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Fuzzy Soft Filter Convergence

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Abstract. In this paper, we introduce the concept of fuzzy soft filters and study some of their properties. Also, we study the notion of convergence of fuzzy soft filters in fuzzy soft topological spaces. We prove the existence of product fuzzy soft filters.

1. Introduction and Preliminaries

It is well known that the notion of filter on a set is a basic concept in topology and plays an important role in it. The basic theory of filters can also be found in [18]. Some applications of filter convergence in topological spaces can be found in [10]. The notion of fuzzy filter appeared for the first time in [12] by Höhle and Šostak. However, similar notions with slight changes already appeared in [8, 9, 13]. One of the recent directions is the study of generalized filters [4–5] and its applications. In [12], Höhle and Šostak studied the convergence of fuzzy topological spaces using neighborhood systems of a point. The aim of this paper, to introduce and study the concept of fuzzy soft filters and show some of its properties. Also, studying the convergence of that fuzzy soft filter in fuzzy soft topological spaces. The existence of product fuzzy soft filter is also proved.

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X, I^X is the set of all fuzzy sets on X (where $I = [0, 1]$) and $I_0 = (0, 1]$, and for any $\alpha \in I_0$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$, $\widetilde{(X, E)}$ denotes the collection of all fuzzy soft sets on X. A fuzzy soft set $f_A \in \widetilde{(X, E)}$ is called a fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in X i.e. there exists $x \in X$ such that $f_A(e)(x) = t$, $t \in I_0$ and $f_A(e)(y) = 0\ \forall y \neq x$, it is denoted by e^t_x, and the set of all fuzzy soft points in X is denoted by $P_t\widetilde{(X, E)}$. Let $f_A, g_B \in \widetilde{(X, E)}$. Then, f_A is said to be fuzzy soft quasi-coincident with g_B, denoted by $f_A\ q\ g_B$ if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$. If f_A is not fuzzy soft quasi-coincident with g_B, then it is denoted by $f_A\ \hat{q}\ g_B$. Most of definitions and properties of fuzzy soft sets on a set X are found in [1, 2, 3, 6, 7, 11, 14, 16, 17].

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Definition 1.1. ([1]) A mapping $\tau : E \to I^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X if it satisfies the following conditions for each $e \in E$:

(O1) $\tau_e(\tilde{\Phi}) = \tau_e(\tilde{E}) = 1$,

(O2) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$ for all $f_A, g_B \in \widetilde{(X,E)}$,

(O3) $\tau_e(\bigsqcup_{j\in J}(f_{A_j})) \geq \bigwedge_{j\in J} \tau_e((f_{A_j}))$ for all $(f_{A_j}) \in \widetilde{(X,E)},\ j \in J$.

(Note that: $\sqcap$ and $\sqcup$ in the definition is explained in [15]). The pair (X, τ) is called a fuzzy soft topological space. The value $\tau_e(f_A)$ is interpreted as the degree of openness of a fuzzy set f_A with respect to that parameter $e \in E$.

2. Fuzzy Soft Filters

Definition 2.1. A mapping $\mathcal{F} : E \to I^{\widetilde{(X,E)}}$ is called a fuzzy soft filter on X if it satisfies, for all $e \in E$, the following conditions:

(F1) $\mathcal{F}_e(\tilde{\Phi}) = 0$ and $\mathcal{F}_e(\tilde{E}) = 1$,

(F2) $\mathcal{F}_e(f_A \sqcap g_B) \geq \mathcal{F}_e(f_A) \wedge \mathcal{F}_e(g_B)$ for all $f_A, g_B \in \widetilde{(X,E)}$,

(F3) If $f_A \sqsubseteq g_B$, then $\mathcal{F}_e(f_A) \leq \mathcal{F}_e(g_B)$.

If $\mathcal{F}^1_E$ and $\mathcal{F}^2_E$ are fuzzy soft filters on X, then $\mathcal{F}^1_E$ is finer than $\mathcal{F}^2_E$ (or $\mathcal{F}^2_E$ is coarser than $\mathcal{F}^1_E$), denoted by $\mathcal{F}^2_E \sqsubseteq \mathcal{F}^1_E$ if and only if $\mathcal{F}^2_e(f_A) \leq \mathcal{F}^1_e(f_A)$ for all $e \in E,\ f_A \in \widetilde{(X,E)}$.

Let us define $\mathcal{F}^\circ$ as follows:

$$\mathcal{F}^\circ_E = \{f_A \in \widetilde{(X,E)} : \mathcal{F}_e(f_A) > 0\}.$$

The main properties of fuzzy soft filters are discussed in the following propositions:

Proposition 2.1. *Let $\{\mathcal{F}^j_{E_j},\ j \in J\}$ be a family of fuzzy soft filters on a set X. Then, the mapping $\mathcal{F} = \sqcap_{j\in J}\mathcal{F}^j_{E_j} : E \to I^{\widetilde{(X,E)}}$ defined, for each $e \in E,\ f_A \in \widetilde{(X,E)}$, by:*

$$\mathcal{F}_e(f_A) = \bigwedge_{j\in J} (\mathcal{F}^j)_e(f_A)$$

is a fuzzy soft filter on X.

Proof. (F1) $\mathcal{F}_e(\tilde{\Phi}) = \bigwedge_{j\in J}(\mathcal{F}^j)_e(\tilde{\Phi}) = 0$ and $\mathcal{F}_e(\tilde{E}) = \bigwedge_{j\in J}(\mathcal{F}^j)_e(\tilde{E}) = 1$.

(F2) For all $f_A, g_B \in \widetilde{(X,E)}$, we have

$$\begin{aligned}
\mathcal{F}_e(f_A) \wedge \mathcal{F}_e(g_B) &= \bigwedge_{j\in J}(\mathcal{F}^j)_e(f_A) \wedge \bigwedge_{j\in J}(\mathcal{F}^j)_e(g_B) \\
&\leq \bigwedge_{j\in J} ((\mathcal{F}^j)_e(f_A) \wedge (\mathcal{F}^j)_e(g_B)) \\
&\leq \bigwedge_{j\in J}(\mathcal{F}^j)_e(f_A \sqcap g_B) \\
&= \mathcal{F}_e(f_A \sqcap g_B).
\end{aligned}$$

(F3) If $f_A \sqsubseteq g_B$, then $(\mathcal{F}^j)_e(f_A) \leq (\mathcal{F}^j)_e(g_B)$ for each $j \in J$, $e \in E$, and therefore

$$\mathcal{F}_e(f_A) = \bigwedge_{j\in J}(\mathcal{F}^j)_e(f_A) \leq \bigwedge_{j\in J}(\mathcal{F}^j)_e(g_B) = \mathcal{F}_e(g_A).$$

□

From a fuzzy soft filter $\mathcal{F} : E \to I^{\widetilde{(X,E)}}$, we can obtain a fuzzy soft topology $\tau_{\mathcal{F}}$ on X as follows:

Proposition 2.2. *Let $\mathcal{F}_E$ be a fuzzy soft filter on X and a map $\tau_{\mathcal{F}} : E \to I^{\widetilde{(X,E)}}$ defined by*

$$(\tau_{\mathcal{F}})_e(f_A) = \begin{cases} \mathcal{F}_e(f_A) & \text{if } f_A \neq \tilde{\Phi}, \\ 1 & \text{if } f_A = \tilde{\Phi}. \end{cases}$$

Then, $(X, (\tau_{\mathcal{F}})_E)$ is a fuzzy soft topological space.

