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Non-Archimedean Stability of a Generalized Reciprocal-Quadratic Functional Equation in Several Variables by Direct and Fixed Point Methods

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Abstract. This study is aimed to determine various stabilities of a generalized reciprocal-quadratic functional equation of the form

$$r\left(\sum_{j=1}^{p}\beta_j u_j\right)=\frac{\prod_{j=1}^{p} r\left(u_j\right)}{\left[\sum_{j=1}^{p}\beta_j\prod_{k=1,k\neq j}^{p}\sqrt{r\left(u_k\right)}\right]^2}$$

connected with Ulam, Hyers, T. M. Rassias, J. M. Rassias and Gavruta in non-Archimedean fields, where $\beta_j \neq 0; j = 1, 2, \ldots, p$ are arbitrary real numbers and $0 < \beta_1+\beta_2+\cdots+\beta_p = \sum_{j=1}^{p}\beta_j = \beta \neq 1$ in non-Archimedean fields by direct and fixed point methods.

1. Introduction

The issue created by Ulam [43] in 1940 is the source for the speculation of stability of functional equations. The question devised by Ulam was responded by Hyers [9] which made a cornerstone in the conjecture of stability of functional equation. The result proved by Hyers is called as Hyers–Ulam stability or ϵ-stability of functional equation. Then, Hyers' result was simplified by Aoki [1]. Also, Hyers' result was modified by T. M. Rassias [34] considering the upper bound as sum of powers of norms. The result obtained by T. M. Rassias is called as Hyers–Ulam–Rassias stability of functional equation. Later, J. M. Rassias [32] established Hyers' result by taking the upper bound as product of powers of norms. This theorem is called as Ulam–Gavruta–Rassias stability of functional equation. In 1994, to promote the stability result into simple form, Gavruta [7] reinstated the upper bound by a general control function. This type of stability result accomplished by Gavruta is known as generalized Hyers–Ulam stability of functional equation.

The p-adic numbers were discovered by Hensel [8] in 1897 as a number theoretical analogue of power series in complex analysis. In fact, he introduced a field with a valuation normed which does not have the Archimedean property. Even though there are many classical results in the normed space theory with

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non-Archimedean property, but their proofs are different and require a relatively new kind of perception. It may be noted that $|n| \leq 1$ in every valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space [29]. These facts formulate the non-Archimedean structure is of exceptional attention.

The stability of functional equations in non-Archimedean was firstly obtained by Arriola and Beyer [2]. They investigated the stability of Cauchy functional equations over p-adic fields. The stability of some other functional equations in non-Archimedean normed spaces have been investigated by many mathematicians [12], [19], [20], [22], [24], [25].

Isac and Rassias [10] were the foremost mathematicians to present the applications of stability problem of functional equations via fixed point theorems. Generally, the investigation of stability of functional equations is carried out through direct method in which the precise solution of the functional equation is unambiguously derived as a limit of a (Hyers) sequence, starting from the given approximate solution. The stability of Cauchy additive functional equation was proved by applying fixed point method by Radu [35] in 2003. There are many results available on the stability of various functional equations using fixed point method [6], [11], [23], [33].

For the first time, Ravi and Senthil Kumar [39] obtained the generaliized Hyers–Ulam stability for a rational functional equation

$$h(u+v) = \frac{h(u)h(v)}{h(u)+h(v)}. \tag{1}$$

It is easy to verify that the rational function $h(u) = \frac{k}{u}$ is a solution of the functional equation (1).

Bodaghi and Kim [5] investigated Ulam stability for the quadratic reciprocal functional equation

$$Q_r(2p+q) + Q_r(2p-q) = \frac{2Q_r(p)Q_r(q)[4Q_r(q)+Q_r(p)]}{(4Q_r(q)-Q_r(p))^2}. \tag{2}$$

The quadratic reciprocal function $Q_r(p) = \frac{k}{p^2}$ is a solution of the functional equation (2).

