operators associated with the non-isotropic homogeneity if $T_2 f(x) = p.v.(K_2 * f)(x)$, where K_2 fulfills condition $(v) - (viii)$.

We now state the main result of this paper.

Theorem 1.5. *Suppose that T_1 and T_2 are the Calderón-Zygmund singular integral operators associated with the isotropic and non-isotropic homogeneity, respectively. Then both T_1 and T_2 extend to bounded operators on Lip^{α}_{com} with $\alpha = (\alpha_1, \alpha_2)$ for $0 < \alpha_1, \alpha_2 < \infty$. Particularly, the composition operator $T = T_1 \circ T_2$ is bounded on Lip^{α}_{com} with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$.*

2.Proof of Theorem 1.1

Proof. First we prove that if $f \in Lip^{\alpha}_{com}$ with $0 < \alpha_1, \alpha_2 < 1$, then $f \in \mathcal{S}'$. To do this, it suffices to show that $< f, g >$ is well defined for any $g \in \mathcal{S}$. In fact, we have

$$g(x) = \sum_{j,k=0}^{\infty} \psi_{j,k} * \psi_{j,k} * g(x),$$

where the series converges in $\mathcal{S}$. Then we only need to prove that $\sum_{j,k=0}^{\infty} < f, \psi_{j,k} * \psi_{j,k} * g >$ is well defined for $g \in \mathcal{S}$. To this end, for all $j, k \ge 0$ we estimate $< \psi_{j,k} * f, \psi_{j,k} * g >$ as follows.
Case 1: $j = k = 0$.

$$|\psi_{0,0} * f(x)| = |\int\int \psi_0^{(1)}(u)\psi_0^{(2)}(v)f(x-u-v)dudv| \le C\|f\|_\infty \le C\|f\|_{Lip^{\alpha}_{com}}.$$

This implies that

$$| < \psi_{0,0} * f, \psi_{0,0} * g > | \le C\|f\|_{Lip^{\alpha}_{com}}\|g\|_{\mathcal{S}}.$$

Case 2: $j \geq 1; k = 0$.
Applying the cancellations conditions on $\psi_j^{(1)}$, we have

$$\begin{aligned}|\psi_{j,0} * f(x)| &= |\int\int \psi_j^{(1)}(u)\psi_0^{(2)}(v)[f(x-u-v) - f(x-v)]dudv| \\ &\leq \sup_{u\neq 0, x\in\mathbb{R}^n} \frac{|f(x-u) - f(x)|}{|u|_e^{\alpha_1}} \int\int |\psi_j^{(1)}(u)\psi_0^{(2)}(v)||u|_e^{\alpha_1} dudv \\ &\leq C\|f\|_{Lip_{com}^\alpha} \int\int \frac{2^{jn}}{(1+2^j|u|_e)^M} \frac{1}{(1+|v|_h)^M} |u|_e^{\alpha_1} dudv \\ &\leq C2^{-j\alpha_1}\|f\|_{Lip_{com}^\alpha}.\end{aligned}$$

Applying the almost orthogonal estimate, we get that

$$|\psi_{j,0} * g(x)| = |\psi_j^{(1)} * \psi_0^{(2)} * g(x)| \leq C2^{-jL} \frac{1}{(1+|x'|)^{n+M-1}} \frac{1}{(1+|x_n|)^{1+M}} \|g\|_{\mathcal{S}}$$

for any $L, M \geq 0$, where $j \in \mathbb{Z}_+$.
Therefore, we obtain that

$$| < \psi_{j,0} * f, \psi_{j,0} * g > | \leq C2^{-j(L+\alpha_1)}\|f\|_{Lip_{com}^\alpha}\|g\|_{\mathcal{S}}.$$

Case 3: $j = 0; k \geq 1$.
Applying the similar estimate, we have

$$| < \psi_{0,k} * f, \psi_{0,k} * g > | \leq C2^{-k(L+\alpha_2)}\|f\|_{Lip_{com}^\alpha}\|g\|_{\mathcal{S}}.$$