Proof. It is straightforward and thus, it is omitted. □

Consider the mapping $\phi : X \to Y$ between two sets, and the mapping $\psi : E \to F$ between two sets of parameters.

Proposition 2.3. *Let $\phi_\psi : \widetilde{(X,E)} \to \widetilde{(Y,F)}$ be a mapping and $\mathcal{F}_E$ a fuzzy soft filter on X. Then, we can define the mapping $\phi_\psi(\mathcal{F}_E) : F \to I^{\widetilde{(Y,F)}}$ by*

$$\phi_\psi(\mathcal{F}_e)(g_B) = \mathcal{F}_e(\phi_\psi^{-1}(g_B)),\ \forall e \in E,\ \forall g_B \in \widetilde{(Y,F)}$$

so that $\phi_\psi(\mathcal{F}_E)$ is a fuzzy soft filter on Y.

Proof. (F1) $\phi_\psi(\mathcal{F}_e)(\tilde{\Phi}) = 0$ and $\phi_\psi(\mathcal{F}_e)(\tilde{F}) = 1$.
(F2) For all $f_A, g_B \in \widetilde{(Y,F)}$, we have

$$\begin{aligned} \phi_\psi(\mathcal{F}_e)(f_A) \wedge \phi_\psi(\mathcal{F}_e)(g_B) &= \mathcal{F}_e(\phi_\psi^{-1}(f_A)) \wedge \mathcal{F}_e(\phi_\psi^{-1}(g_B)) \\ &\leq \mathcal{F}_e(\phi_\psi^{-1}(f_A) \sqcap \phi_\psi^{-1}(g_B)) \\ &= \mathcal{F}_e(\phi_\psi^{-1}(f_A \sqcap g_B)) \\ &= \phi_\psi(\mathcal{F}_e)(f_A \sqcap g_B). \end{aligned}$$

(F3) If $f_A \sqsubseteq g_B$, then

$$\phi_\psi(\mathcal{F}_e)(f_A) = \mathcal{F}_e(\phi_\psi^{-1}(f_A)) \leq \mathcal{F}_e(\phi_\psi^{-1}(g_B)) = \phi_\psi(\mathcal{F}_e)(g_B).$$

□

Let $\mathcal{F}_E^1$ and $\mathcal{F}_F^2$ be two fuzzy soft filters on X and Y respectively, and $\phi_\psi : \widetilde{(X,E)} \to \widetilde{(Y,F)}$ a mapping. Then, ϕ_ψ is said to be a fuzzy soft filter mapping if and only if

$$\mathcal{F}^2_{\psi(e)}(g_B) \leq \mathcal{F}^1_e(\phi_\psi^{-1}(g_B))\ \forall g_B \in \widetilde{(Y,F)},\ e \in E.$$

In this way, we obtain the category **FSFIL** with objects all the fuzzy soft filters $(X, \mathcal{F}_E)$ and arrows all the fuzzy soft filter mappings ϕ_ψ as given above. Furthermore, if $\phi_\psi : (X, \mathcal{F}_E^1) \to (Y, \mathcal{F}_F^2)$ the fuzzy soft filter mapping is an arrow in **FSFIL**, then the same mapping ϕ_ψ is also an arrow in **FSTOP** with objects all the fuzzy topological spaces (X, τ) and arrows all the fuzzy continuous mappings and the diagram

$$\begin{array}{ccc} (X, \mathcal{F}^1_E) & \longrightarrow & (X, (\tau_{\mathcal{F}^1})_E) \\ \big\downarrow & & \big\downarrow \\ (Y, \mathcal{F}^2_F) & \longrightarrow & (Y, (\tau_{\mathcal{F}^2})_F) \end{array}$$

is commutative.

Theorem 2.1. *Let $\{\mathcal{F}^j_E,\ j \in J\}$ be a family of fuzzy soft filters on X satisfying the following condition:*

(C) *If $(f_A)_j \in (\mathcal{F}^j_E)^\circ$ for all $j \in J$, then we have $\sqcap_{j\in J_0}(f_A)_j \neq \tilde{\Phi}$ for every finite subset J_0 of J.*

If we defined a mapping $\bigsqcup_{j\in J} \mathcal{F}^j : E \to I^{\widetilde{(X,E)}}$ as follows:

$$(\bigsqcup_{j\in J}(\mathcal{F}^j))_e\,(g_B) = \begin{cases} \bigvee\{\bigwedge_{j\in J_0} (\mathcal{F}^j)_e((g_B)_j) : g_B = \sqcap_{j\in J_0}(g_B)_j\} & \text{if } (g_B)_j \in (\mathcal{F}^j_E)^\circ,\ e \in E, \\ 0 & \text{otherwise,} \end{cases}$$

where the supremum $\bigvee$ is taken for every finite index subset J_0 of J such that $g_B = \sqcap_{j\in J_0}(g_B)_j$. Then $\bigsqcup_{j\in J}(\mathcal{F}^j)_E$ is the coarsest fuzzy soft filter finer than $(\mathcal{F}^j)_E$ for each $j \in J$.

Proof. Firstly; we will show that $\mathcal{H}_E = \bigsqcup_{j\in J}(\mathcal{F}^j)_E$ is a fuzzy soft filter on X.

(F1) It is clear that $\mathcal{H}_e(\tilde{\Phi}) = 0$ and $\mathcal{H}_e(\tilde{E}) = 1$ for each $e \in E$.

(F2) For every finite index subsets K and L of J such that $f_A = \sqcap_{i\in K}(f_A)_i$ and $g_B = \sqcap_{l\in L}(g_B)_l$, we have

$$f_A \sqcap g_B = (\sqcap_{i\in K}(f_A)_i) \sqcap (\sqcap_{l\in L}(g_B)_l).$$

Furthermore, for each $m \in K \cap L$, put $f_A \sqcap g_B = \sqcap_{m\in K\cup L}(h_C)_m$, $C = A \cap B$, where

$$(h_C)_m = \begin{cases} (f_A)_m & \text{if } m \in K - (K\cap L), \\ (g_B)_m & \text{if } m \in L - (K\cap L), \\ (f_A)_m \sqcap (g_B)_m & \text{if } m \in (K\cap L), \end{cases}$$

which means that

$$\begin{aligned} \mathcal{H}_e(f_A \sqcap g_B) &\geq \bigwedge_{m\in K\cap L} (\mathcal{F}^m_e)((h_C)_m) \\ &\geq \bigwedge_{i\in K} (\mathcal{F}^i_e)((f_A)_i) \wedge \bigwedge_{m\in L} (\mathcal{F}^l_e)((g_B)_l). \end{aligned}$$

Then, $\mathcal{H}_e(f_A \sqcap g_B) \geq \mathcal{H}_e(f_A) \wedge \mathcal{H}_e(g_B)$.

(F3) Suppose that $f_A \sqsubseteq g_B$, by the definition of $\mathcal{H}$, there exists a finite index set K with $f_A = \sqcap_{i\in K}(f_A)_i$ so that $\mathcal{H}_e(f_A) \geq \bigwedge_{i\in K} (\mathcal{F}^i_e)((f_A)_i)$. On the other hand, since $g_B = f_A \sqcup g_B = \sqcap_{i\in K}((f_A)_i \sqcup g_B)$, then we have $\mathcal{H}_e(g_B) \geq \bigwedge_{i\in K} (\mathcal{F}_e^i)((f_A)_i \sqcup g_B) \geq \bigwedge_{i\in K} (\mathcal{F}_e^i)((f_A)_i)$. Thus, $\mathcal{H}_e(g_B) \geq \mathcal{H}_e(f_A)$. Now, we will show that $\mathcal{H}_e(f_A) \geq \mathcal{F}_e^j(f_A)$ for each $j \in J$ from the following:

If $\mathcal{F}_e^j(f_A) = 0$, then it is trivial.

If $\mathcal{F}_e^j(f_A) > 0$, then for $f_A = f_A \sqcap \tilde{E}$, we have $\mathcal{H}_e(f_A) \geq \mathcal{F}_e^j(f_A) \wedge \mathcal{F}_e^j(\tilde{E}) = \mathcal{F}_e^j(f_A)$.

If $\mathcal{K}_E \sqsupseteq \mathcal{F}_E^j$ for each $j \in J$, we will show that $\mathcal{K}_E \sqsupseteq \mathcal{H}_E$. By the definition of $\mathcal{H}$, there exists a finite index set K with $f_A = \sqcap_{i\in K}(f_A)_i$ so that $\mathcal{H}_e(f_A) \geq \bigwedge_{i\in K} (\mathcal{F}_e^i)((f_A)_i)$. On the other hand, since $\mathcal{K}_E \sqsupseteq \mathcal{F}_E^i$ for each $i \in K$, then we have $\mathcal{K}_e(f_A) \geq \bigwedge_{i\in K} \mathcal{K}_e((f_A)_i) \geq \bigwedge_{i\in K} (\mathcal{F}_e^i)((f_A)_i)$. Hence, $\mathcal{K}_e(f_A) \geq \mathcal{H}_e(f_A)$. □

Theorem 2.2. *Let $\phi_\psi : \widetilde{(X,E)} \to \widetilde{(Y,F)}$ be a mapping and $\mathcal{F}$ a fuzzy soft filter on Y. Then, we can define the mapping $\phi_\psi^{-1}(\mathcal{F}_F) : E \to I^{\widetilde{(X,E)}}$ for each $k \in F$ by:*

$$\phi_\psi^{-1}(\mathcal{F}_k)\,(f_A) = \begin{cases} \bigvee\{\mathcal{F}_k(g_B) : f_A = \phi_\psi^{-1}(g_B)\} & \text{if } g_B \neq \tilde{\Phi}, \\ 0 & \text{if } g_B = \tilde{\Phi}. \end{cases}$$

so that $\phi_\psi^{-1}(\mathcal{F})$ is a fuzzy soft filter on X.

Proof. (F1) $\phi_\psi^{-1}(\mathcal{F}_k)\,(\tilde{\Phi}) = 0$ and $\phi_\psi^{-1}(\mathcal{F}_k)\,(\tilde{E}) = 1$.

(F2) It is proved from that:

$$\begin{aligned} \phi_\psi^{-1}(\mathcal{F}_k)\,(f_A) \wedge \phi_\psi^{-1}(\mathcal{F}_k)\,(g_B) &= (\bigvee\{\mathcal{F}_k(o_D) : f_A = \phi_\psi^{-1}(o_D)\}) \wedge \\ &\quad (\bigvee\{\mathcal{F}_k(h_C) : g_B = \phi_\psi^{-1}(h_C)\}) \\ &= \bigvee\{\mathcal{F}_k(o_D) \wedge \mathcal{F}_k(h_C) : f_A \sqcap g_B = \phi_\psi^{-1}(o_D) \sqcap \\ &\quad \phi_\psi^{-1}(h_C)\} \\ &\leq \bigvee\{\mathcal{F}_k(o_D \sqcap (h_C) : f_A \sqcap g_B = \phi_\psi^{-1}(o_D) \sqcap \\ &\quad \phi_\psi^{-1}(h_C)\} \\ &\leq \phi_\psi^{-1}(\mathcal{F}_k)\,(f_A \sqcap g_B). \end{aligned}$$

(F3) If $f_A \sqsubseteq g_B$, then

$$\phi_\psi^{-1}(\mathcal{F}_k)\,(f_A) = \mathcal{F}_k(\phi_\psi(f_A)) \leq \mathcal{F}_k(\phi_\psi(g_B)) = \phi_\psi^{-1}(\mathcal{F}_k)\,(g_B).$$

□

Theorem 2.3. *Let $\{\mathcal{F}_{E_j}^j,\ j \in J\}$ be a family of fuzzy soft filters on X_j, E_j are parameter sets, and $E = \prod_{j\in J} E_j$. Let $X = \prod_{j\in J} X_j$ be the product space, $\pi_j : X \to X_j$, $\psi_j : E \to E_j$ are the projection maps for each $j \in J$ and $(\pi_\psi)_j : \widetilde{(X,E)} \to \widetilde{(X_j,E_j)}$. Then, we can define a mapping $\mathcal{F} : E \to I^{\widetilde{(X,E)}}$ by:*

$$\mathcal{F}_e(f_A) = \begin{cases} \bigvee\{\bigwedge_{j\in K} (\mathcal{F}^j)_e((g_B)_j) : f_A = \sqcap_{j\in K}((\pi_\psi^{-1})_j(g_B)_j)\} & \text{if } (g_B)_j \in (\mathcal{F}_{E_j}^j)^\circ,\ e \in E, \\ 0 & \text{otherwise}, \end{cases}$$

where the supremum $\bigvee$ is taken for every finite index subset K of J such that $f_A = \sqcap_{j\in K}(\pi_\psi^{-1})_j((g_B)_j)$. Then

(1) *For $f_A \in \widetilde{(X,E)}$, we have $\mathcal{F}_e(f_A) = \bigvee_{j\in J}(\pi_\psi^{-1})_j(\mathcal{F}_{e_j}^j)\,(f_A)$.*

(2) *$\mathcal{F}$ is the coarsest fuzzy soft filter on X for which each projection map $(\pi_\psi)_j : \widetilde{(X,E)} \to \widetilde{(X_j, E_j)}$ is a fuzzy soft filter mapping.*

(3) *A map $\kappa_\psi : (Y, \mathcal{H}_F) \to (X, \mathcal{F}_E)$ is a fuzzy soft filter map if and only if for each $j \in J$, we have $(\pi_\psi)_j \circ \kappa_\psi : (Y, \mathcal{H}_F) \to (X_j, \mathcal{F}^j_{E_j})$ is a fuzzy soft filter map.*

Proof. (1) From Theorem 2.2, each $(\pi_\psi^{-1})_j(\mathcal{F}^j_{E_j})$ is a fuzzy soft filter on X_j. Firstly, we will show that $\bigvee_{j\in J}(\pi_\psi^{-1})_j(\mathcal{F}^j_{E_j})$ exists, that is, it satisfies the condition (C) of Theorem 2.1.