Case 4: $j \geq 1; k \geq 1$.
Applying the cancellations conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$, we have

$$\begin{aligned}&|\psi_{j,k} * f(x)| \\ &= |\int\int \psi_j^{(1)}(u)\psi_k^{(2)}(v)[f(x-u-v) - f(x-u) - f(x-v) + f(x)]dudv| \\ &\leq C \sup_{u,v\neq 0, x\in\mathbb{R}^n} \frac{|f(x-u-v) - f(x-u) - f(x-v) + f(x)|}{|u|_e^{\alpha_1}|v|_e^{\alpha_2}} \int\int |\psi_j^{(1)}(u)\psi_k^{(2)}(v)||u|_e^{\alpha_1}|v|_e^{\alpha_2} dudv \\ &\leq C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{Lip_{com}^\alpha}.\end{aligned}$$

Similarly, applying the almost orthogonal estimate, we get that

$$|\psi_{j,k} * g(x)| = |\psi_j^{(1)} * \psi_k^{(2)} * g(x)| \leq C2^{-jL}2^{-kL} \frac{1}{(1+|x'|)^{n+M-1}} \frac{1}{(1+|x_n|)^{1+M}} \|g\|_{\mathcal{S}}$$

for any $L, M \geq 0$, where $j, k \in \mathbb{Z}_+$.
We obtain that

$$| < \psi_{j,k} * f, \psi_{j,k} * g > | \leq C2^{-j(L+\alpha_1)}2^{-k(L+\alpha_2)}\|f\|_{Lip_{com}^\alpha}\|g\|_{\mathcal{S}}$$

and thus, $< f, g >$ is well defined. Moreover, if $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1, \alpha_2 < 1$, we have

$$\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * f\|_\infty \leq C\|f\|_{Lip_{com}^\alpha}.$$

When $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = 1$, $0 < \alpha_2 < 1$, we only need to consider the cases where $j \geq 1; k = 0$ and $j \geq 1; k \geq 1$ since the other two cases are similar to the case where $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = 1, \alpha_2 < 1$. Indeed, if $j \geq 1; k = 0$, noting that $\psi_j^{(1)}$ is a radial function, we have

$$\begin{aligned}
|\psi_{j,0} * f(x)| &= |\int\int \psi_j^{(1)}(u)\psi_0^{(2)}(v)[f(x-u-v) - f(x-v)]dudv| \\
&= \frac{1}{2}|\int\int \psi_j^{(1)}(u)\psi_0^{(2)}(v)[f(x+u-v) + f(x-u-v) - 2f(x-v)]dudv| \\
&\leq C \sup_{u\neq 0, x\in\mathbb{R}^n} \frac{|f(x+u-v) + f(x-u-v) - 2f(x-v)|}{|u|_e} \int\int |\psi_j^{(1)}(u)\psi_0^{(2)}(v)||u|_e dudv \\
&\leq C2^{-j}\|f\|_{Lip^\alpha_{com}}.
\end{aligned}$$

If $j \geq 1; k \geq 1$, then

$$\begin{aligned}
|\psi_{j,k} * f(x)| &= \frac{1}{2}|\int\int \psi_j^{(1)}(u)\psi_k^{(2)}(v)[f(x-u-v) + f(x+u-v)]dudv| \\
=&\frac{1}{2}|\int\int \psi_j^{(1)}(u)\psi_k^{(2)}(v)\{[f(x-u-v) + f(x+u-v) - 2f(x-v)] \\
&- 2[f(x-u) + f(x+u) - 2f(x)]\}dudv| \\
\leq& C \sup_{u,v\neq 0, x\in\mathbb{R}^n} \frac{|[f(x-u-v) + f(x+u-v) - 2f(x-v)] - 2[f(x-u) + f(x+u) - 2f(x)]|}{|u|_e|v|_e^{\alpha_2}} \\
&\int\int |\psi_j^{(1)}(u)\psi_k^{(2)}(v)||u|_e|v|_e^{\alpha_2} dudv \\
\leq& C2^{-j}2^{-k\alpha_2}\|f\|_{Lip^\alpha_{com}}.
\end{aligned}$$