(C) If $(f_A)_j \in (\pi_\psi^{-1})_j(\mathcal{F}^j_{E_j})^\circ$ for all $j \in J$, there exists $(g_B)_j \in \widetilde{(Y,F)}$ with $(f_A)_j = (\pi_\psi^{-1})_j((g_B)_j)$ such that $(\mathcal{F}^j)_{e_j}((g_B)_j) > 0$. It implies that $(g_B)_j \neq \tilde{\Phi}$, that is, there exists $x_j \in X_j$ with $(g_B)_j(x_j) > 0$. For every finite index subset K of J, put

$$x = \begin{cases} \pi_i^{-1}(x_i) & \text{if } x_i \in X_i \text{ for each } i \in K, \\ \pi_j^{-1}(x_j) & \text{if } x_j \in X_j \text{ for each } j \in J - K. \end{cases}$$

Then, we have

$$\bigwedge_{j\in K}(f_e)_j(x) = \bigwedge_{j\in K}(\pi_\psi^{-1})_j(g_{\psi(e)})_j(x) = \bigwedge_{j\in K}(g_{\psi(e)})_j(x_j) > 0.$$

We will show that $\mathcal{F} = \bigvee_{j\in J}(\pi_\psi^{-1})_j(\mathcal{F}^j)$. By the definition of $\mathcal{F}_{E_i}$, there exists a finite index set $K \subseteq J$ with $f_A = \sqcap_{i\in K}(\pi_\psi^{-1})_i((g_B)_i)$ such that $\mathcal{F}_e(f_A) \geq \bigwedge_{i\in K}(\mathcal{F}^i_{\psi(e)})((g_{\psi(e)})_i)$, putting $(f_A)_i = (\pi_\psi^{-1})_i((g_B)_i)$ for each $i \in K$, then, for

$$f_A = \sqcap_{i\in K}(f_A)_i = \sqcap_{i\in K}(\pi_\psi^{-1})_i(\mathcal{F}^i_{\psi(e)})((g_B)_i),$$

we have $\bigvee_{j\in J}(\pi_\psi^{-1})_j(\mathcal{F}^j)_{e_j}(g_B) \geq \bigwedge_{i\in K}(\pi_\psi^{-1})_i(\mathcal{F}^i)_{e_i}((f_A)_i)$. Hence, $\bigvee_{j\in J}(\pi_\psi^{-1})_j(\mathcal{F}^j)_{E_j} \sqsupseteq \mathcal{F}_E$.

For every finite index set $L \subseteq J$ with $h_C = \sqcap_{l\in L}(f_A)_l$, we have $\bigvee_{j\in J}(\pi_\psi^{-1})_j(\mathcal{F}^j)_{e_j}(h_C) \geq \bigwedge_{l\in L}(\pi_\psi^{-1})_l(\mathcal{F}^l)_{e_l}((f_A)_l)$, and there exists $(g_B)_l \in \widetilde{(X_l, E_l)}$ with $(f_A)_l = (\pi_\psi^{-1})_l((g_B)_l)$ such that $\bigwedge_{l\in L}(\pi_\psi^{-1})_l(\mathcal{F}^l)_{e_l}((f_A)_l) \geq \bigwedge_{l\in L}(\mathcal{F}^l)_{\psi(e)}((g_B)_l)$. On the other hand, for $h_C = \sqcap_{l\in L}(f_A)_l = \sqcap_{l\in L}(\pi_\psi^{-1})_l((g_B)_l)$, $\mathcal{F}_e(h_C) \geq \bigwedge_{l\in L}(\mathcal{F}^l)_{\psi(e)}((g_B)_l)$. Then $(\pi_\psi^{-1})_j(\mathcal{F}^j)_{E_j} \sqsubseteq \mathcal{F}_E$, and thus, $(\pi_\psi^{-1})_j(\mathcal{F}^j)_{E_j} = \mathcal{F}_E$.

(2) From (1) above, Theorem 2.1 and Theorem 2.2, we get that $\mathcal{F}_E$ is a fuzzy soft filter on X. For each $j \in J$, and $(g_B)_j \in \widetilde{(X_j, E_j)}$, and by the definition of $\mathcal{F}_E$, we then have $\mathcal{F}_e((\pi_\psi^{-1})_j((g_B)_j)) \geq \mathcal{F}^j_{\psi(e)}((g_B)_j)$. Hence, $(\pi_\psi)_j : (X, \mathcal{F}_E) \to (X_j, \mathcal{F}_{E_j})$ is a fuzzy soft filter mapping.

Let $(\pi_\psi)_j : (X, \mathcal{G}_E) \to (X_j, \mathcal{F}_{E_j})$ is a fuzzy soft filter mapping for each $j \in J$, that is, $\mathcal{G}_e((\pi_\psi^{-1})_i((g_B)_j)) \geq \mathcal{F}^j_{\psi(e)}((g_B)_j)$ for all finite index set K with $f_A = \sqcap_{i\in K}(\pi_\psi^{-1})_i((g_B)_i)$, and thus,

$$\mathcal{G}_e(f_A) \geq \bigwedge_{i\in K}\mathcal{G}_e((\pi_\psi^{-1})_i((g_B)_i)) \geq \bigwedge_{i\in K}(\mathcal{F}^i_{\psi(e)})((g_B)_j)),$$

which implies that $\mathcal{G}_e(f_A) \geq \mathcal{F}_e(f_A)$ for each $f_A \in \widetilde{(X,E)}$.

(3) Necessity of the composition condition is clear since the composition of fuzzy soft filter mappings is a fuzzy soft filter mapping. Conversely, suppose $\kappa_\psi : (Y, \mathcal{H}_F) \to (X, \mathcal{F}_E)$ is just a fuzzy soft map. For every finite index set K with $f_A = \sqcap_{i\in K}(\pi_\psi^{-1})_i((g_B)_i)$. Since for each $j \in J$, we have $((\pi_\psi)_j \circ \kappa_\psi) : (Y, \mathcal{H}_F) \to (X_j, \mathcal{F}^j_{E_j})$

is a fuzzy soft filter map, and $\mathcal{F}^j_{\psi(e)}((g_B)_j) \le \mathcal{H}_e(\kappa^{-1}_\psi((\pi^{-1}_\psi)_j((g_B)_j)))$. It follows that $\mathcal{H}_e(\kappa^{-1}_\psi((\pi^{-1}_\psi)_i((g_B)_i))) \ge \mathcal{F}_{\psi(e)}((g_B)_i)$. Hence, we have

$$\mathcal{H}_e((\kappa^{-1}_\psi)(f_A)) \ge \bigwedge_{i\in K} \mathcal{H}_e((\kappa^{-1}_\psi)((\pi^{-1}_\psi)_i((g_B)_i))) \ge \bigwedge_{i\in K} (\mathcal{F}^i_{\psi(e)})((g_B)_i).$$