Applying the almost orthogonal estimate, namely

$$|\psi_{j,k} * g(x)| = |\psi_j^{(1)} * \psi_k^{(2)} * g(x)| \leq C2^{-jL}2^{-kL}\frac{1}{(1+|x'|)^{n+M-1}}\frac{1}{(1+|x_n|)^{1+M}}\|g\|_{\mathcal{S}}$$

for any $L, M \geq 0$, where $j, k \in \mathbb{Z}_+$, yields

$$| < \psi_{j,k} * f, \psi_{j,k} * g > | \leq C2^{-j(L+1)}2^{-k(L+\alpha_2)}\|f\|_{Lip^\alpha_{com}}\|g\|_{\mathcal{S}}.$$

Thus, $< f, g >$ is well defined and

$$\sup_{j,k\geq 0} 2^j 2^{k\alpha_2}\|\psi_{j,k} * f\|_\infty \leq C\|f\|_{Lip^\alpha_{com}}.$$

Similarly, we can deal with other cases $\alpha = (\alpha_1, \alpha_2)$ where $0 < \alpha_1 < 1$, $\alpha_2 = 1$ or $\alpha_1 = \alpha_2 = 1$. Finally, we consider the case where $\alpha = (\alpha_1, \alpha_2)$ with $1 < \alpha_1 = m_1 + r_1, 1 < \alpha_2 = m_2 + r_2$ with $0 < r_1, r_2 \leq 1, r = (r_1, r_2)$. We write $\beta = (\beta_1, \beta_2)$, $|\beta_1| = m_1$, $|\beta_2| = m_2$, $\widehat{\tilde{\psi}_j^{(1)}}(\xi) = \frac{\widehat{\psi_j^{(1)}}(\xi)}{(-2\pi i\xi)^{\beta_1}}$ and $\widehat{\tilde{\psi}_k^{(2)}}(\xi) = \frac{\widehat{\psi_k^{(2)}}(\xi)}{(2\pi i\xi)^{\beta_2}}$ for $j, k \geq 0$. Then $\psi_{j,k} * f = D^\beta\tilde{\psi}_{j,k} * f = (-1)^{m_1+m_2}\tilde{\psi}_{j,k} * D^\beta f$, where $\tilde{\psi}_{j,k} = \tilde{\psi}_j^{(1)} * \tilde{\psi}_k^{(2)}$. Note that $2^{jm_1}2^{km_2}\tilde{\psi}_{j,k}$ satisfy the similar smoothness, size and cancellation conditions as $\psi_{j,k}$. Therefore, the similar argument yields that for any $j, k \geq 0$, $|\beta| = m_1 + m_2$

$$\begin{aligned}
\|\psi_{j,k} * f\|_\infty &= \|2^{-jm_1}2^{-km_2}(2^{jm_1}2^{km_2}\varphi_{j,k}) * D^\beta f\|_\infty \\
&\leq C2^{-jm_1}2^{-km_2}2^{-jr_1}2^{-kr_2}\|D^\beta f\|_{Lip^r_{com}} = C2^{-j\alpha_1}2^{-k\alpha_2}\|D^\beta f\|_{Lip^r_{com}}.
\end{aligned}$$

That is,

$$\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty \leq C\|f\|_{Lip^\alpha_{com}}.$$

To prove the converse statement, we first show that every distribution $f \in \mathcal{S}'$ satisfying

$$\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty \leq C$$

coincide with a bounded continuous function in $\mathbb{R}^n$.

To see this, as mentioned, $f(x) = \sum_{j,k\geq 0}\psi_{j,k}*\psi_{j,k}*f(x)$ in $\mathcal{S}'$. Observe that

$$|\psi_{j,k}*\psi_{j,k}*f| \leq \|\psi_{j,k}*f\|_\infty\|\psi_{j,k}\|_{L^1} \leq \Big(\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty\Big)2^{-j\alpha_1}2^{-k\alpha_2}.$$

Thus, the series $\sum_{j,k\geq 0}\psi_{j,k}*\psi_{j,k}*f$ converges uniformly in x. Since $\psi_{j,k}*\psi_{j,k}*f$ is continuous in $\mathbb{R}^n$, the sum function f is also continuous in $\mathbb{R}^n$. Moreover,

$$\|f\|_\infty \leq C\Big(\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty\Big).$$

Now we estimate $\|f\|_{Lip^\alpha_{com}}$ as follows.
Case 1: if $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1, \alpha_2 < 1$.

We show that

$$|f(x-u) - f(x)| \leq C\Big(\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty\Big)|u|_e^{\alpha_1}.$$

To do this, write

$$\begin{aligned}
&|f(x-u) - f(x)|\\
&= |\sum_{j,k\geq 0}\int[\psi_{j,k}(x-u-w) - \psi_{j,k}(x-w)]\psi_{j,k}*f(w)dw|\\
&\leq \Big(\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty\Big)\sum_{j,k\geq 0}2^{-j\alpha_1}2^{-k\alpha_2}\int|\psi_{j,k}(x-u-w) - \psi_{j,k}(x-w)|dw.
\end{aligned}$$

Therefore, we only need to consider the case where $|u|_e \leq 1$. Let n_1 be the unique nonnegative integer such that $2^{-n_1-1} \leq |u|_e < 2^{-n_1}$ and denote $A := \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k}*f\|_\infty$. Then we have

$$\begin{aligned}
&|f(x-u) - f(x)|\\
\leq & A\Big(\sum_{j=0}^{n_1}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\int|\psi_{j,k}(x-u-w) - \psi_{j,k}(x-w)|dw\\
&+\sum_{j=n_1}^{\infty}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\int|\psi_{j,k}(x-u-w)| + |\psi_{j,k}(x-w)|dw\Big)\\
=: & I + II.
\end{aligned}$$

For I, applying the mean value theorem, for any $M > 0$ there exist an $\theta \in (0, 1)$ such that

$$
\begin{aligned}
I &= A\Big(\sum_{j=0}^{n_1}\sum_{k=0}^{\infty} 2^{-j\alpha_1}2^{-k\alpha_2}\int |\psi_{j,k}(u+w)-\psi_{j,k}(w)|dw\Big)\\
&\leq A\Big(\sum_{j=0}^{n_1}\sum_{k=0}^{\infty} 2^{-j\alpha_1}2^{-k\alpha_2}\int\int |\psi_j^{(1)}(u+w-z)-\psi_j^{(1)}(w-z)||\psi_k^{(2)}(z)|dzdw\Big)\\
&\leq A\Big(\sum_{j=0}^{n_1}\sum_{k=0}^{\infty} 2^{-j\alpha_1}2^{-k\alpha_2}\int\int 2^j|u|_e\frac{2^{jn}}{(1+2^j|w-z+\theta u|_e)^M}\frac{2^{k(n+1)}}{(1+2^k|z|_h)^M}dzdw\Big)\\
&\leq A\Big(\sum_{j=0}^{n_1} 2^j2^{-n_1}2^{-j\alpha_1}\Big) \leq CA2^{-n_1}2^{n_1(1-\alpha_1)} \sim A2^{-n_1\alpha_1} \sim A|u|_e^{\alpha_1}.
\end{aligned}
$$