It implies $\mathcal{H}_e((\kappa^{-1}_\psi)(f_A)) \ge \mathcal{F}_e(f_A)$ for all $f_A \in \widetilde{(X,E)}$. Hence, $\kappa_\psi : (Y,\mathcal{H}_F) \to (X,\mathcal{F}_E)$ is a fuzzy soft filter map. □

Definition 2.2. Let $\{\mathcal{F}^j_{E_j},\ j \in J\}$ be a family of fuzzy soft filters on $X_j,\ j \in J$, and $X = \prod_{j\in J} X_j,\ E = \prod_{j\in J} E_j$ are product sets, $\pi_j : X \to X_j,\ \psi_j : E \to E_j$ are the projection mappings. The product of fuzzy soft filters is the coarsest fuzzy soft filter on X for which all $(\pi_\psi)_j : (X,\mathcal{F}_E) \to (X_j,\mathcal{F}^j_{E_j}),\ j \in J$ are fuzzy soft filter mappings.

3. Fuzzy Soft Quasi-Coincident Neighborhood Spaces

In this section, we introduce the notion of fuzzy soft quasi-coincident neighborhood spaces.

Definition 3.1. A fuzzy soft quasi-coincident neighborhood system on X is a set $\mathcal{Q} = \{\mathcal{Q}_{e^t_x} : e^t_x \in P_t\widetilde{(X,E)}\}$ of maps $\mathcal{Q}_{e^t_x} : E \to I^{\widetilde{(X,E)}}$ such that for each $f_A, g_B \in \widetilde{(X,E)}$, we have

(N1) $\mathcal{Q}_{e^t_x}$ is fuzzy soft filter on X,

(N2) $(\mathcal{Q}_{e^t_x})_e\,(f_A) > 0$ implies that $e^t_x\ q\ f_A$,

(N3) $(\mathcal{Q}_{e^t_x})_e\,(f_A) = \bigvee_{e^t_x\ q\ g_B,\ g_B \sqsubseteq f_A} [\bigwedge_{e^t_y\ q\ g_B} (\mathcal{Q}_{e^t_y})_e\,(g_B)]$.

The pair $(X,\mathcal{Q}_E)$ is called a fuzzy soft quasi-coincident neighborhood space. $(\mathcal{Q}_{e^t_x})_e\,(f_A)$ can be interpreted as the degree to which f_A is a quasi-coincident neighborhood of e^t_x.

An N-map between fuzzy soft quasi-coincident neighborhood spaces $(X,\mathcal{Q}^1_E)$ and $(Y,\mathcal{Q}^2_F)$ is a map ϕ_ψ : $(X,\mathcal{Q}^1_E) \to (Y,\mathcal{Q}^2_F)$ such that

$$(\mathcal{Q}^1_{e^t_x})_e\,(\phi^{-1}_\psi(f_A)) \ \ge\ (\mathcal{Q}^2_{\psi(e)^t_{\phi(x)}})_{\psi(e)}\,(f_A)$$

for all $f_A \in \widetilde{(Y,F)},\ e \in E$ and for all $e^t_x \in P_t\widetilde{(X,E)}$.

Theorem 3.1. *Let (X,τ_E) be a fuzzy soft topological space and $e^t_x \in P_t\widetilde{(X,E)}$. Define a map $\mathcal{Q}^\tau_{e^t_x} : E \to I^{\widetilde{(X,E)}}$ as:*

$$(\mathcal{Q}^\tau_{e^t_x})_e\,(f_A) = \begin{cases} \bigvee\{\tau_e(g_B) : e^t_x\ q\ g_B,\ g_B \sqsubseteq f_A\} & \text{if } e^t_x\ q\ f_A, \\ 0 & \text{if } e^t_x\ \hat{q}\ f_A. \end{cases}$$

Then,

(1) $\mathcal{Q}^\tau = \{\mathcal{Q}^\tau_{e^t_x} : e^t_x \in P_t\widetilde{(X,E)}\}$ *is a fuzzy soft quasi-coincident neighborhood system on X,*

(2) *If $t < s$ for $t,s \in I$, then $(\mathcal{Q}^\tau_{e^t_x})_e\,(f_A) \le (\mathcal{Q}^\tau_{e^s_x})_e\,(f_A)$.*

Proof. To prove (1), we need to prove all conditions (N1) - (N3).

Firstly, we prove (N1). (F1) and (F3) are easily proved.

For (F2), suppose there exist $f_A, g_B \in \widetilde{(X,E)}$ such that

$$(\mathcal{Q}^\tau_{e^t_x})_e\,(f_A \sqcap g_B) \ \not\ge\ (\mathcal{Q}^\tau_{e^t_x})_e\,(f_A) \ \wedge\ (\mathcal{Q}^\tau_{e^t_x})_e\,(g_B).$$

By the definition of $(Q^{\tau}_{e^t_x})_e(f_A)$, there exists $(f_A)_1 \in \widetilde{(X,E)}$ with $e^t_x\, q\, (f_A)_1,\ (f_A)_1 \sqsubseteq f_A$ such that

$$(Q^{\tau}_{e^t_x})_e(f_A \sqcap g_B) \ \not\geq\ \tau_e((f_A)_1) \wedge (Q^{\tau}_{e^t_x})_e(g_B).$$

Again by the definition of $(Q^{\tau}_{e^t_x})_e(g_B)$, there exists $(g_B)_1 \in \widetilde{(X,E)}$ with $e^t_x\, q\, (g_B)_1,\ (g_B)_1 \sqsubseteq g_B$ such that

$$(Q^{\tau}_{e^t_x})_e(f_A \sqcap g_B) \ \not\geq\ \tau_e((f_A)_1) \wedge \tau_e((g_B)_1).$$

Since $e^t_x\, q\, ((f_A)_1 \sqcap (g_B)_1),\ ((f_A)_1 \sqcap (g_B)_1) \sqsubseteq f_A \sqcap g_B$, we have

$$(Q^{\tau}_{e^t_x})_e(f_A \sqcap g_B) \ \geq\ \tau_e((f_A)_1 \sqcap (g_B)_1) \ \geq\ \tau_e((f_A)_1) \wedge \tau_e((g_B)_1).$$

It is a contradiction. Hence, (F2) holds and thus, $Q_{e^t_x}$ is a fuzzy soft filter on X.

For (N2), it is easy from the definition.