For II, the size conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$ yield

$$
\begin{aligned}
II &\leq CA\Big(\sum_{j=n_1}^{\infty}\sum_{k=0}^{\infty} 2^{-j\alpha_1}2^{-k\alpha_2}\int |\psi_{j,k}(w)|dw\Big)\\
&\leq CA\sum_{j=n_1}^{\infty} 2^{-j\alpha_1} \sim A2^{-n_1\alpha_1} \sim A|u|_e^{\alpha_1}.
\end{aligned}
$$

Thus, we obtain that for any $u \neq 0$ and $x \in \mathbb{R}^n$,

$$\frac{|f(x-u)-f(x)|}{|u|_e^{\alpha_1}} \leq \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * f\|_\infty.$$

Similarly,

$$\sup_{u\neq 0, x\in\mathbb{R}^n}\frac{|f(x-v)-f(x)|}{|v|_h^{\alpha_2}} \leq \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * f\|_\infty.$$

Finally, we show that

$$\sup_{u,v\neq 0}\frac{|f(x-u-v)-f(x-u)-f(x-v)+f(x)|}{|u|_e^{\alpha_1}|v|_h^{\alpha_2}} \leq \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * f\|_\infty.$$

In fact,

$$
\begin{aligned}
&|f(x-u-v)-f(x-u)-f(x-v)+f(x)|\\
=&|\sum_{j,k\geq 0}\int [\psi_{j,k}(x-u-v-w)-\psi_{j,k}(x-u-w)-\psi_{j,k}(x-v-w)+\psi_{j,k}(x-w)]\\
&\times \psi_{j,k} * f(w)dw|\\
=&|\sum_{j,k\geq 0}\int [\psi_j^{(1)}(x-u-z-w)-\psi_j^{(1)}(x-z-w)][\psi_k^{(2)}(z-v)-\psi_k^{(2)}(z)]\psi_{j,k} * f(w)dzdw|\\
\leq& A\sum_{j,k\geq 0} 2^{-j\alpha_1}2^{-k\alpha_2}\int |\psi_j^{(1)}(x-u-z-w)-\psi_j^{(1)}(x-z-w)||\psi_k^{(2)}(z-v)-\psi_k^{(2)}(z)|dzdw.
\end{aligned}
$$

We only consider the case where $|u|_e \leq 1$ and $|v|_h \leq 1$ since other cases are similar and easier. Let n_1, n_2 be the unique nonnegative integer such that $2^{-n_1-1} \leq |u|_e < 2^{-n_1}$ and $2^{-n_2-1} \leq |v|_h < 2^{-n_2}$.

Now we split the above series by

$$\begin{aligned} & A\sum_{j\geq n_1}\sum_{k\geq n_2}2^{-j\alpha_1}2^{-k\alpha_2}B + A\sum_{j<n_1}\sum_{k\geq n_2}2^{-j\alpha_1}2^{-k\alpha_2}B \\ + & \sum_{j\geq n_1}\sum_{k<n_2}2^{-j\alpha_1}2^{-k\alpha_2}B + \sum_{j<n_1}\sum_{k<n_2}2^{-j\alpha_1}2^{-k\alpha_2}B \\ & := B_1 + B_2 + B_3 + B_4, \end{aligned}$$

where $B = \int |\psi_j^{(1)}(x-u-z-w) - \psi_j^{(1)}(x-z-w)||\psi_k^{(2)}(z-v) - \psi_k^{(2)}(z)|dzdw$.

To deal with the first series, applying the size conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$ yields that B_1 is dominated by

$$\Big|B_1\Big| \leq C\sum_{j\geq n_1}\sum_{k\geq n_2}2^{-j\alpha_1}2^{-k\alpha_2} \leq C2^{-n_1\alpha_1}2^{-n_2\alpha_2} \leq C|u|_e^{\alpha_1}|v|_h^{\alpha_2}.$$