Now (N3), for all $f_A \in \widetilde{(X,E)}$ with $e^t_x\, q\, g_B,\ g_B \sqsubseteq f_A$, we have

$$\tau_e(g_B) \ \leq\ \bigwedge\{Q^{\tau}_{e^s_y}(g_B) : e^s_y\, q\, g_B\} \ \leq\ (Q^{\tau}_{e^t_x})_e(g_B) \ \leq\ (Q^{\tau}_{e^t_x})_e(f_A).$$

Therefore,

$$\begin{aligned}(Q^{\tau}_{e^t_x})_e(f_A) &= \bigvee_{e^t_x\, q\, g_B,\ g_B \sqsubseteq f_A} \tau_e(g_B)\\ &\leq \bigvee_{e^t_x\, q\, g_B,\ g_B \sqsubseteq f_A} \Big[\bigwedge_{e^s_y\, q\, g_B} (Q^{\tau}_{e^s_y})_e(g_B)\Big]\\ &\leq (Q^{\tau}_{e^t_x})_e(f_A).\end{aligned}$$

This means that $(Q^{\tau}_{e^t_x})_e(f_A) = \bigvee_{e^t_x\, q\, g_B,\ g_B \sqsubseteq f_A} \big[\bigwedge_{e^s_y\, q\, g_B} (Q^{\tau}_{e^s_y})_e(g_B)\big]$. Hence, (1) is fulfilled.

Now (2), for $t < s$ with $t, s \in I$ and $f_A \in \widetilde{(X,E)}$ since

$$\{g_B \in \widetilde{(X,E)} : e^t_x\, q\, g_B,\ g_B \sqsubseteq f_A\} \sqsubseteq \{h_C \in \widetilde{(X,E)} : e^s_x\, q\, h_C,\ h_C \sqsubseteq f_A\},$$

then we have $(Q^{\tau}_{e^t_x})_e(f_A) \ \leq\ (Q^{\tau}_{e^s_x})_e(f_A)$. □

Theorem 3.2. *Let* $Q = \{Q_{e^t_x} : e^t_x \in P_t\widetilde{(X,E)}\}$ *be a family of* $Q_{e^t_x} : E \to I^{\widetilde{(X,E)}}$ *satisfying* (N1) *and* (N2) *of Definition 3.1. We define a map* $\tau^Q : E \to I^{\widetilde{(X,E)}}$ *as follows:*

$$(\tau^Q)_e(f_A) = \begin{cases} \bigwedge\{(Q_{e^t_x})_e(f_A) : e^t_x\, q\, f_A\} & \text{if } f_A \neq \tilde{\Phi},\\ 1 & \text{if } f_A = \tilde{\Phi}.\end{cases}$$

Then, we have the following properties:

(1) $(\tau^Q)_E$ *is a fuzzy soft topology on* X*,*

(2) *If* Q_E *is a fuzzy soft quasi-coincident neighborhood system on* X*, then* $Q^{\tau^Q}_{e^t_x} = Q_{e^t_x}$ *for all* $e^t_x \in P_t\widetilde{(X,E)}$*,*

(3) *If* Q^1_E *and* Q^2_E *are fuzzy soft quasi-coincident neighborhood systems on* X *such that* $(\tau^{Q^1})_E = (\tau^{Q^2})_E$ *then* $Q^1_E = Q^2_E$*.*

Proof. To prove (1), we need to prove all conditions (O1)–(O3).

(O1) is trivial.

(O2) For $f_A, g_B \in \widetilde{(X,E)}$ we have

$$\begin{aligned}
(\tau^Q)_e\,(f_A \sqcap g_B) &= \bigwedge\{(Q_{e_x^t})_e\,(f_A \sqcap g_B)\ :\ e_x^t\, q\, (f_A \sqcap g_B)\} \\
&\geq \bigwedge\{(Q_{e_x^t})_e\,(f_A)\ \wedge\ (Q_{e_x^t})_e\,(g_B)\ :\ e_x^t\, q\, (f_A \sqcap g_B)\} \\
&= (\bigwedge\{(Q_{e_x^t})_e\,(f_A)\ :\ e_x^t\, q\, (f_A \sqcap g_B)\})\ \wedge \\
&\quad\ (\bigwedge\{(Q_{e_x^t})_e\,(g_B)\ :\ e_x^t\, q\, (f_A \sqcap g_B)\}) \\
&\geq (\bigwedge\{(Q_{e_x^t})_e\,(f_A)\ :\ e_x^t\, q\, (f_A)\})\ \wedge \\
&\quad\ (\bigwedge\{(Q_{e_x^t})_e\,(g_B)\ :\ e_x^t\, q\, (g_B)\}) \\
&= (\tau^Q)_e\,(f_A)\ \wedge\ (\tau^Q)_e\,(g_B).
\end{aligned}$$

(O3) Since $(Q_{e_x^t})_e\,(\bigsqcup_{j\in J}(g_{B_j}))\ \geq\ \bigwedge_{j\in J}(Q_{e_x^t})_e\,((g_{B_j}))$, then

$$\begin{aligned}
(\tau^Q)_e\,(\bigsqcup_{j\in J}(g_{B_j})) &= \bigwedge\{(Q_{e_x^t})_e\,(\bigsqcup_{j\in J}(g_{B_j}))\ :\ e_x^t\, q\, (\bigsqcup_{j\in J}(g_{B_j}))\} \\
&\geq \bigwedge\{\bigwedge_{j\in J}(Q_{e_x^t})_e\,((g_{B_j}))\ :\ e_x^t\, q\, ((g_{B_j}))\} \\
&= \bigwedge_{j\in J}\{\bigwedge(Q_{e_x^t})_e\,((g_{B_j}))\ :\ e_x^t\, q\, ((g_{B_j}))\} \\
&= \bigwedge_{j\in J}(\tau^Q)_e\,((g_{B_j})).
\end{aligned}$$

Hence, $(\tau^Q)_E$ is a fuzzy soft topology on X.

For (2), it is proved by (N3) so that

$$\begin{aligned}
(Q^{\tau^Q}_{e_x^t})_e\,(f_A) &= \bigvee_{e_x^t\, q\, g_B,\ g_B \sqsubseteq f_A} (\tau^Q)_e\,(g_B) \\
&= \bigvee_{e_x^t\, q\, g_B,\ g_B \sqsubseteq f_A} \bigwedge\{(Q_{e_y^s})_e\,(g_B)\ :\ e_y^s\, q\, g_B\} \\
&= (Q_{e_x^t})_e\,(f_A).
\end{aligned}$$

For (3), since $(\tau^{Q^1})_E\ =\ (\tau^{Q^2})_E$, then for $f_A \in \widetilde{(X,E)}$ and $e_x^t \in P_t\widetilde{(X,E)}$, we have

$$\begin{aligned}
(Q^1_{e_x^t})_e\,(f_A) &= \bigvee_{e_x^t\, q\, g_B,\ g_B \sqsubseteq f_A} \bigwedge\{(Q^1_{e_y^s})_e\,(g_B)\ :\ e_y^s\, q\, g_B\} \\
&= \bigvee_{e_x^t\, q\, g_B,\ g_B \sqsubseteq f_A} (\tau^{Q^1})_e\,(g_B) \\
&= \bigvee_{e_x^t\, q\, g_B,\ g_B \sqsubseteq f_A} (\tau^{Q^2})_e\,(g_B) \\
&= \bigvee_{e_x^t\, q\, g_B,\ g_B \sqsubseteq f_A} \bigwedge\{(Q^2_{e_y^s})_e\,(g_B)\ :\ e_y^s\, q\, g_B\} \\
&= (Q^2_{e_x^t})_e\,(f_A).
\end{aligned}$$

Hence, $Q_E^1 = Q_E^2$. □

Theorem 3.3. *Let* (X, τ_E) *be a fuzzy soft topological space and* Q_E^τ *a fuzzy soft quasi-coincident neighborhood system on* (X, τ_E). *Then,* $\tau_E = (\tau^{Q^\tau})_E$.