To estimate the second series B_2, applying the smoothness condition on $\psi_j^{(1)}$ and the size condition on $\psi_k^{(2)}$ implies that B_2 is bounded by

$$\Big|B_2\Big| \leq C\sum_{j<n_1}\sum_{k\geq n_2}2^{j(1-\alpha_1)}2^{-k\alpha_2}|u|_e \lesssim 2^{n_1(1-\alpha_1)}2^{-n_2\alpha_2}|u|_e \lesssim |u|_e^{\alpha_1-1}|v|_h^{\alpha_2}|u|_e \lesssim |u|_e^{\alpha_1}|v|_h^{\alpha_2}.$$

The estimate for third series B_3 is similar to the estimate for B_2. Finally, to handle with the last series B_4, applying the smoothness conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$ we obtain that B_4 is dominated by

$$\Big|B_4\Big| \leq C\sum_{j<n_1}\sum_{k<n_2}2^j|u|_e2^k|v|_h2^{-j\alpha_1}2^{-k\alpha_2} \lesssim 2^{n_1(1-\alpha_1)}2^{n_2(1-\alpha_2)}|u|_e|v|_h \lesssim |u|_e^{\alpha_1}|v|_h^{\alpha_2}.$$

These estimates imply that

$$|f(x-u-v) - f(x-u) - f(x-v) + f(x)| \leq A|u|_e|^{\alpha_1}|v|_h^{\alpha_2}.$$

Next we consider the case where $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = \alpha_2 = 1$ and only show the following estimate for any $u, v \in \mathbb{R}^n$

$$\begin{aligned} & [f(x-u-v) + f(x+u-v) - 2f(x-v)] + [f(x-u+v) + f(x+u+v) - 2f(x+v)] \\ & -2[f(x-u) + f(x+u) - 2f(x)] \\ & \leq C\Big(\sup_{j,k\geq 0}2^j2^k\|\psi_{j,k} * f(x)\|_\infty\Big)|u|_e|v|_h. \end{aligned}$$

The other estimates are similar and easier. Observe that

$$\begin{aligned} & [f(x-u-v) + f(x+u-v) - 2f(x-v)] + [f(x-u+v) + f(x+u+v) - 2f(x+v)] \\ & -2[f(x-u) + f(x+u) - 2f(x)] \\ & = \sum_{j,k\geq 0}\int [\psi_j^{(1)}(x-u-z-w) + \psi_j^{(1)}(x+u-z-w) - 2\psi_j^{(1)}(x-z-w)] \\ & \quad \times [\psi_k^{(2)}(z-v) + \psi_k^{(2)}(z+v) - 2\psi_k^{(2)}(z)]\psi_{j,k} * f(w)dzdw. \end{aligned}$$

Repeating a similar calculation gives the desired result for this case.

By repeating the similar calculation, we can handle the other cases where $\alpha_1 = 1$, $0 < \alpha_2 < 1$ and $0 < \alpha_1 < 1$, $\alpha_2 = 1$ similarly. To end the whole proof, we need to consider the case where $\alpha_1, \alpha_2 > 1$. We denote that $\beta = (\beta_1, \beta_2)$, $|\beta_i| = m_i$ and $\alpha_i = m_i + r_2$, $i = 1, 2$. Here we will prove that for any $|\beta| = m_1 + m_2$ and $0 < r_1, r_2 < 1$

$$\|D^\beta f\|_{Lip^r_{com}} \leq C \sup_{j,k\geq 0} 2^{j\alpha_1} 2^{\alpha_2} \|\psi_{j,k} * f\|_\infty.$$

Note that

$$\begin{aligned} & D^\beta f(x-u-v) - D^\beta f(x-u) - D^\beta f(x-v) + D^\beta f(x) \\ \leq & \sum_{j,k\geq 0} \int\int [D^{\beta_1}\psi_j^{(1)}(x-u-z-w) - D^{\beta_1}\psi_j^{(1)}(x-z-w)] \\ & \times [D^{\beta_2}\psi_k^{(2)}(z-v) - D^{\beta_2}\psi_k^{(2)}(z)]\psi_{j,k} * f(w)dwdz. \end{aligned}$$

By analogous argument we can obtain the desired results. We leave the details to the reader.

Therefore, the proof of Theorem 1.1 is concluded. □

3.Proof of Theorem 1.2

In order to prove Theorem 1.5, we first show the following

Proposition 3.1. *If $f \in Lip^\alpha_{com}$, then there exist a sequence $\{f_n\} \in L^2 \cap Lip^\alpha_{com}$ such that f_n converges to f in the distribution sense. Furthermore,*

$$\|f_n\|_{Lip^\alpha_{com}} \leq C\|f\|_{Lip^\alpha_{com}},$$

where the constant C is independent of f_n and f.

Proof. To show this proposition, note that

$$f = \sum_{j,k\geq 0} \psi_{j,k} * \psi_{j,k} * f(x)$$

in the distribution sense.

Set

$$f_n = \sum_{|j|,|k|\leq n} \psi_{j,k} * \psi_{j,k} * f(x).$$

Obviously, $f_n \in L^2$ and converges to f in the distribution sense. To see that $f_n \in Lip^\alpha_{com}$, by Theorem 1.1,

$$\|f_n\|_{Lip^\alpha_{com}} \leq C \sup_{j,k\geq 0, x\in\mathbb{R}^n} 2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f_n(x)|.$$

Observe that

$$\psi_{j,k} * f_n(x) = \psi_{j,k} * \sum_{|j'|,|k'|\leq n} \psi_{j',k'} * \psi_{j',k'} * f(x) = \sum_{|j'|,|k'|\leq n} \psi_{j,k} * \psi_{j',k'} * \psi_{j',k'} * f(x).$$

By the classical almost orthogonal estimate, that is, there exist two positive integers $L, M > \alpha_1 + \alpha_2 + 1$, such that

$$|\psi_{j,k} * \psi_{j',k'}(x)| \lesssim 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{(j\wedge j'\wedge k\wedge k')(n-1)}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|x'|)^{n+M-1}} \frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1 + 2^{j\wedge j'\wedge 2(k\wedge k')}|x_n|)^{1+M}}.$$

Therefore, again by Theorem 1.1, it follows that

$$2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f_n(x)| \quad \lesssim \quad \sup_{j',k',x} 2^{j'\alpha_1} 2^{k'\alpha_2} |\psi_{j',k'} * f(x)| \lesssim \|f\|_{Lip^\alpha_{com}}.$$

□

We now prove Theorem 1.2.

Proof of Theorem 1.5. First we prove that for any $f \in L^2 \cap Lip^{\alpha}_{com}$,

$$\|T_2(f)\|_{Lip^{\alpha}_{com}} \le C\|f\|_{Lip^{\alpha}_{com}}.$$

To see this, by Theorem 1.1,

$$\|T_2(f)\|_{Lip^{\alpha}_{com}} \le C \sup_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * T_2(f)\|_\infty.$$

Noting that T_2 is bounded on L^2, $f \in L^2$, and applying the Calderón identity yields

$$\psi_{j,k} * T_2 f(x) = \sum_{j',k'\ge 0} (\psi_{j,k} * K_2 * \psi_{j',k'}) * \psi_{j',k'} * f(x). \tag{1}$$

By the following almost orthogonal estimate

$$|\psi_{j,k} * K_2 * \psi_{j',k'}(x', x_n)| \le C \frac{2^{-|j-j'|L}2^{-|k-k'|L}}{[2^{-(j\wedge j'\wedge k\wedge k')} + |x'|]^{n+M-1}} \frac{2^{-(j\wedge j'\wedge k\wedge k')M}2^{-(j\wedge j'\wedge 2k\wedge 2k')M}}{[2^{-(j\wedge j'\wedge 2k\wedge 2k')} + |x_n|]^{1+M}}, \tag{2}$$