Proof. Since

$$(Q^\tau_{e_x^t})_e\,(f_A) = \bigvee_{e_x^t\ q\ g_B,\ g_B \sqsubseteq f_A} \tau_e(g_B) \geq \tau_e(f_A)$$

for all $e \in E$, $e_x^t\ q\ f_A$, then we have

$$\bigwedge_{e_x^t\ q\ f_A} (Q_{e_x^t})_e\,(f_A) \geq \tau_e(f_A).$$

So, $\tau_E \sqsubseteq (\tau^{Q^\tau})_E$.

Conversely, suppose that there exists $f_A \in \widetilde{(X,E)}$, $e \in E$ such that

$$\tau_e(f_A) \ngeq (\tau^{Q^\tau})_e(f_A).$$

For each $e_x^t \in P_t\widetilde{(X,E)}$ with $e_x^t\ q\ f_A$. If $e_x^t\ q\ (g_B)_{e_x^t}$, $(g_B)_{e_x^t} \sqsubseteq f_A$, then $f_A = \bigsqcup_{e_x^t\ q\ f_A} (g_B)_{e_x^t}$. That is,

$$\tau_e(f_A) = \tau_e\Big(\bigsqcup_{e_x^t\ q\ f_A} (g_B)_{e_x^t}\Big) \geq \bigwedge_{e_x^t\ q\ f_A} \tau_e\,((g_B)_{e_x^t}),$$

which means that

$$\bigwedge_{e_x^t\ q\ f_A} \tau_e\,((g_B)_{e_x^t}) \ngeq \tau_e^{Q^\tau}\,(f_A) = \bigwedge_{e_x^t\ q\ f_A} (Q^\tau_{e_x^t})_e\,(f_A).$$

Hence, there exists $(g_B)_{e_x^t})$ with $e_x^t\ q\ (g_B)_{e_x^t}$, $(g_B)_{e_x^t} \sqsubseteq f_A$ such that

$$\tau_e((g_B)_{e_x^t}) \ngeq \bigwedge_{e_x^t\ q\ f_A} (Q^\tau_{e_x^t})_e\,(f_A).$$

It is a contradiction. Thus, $\tau_E \sqsupseteq (\tau^{Q^\tau})_E$. □

Theorem 3.4. *Let* (X, Q_E^1) *and* (Y, Q_F^2) *be two fuzzy soft quasi-coincident neighborhood spaces. A mapping* $\phi_\psi : (X, Q_E^1) \to (Y, Q_F^2)$ *is N-map iff* $\phi_\psi : (X, \tau_E^{Q^1}) \to (Y, \tau_F^{Q^2})$ *is fuzzy soft continuous.*

Proof. Since for all $f_A \in \widetilde{(Y,F)}$, $e_x^t \in P_t\widetilde{(X,E)}$

$$e_x^t\ q\ \phi_\psi^{-1}(f_A) \text{ iff } (\phi_\psi(e_x^t) = \psi(e)^t_{\phi(x)})\ q\ f_A \text{ and}$$

$$\{\psi(e)_y^t \in P_t\widetilde{(Y,F)} : \psi(e)_y^t\ q\ f_A\} \sqsupseteq \{\psi(e)^t_{\phi(x)} \in P_t\widetilde{(Y,F)} : e_x^t \in P_t\widetilde{(X,E)},\ \psi(e)^t_{\phi(x)}\ q\ f_A\}.$$

Then, we have

$$\begin{aligned}
\tau^{Q^2}_{\psi(e)}(f_A) &= \bigwedge\{(Q^2_{\psi(e)_y^t})_{\psi(e)}(f_A) : \psi(e)_y^t\ q\ f_A\} \\
&\leq \bigwedge\{(Q^2_{\psi(e)^t_{\phi(x)}})_{\psi(e)}(f_A) : \psi(e)^t_{\phi(x)}\ q\ f_A\} \\
&\leq \bigwedge\{(Q^1_{e_x^t})_e(\phi_\psi^{-1}(f_A)) : e_x^t\ q\ \phi_\psi^{-1}(f_A)\} \\
&= \tau_e^{Q^1}(\phi_\psi^{-1}(f_A)).
\end{aligned}$$

Thus, $\phi_\psi : (X, \tau_E^{Q^1}) \to (Y, \tau_F^{Q^2})$ is fuzzy soft continuous.

Conversely, since for all $f_A \in \widetilde{(Y,F)}$, $\tau^{Q^2}_{\psi(e)}(f_A) \leq \tau^{Q^1}_e(\phi_\psi^{-1}(f_A))$, $Q^1_E = Q_E^{\tau^{Q^1}}$ and $Q^2_F = Q_F^{\tau^{Q^2}}$, then we have

$$\begin{aligned}((Q^2)_{\psi(e)^t_{\phi(x)}})_{\psi(e)}(f_A) &= \bigvee\{\tau^{Q^2}_{\psi(e)}(g_B) : \psi(e)^t_{\phi(x)}\, q\, g_B, g_B \sqsubseteq f_A\}\\ &\leq \bigvee\{\tau^{Q^2}_{\psi(e)}(g_B) : e^t_x\, q\, \phi_\psi^{-1}(g_B), \phi_\psi^{-1}(g_B) \sqsubseteq \phi_\psi^{-1}(f_A)\}\\ &\leq \bigvee\{\tau^{Q^1}_e(\phi_\psi^{-1}(g_B)) : e^t_x\, q\, \phi_\psi^{-1}(g_B), \phi_\psi^{-1}(g_B) \sqsubseteq \phi_\psi^{-1}(f_A)\}\\ &\leq (Q^1_{e^t_x})_e(\phi_\psi^{-1}(f_A)).\end{aligned}$$

□

Corollary 3.1. *Let (X, τ^1_E) and (Y, τ^2_F) be two fuzzy topological spaces. A map $\phi_\psi : (X, \tau^1_E) \to (Y, \tau^2_F)$ is fuzzy soft continuous* iff *$\phi_\psi : (X, Q_E^{\tau^1}) \to (Y, Q_F^{\tau^2})$ is an N-map.*

4. Fuzzy Soft Filter Convergence

Definition 4.1. Let (X, τ_E) be a fuzzy soft topological space, $\mathcal{F}_E$ a fuzzy soft filter, $f_A, g_B \in \widetilde{(X,E)}$ and $e^t_x \in P_t\widetilde{(X,E)}$.