we obtain that

$$\begin{aligned}
&\|T_2(f)\|_{Lip^{\alpha}_{com}} \\
&\le C \sup_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2} \sum_{j',k'\ge 0} 2^{-|j-j'|L}2^{-|k-k'|L}\|\psi_{j,k} * f\|_\infty \\
&\le C \sup_{j',k'\ge 0} 2^{j'\alpha_1}2^{k'\alpha_2} \sup_{j,k\ge 0} \sum_{j',k'\ge 0} 2^{(j-j')\alpha_1}2^{(k-k')\alpha_2}2^{-|j-j'|L}2^{-|k-k'|L}\|\psi_{j,k} * f\|_\infty \\
&\le C \sup_{j',k'\ge 0} 2^{j'\alpha_1}2^{k'\alpha_2}\|\psi_{j',k'} * f\|_\infty \le C\|f\|_{Lip^{\alpha}_{com}}.
\end{aligned} \tag{3}$$

Next we extend T_2 to Lip^{α}_{com} as follows. By Proposition 3.1, if $f \in Lip^{\alpha}_{com}$, then there exist a sequence $\{f_n\} \in L^2 \cap Lip^{\alpha}_{com}$ such that f_n converges to f in the distribution sense. Furthermore,

$$\|f_n\|_{Lip^{\alpha}_{com}} \le C\|f\|_{Lip^{\alpha}_{com}}.$$

Applying (3) implies that

$$\|T_2(f_n) - T_2(f_m)\|_{Lip^{\alpha}_{com}} \le C\|f_n - f_m\|_{Lip^{\alpha}_{com}}.$$

Thus, $T_2(f_n)$ converges in the distribution sense and we can define

$$T_2(f) = \lim_{n\to\infty} T_2(f_n) \quad \text{in} \quad \mathcal{S}'.$$

To see the existence of this limit, we write $< T_2(f_n - f_m), g > = < f_n - f_m, T_2^*(g) >$ since that $f_n - f_m$ and g belong to L^2, and T_2 is bounded on L^2 as well as T^*. By Proposition 3.1, $< f_n - f_m, T_2^*(g) >$ tends to zero as $n, m \to \infty$.

Applying Theorem 1.1 again, we get that

$$\begin{aligned}
\|T_2(f)\|_{Lip^{\alpha}_{com}} &\le C \sup_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * T_2(f)(x)\|_\infty \\
&\le C \sup_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\|\lim_{n\to\infty} \psi_{j,k} * T_2(f_n)(x)\|_\infty \\
&\le C \liminf_{n\to\infty} \sup_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\|\psi_{j,k} * T_2(f_n)(x)\|_\infty \\
&\le C \liminf_{n\to\infty} \|f_n\|_{Lip^{\alpha}_{com}} \\
&\le C\|f\|_{Lip^{\alpha}_{com}}.
\end{aligned}$$

Similarly, we can also prove that

$$||T_1(f)||_{Lip^{\alpha}_{com}} \le C||f||_{Lip^{\alpha}_{com}}.$$

As a consequence, we obtain that

$$||T(f)||_{Lip^{\alpha}_{com}} \le C||f||_{Lip^{\alpha}_{com}}.$$

Therefore, we conclude the proof of Theorem 1.5. □

Finally, we remark that Theorem 1.1 holds for $0 < \alpha_1 < \epsilon_1$ and $0 < \alpha_2 < \epsilon_2$ if

$$\left|\mathcal{K}_1(x) - \mathcal{K}_1(x')\right| \le A|x - x'|_e^{\epsilon_1}|x|_e^{-n-\epsilon_1} \qquad \text{for } |x - x'|_e \le \frac{1}{2}|x|_e$$

and

$$\left|\mathcal{K}_2(x) - \mathcal{K}_2(x')\right| \le B|x - x'|_h^{\epsilon_2}|x|_h^{-n-1-\epsilon_2} \qquad \text{for all } |x - x'|_h \le \frac{1}{2}|x|_h.$$

We leave the details of the proof to the reader.

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