(1) e^t_x is called a fuzzy soft cluster point of $\mathcal{F}_E$, denoted by $\mathcal{F}_E \rightsquigarrow e^t_x$ if for every $g_B \in (Q_{e^t_x})^\circ_E$ and $f_A \in (\mathcal{F}_E)^\circ$, we have $f_A \sqcap g_B \neq \tilde{\Phi}$.

(2) e^t_x is called a fuzzy soft limit point of $\mathcal{F}_E$, denoted by $\mathcal{F}_E \rightarrowtail e^t_x$ if $(Q_{e^t_x})_E \sqsubseteq \mathcal{F}_E$.

We denote

$$\mathrm{cls}_{\tau_E}(\mathcal{F}_E) = \bigsqcup\{e^t_x \in P_t\widetilde{(X,E)} : e^t_x \text{ is a fuzzy soft cluster point of } \mathcal{F}_E\},$$

$$\mathrm{lim}_{\tau_E}(\mathcal{F}_E) = \bigsqcup\{e^t_x \in P_t\widetilde{(X,E)} : e^t_x \text{ is a fuzzy soft limit point of } \mathcal{F}_E\}.$$

Theorem 4.1. *Let (X, τ_E) be a fuzzy soft topological space and $\mathcal{F}_E, \mathcal{G}_E$ are two fuzzy soft filters on X such that $\mathcal{F}_E$ is coarser than $\mathcal{G}_E$. Then the following properties hold.*

(1) $\mathcal{F}_E \rightarrowtail e^t_x \Rightarrow \mathcal{F}_E \rightsquigarrow e^t_x$.

(2) $\mathrm{lim}_{\tau_E}(\mathcal{F}_E) \sqsubseteq \mathrm{cls}_{\tau_E}(\mathcal{F}_E)$.

(3) $\mathcal{F}_E \rightsquigarrow e^t_x,\ e^s_x \sqsubseteq e^t_x \Rightarrow \mathcal{F}_E \rightsquigarrow e^s_x$.

(4) $\mathcal{F}_E \rightarrowtail e^t_x,\ e^s_x \sqsubseteq e^t_x \Rightarrow \mathcal{F}_E \rightarrowtail e^s_x$.

(5) $\mathcal{F}_E \rightsquigarrow e^t_x \Leftrightarrow e^t_x \sqsubseteq \mathrm{cls}_{\tau_E}(\mathcal{F}_E)$.

(6) $\mathcal{F}_E \rightarrowtail e^t_x \Leftrightarrow e^t_x \sqsubseteq \mathrm{lim}_{\tau_E}(\mathcal{F}_E)$.

(7) $\mathcal{F}_E \rightarrowtail e^t_x \Rightarrow \mathcal{G}_E \rightarrowtail e^t_x$.

(8) $\mathrm{lim}_{\tau_E}(\mathcal{F}_E) \sqsubseteq \mathrm{lim}_{\tau_E}(\mathcal{G}_E)$.

(9) $\mathcal{G}_E \rightsquigarrow e^t_x \Rightarrow \mathcal{F}_E \rightsquigarrow e^t_x$.

(10) $\mathrm{cls}_{\tau_E}(\mathcal{G}_E) \sqsubseteq \mathrm{cls}_{\tau_E}(\mathcal{F}_E)$.

Proof. (1) For every $g_B \in (Q_{e_x^t})_E^\circ$ and $f_A \in (\mathcal{F}_E)^\circ$, since $(Q_{e_x^t})_E \sqsubseteq \mathcal{F}_E$, we have $g_B \in (\mathcal{F}_E)^\circ$. Hence, $\mathcal{F}_e(f_A \sqcap g_B) > 0$. It implies that $f_A \sqcap g_B \neq \tilde{\Phi}$.

(2) From (1), it is clear.

(3) Since $e_x^s \sqsubseteq e_x^t$, by Theorem 3.1 (2), $(Q_{e_x^s})_E \sqsubseteq (Q_{e_x^t})_E$. For every $g_B \in (Q_{e_x^s})_E^\circ$, we have $g_B \in (Q_{e_x^t})_E^\circ$. Since $\mathcal{F}_e \rightsquigarrow e_x^t$, for each $f_A \in (\mathcal{F}_E)^\circ$, $f_A \sqcap g_B \neq \tilde{\Phi}$. Hence, $\mathcal{F}_e \rightsquigarrow e_x^s$.

(4) Since $\mathcal{F}_e \rightarrowtail e_x^t$, we have $(Q_{e_x^t})_E \sqsubseteq \mathcal{F}_E$. Since $e_x^s \sqsubseteq e_x^t$, by Theorem 3.1 (2), $(Q_{e_x^s})_E \sqsubseteq (Q_{e_x^t})_E$. Hence, $(Q_{e_x^s})_e \sqsubseteq (Q_{e_x^t})_e \sqsubseteq \mathcal{F}_e$. Thus, $\mathcal{F}_e \rightarrowtail e_x^s$.

(5) If $e_x^t \sqsubseteq \mathrm{cls}_{\tau_E}(\mathcal{F}_E)$, for every $g_B \in (Q_{e_x^t})_E^\circ$ by the definition of $(Q_{e_x^t})_E$, there exists $h_C \in \widetilde{(X,E)}$ such that $e_x^t \, q \, h_C, h_C \sqsubseteq g_B$ and $(Q_{e_x^t})_e(g_B) \geq \tau_e(h_C) > 0$. This implies that $h_C \, q \, \mathrm{cls}_{\tau_e}(\mathcal{F}_e)$.

By the definition of $\mathrm{cls}_{\tau_e}(\mathcal{F}_e)$, there exists a fuzzy soft cluster point $e_x^t \in P_t\widetilde{(X,E)}$ of $\mathcal{F}_E$ such that $e_x^t \, q \, h_C$ implies that $h_C \, q \, \mathrm{cls}_{\tau_e}(\mathcal{F}_e)$. Thus, $e_x^s \, q \, h_C, h_C \sqsubseteq g_B$ and $(Q_{e_x^s})_e(g_B) \geq \tau_e(h_C) > 0$. Hence, $g_B \in (Q_{e_x^s})_E^\circ$ and e_x^s is a fuzzy soft cluster point of $\mathcal{F}_E$. Hence, for each $f_A \in (\mathcal{F}_E)^\circ$, $f_A \sqcap g_B \neq \tilde{\Phi}$. Therefore, $\mathcal{F}_E \rightsquigarrow e_x^t$.

The converse is clear.

(6) It is similar to (5).

(7) It is easily proved from $(Q_{e_x^t})_e \sqsubseteq \mathcal{F}_e \sqsubseteq \mathcal{G}_e$.

(8) From (7), it is clear.

(9) For each $g_B \in (Q_{e_x^t})_E^\circ$ and $f_A \in (\mathcal{F}_E)^\circ$ since $\mathcal{F}_E \sqsubseteq \mathcal{G}_E$, we have $f_A \in (\mathcal{G}_e)_E^\circ$. Since $\mathcal{G}_e \rightsquigarrow e_x^t$, $f_A \sqcap g_B \neq \tilde{\Phi}$. That is, $\mathcal{F}_e \rightsquigarrow e_x^t$.

(10) From (9), it is clear. □

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