Bodaghi and Ebrahimdoost [4] generalized equation (2) as

$$Q_r((a+1)u+av) + Q_r((a+1)u-av) = \frac{2Q_r(u)Q_r(v)\left[(a+1)^2Q_r(v)+a^2Q_r(u)\right]}{\left((a+1)^2Q_r(v)-a^2Q_r(u)\right)^2} \tag{3}$$

where $a \in \mathbb{Z}$ with $a \neq 0$ and attained its generalized Hyers–Ulam–Rassias stability.

Further results associated with the stability of different rational functional equations are available in ([36], [37], [38], [41]).

Recently the theory of Ulam–Hyers stability has developed enormously in investigating various equations such as polynomial equations, different type of functional equations, ordinary differential equations and partial differential equations with interesting and motivating results (see [3], [13], [14], [15], [16], [17], [18], [26], [27], [28], [30], [31], [42], [44], [45], [46]).

In recent times, Ravi and Suresh [40] have investigated the generalized Hyers–Ulam stability of reciprocal-quadratic functional equation in two variables of the form

$$R_q(u+v) = \frac{R_q(u)R_q(v)}{R_q(u)+R_q(v)+2\sqrt{R_q(u)R_q(v)}} \tag{4}$$

in the setting of real numbers. It is easy to verify that the reciprocal-quadratic function $R_q(u) = \frac{k}{u^2}$ is a solution of equation (4).

In this paper, we extend the equation (4) to several variables in the following form

$$r\left(\sum_{j=1}^{p} \beta_j u_j\right) = \frac{\prod_{j=1}^{p} r\left(u_j\right)}{\left[\sum_{j=1}^{p} \beta_j \prod_{k=1,k\neq j}^{p} \sqrt{r\left(u_k\right)}\right]^2} \tag{5}$$

and acquire various stabilities of this equation associated with Ulam, Hyers, T. M. Rassias, J. M. Rassias and Gavruta in non-Archimedean fields, where $\beta_j \neq 0; j = 1, 2, \ldots, p$ are arbitrary real numbers and $0 < \beta_1 + \beta_2 + \cdots + \beta_p = \sum_{j=1}^{p} \beta_j = \beta \neq 1$ in non-Archimedean fields by direct and fixed point methods.

Note that for $j = 2$, we have

$$r(\beta_1 u_1 + \beta_2 u_2) = \frac{r(u_1)r(u_2)}{\left[\beta_1 \sqrt{r(u_2}) + \beta_2 \sqrt{r(u_1)}\right]^2}$$

with $\beta_1 + \beta_2 \neq 1$ and for $j = 3$, we find

$$r(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) = \frac{r(u_1)r(u_2)r(u_3)}{\left[\beta_1 \sqrt{r(u_2)r(u_3)} + \beta_2 \sqrt{r(u_1)r(u_3)} + \beta_3 \sqrt{r(u_1)r(u_2)}\right]^2}$$

with $\beta_1 + \beta_2 + \beta_3 \neq 1$.

2. Preliminaries

In this segment, we summon up the fundamental notions of non-Archimedean field, non-Archimedean norm, non-Archimedean space and non-Archimedean alternative contraction principle which will be useful to establish our main results.

Definition 2.1. *A field K is said to be a non-Archimedean field if it is equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty)$ such that $|u| = 0$ if and only if $u = 0$, $|uv| = |u||v|$ and $|u + v| \leq max\{|u|, |v|\}$ for all $u, v \in K$.*

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is a *non-Archimedean norm* (*valuation*) if it satisfies the following conditions:

(i) $\|u\| = 0$ if and only if $u = 0$;
(ii) $\|\rho u\| = |\rho|\|u\| \qquad (\rho \in \mathbb{K}, u \in X)$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|u + v\| \leq \max\{\|u\|, \|v\|\} \qquad (u, v \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. Due to the fact that

$$\|u_n - u_m\| \leq \max\left\{\left\|u_{j+1} - u_j\right\| : m \leq j \leq n-1\right\} \qquad (n > m),$$

a sequence $\{u_n\}$ is Cauchy if and only if $\{u_{n+1} - u_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean that every Cauchy sequence is convergent in the space.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{A}$ is the mapping

$$|\lambda| = \begin{cases} 0 & \text{if } \lambda = 0 \\ \frac{1}{\lambda} & \text{if } \lambda > 0 \\ -\frac{1}{\lambda} & \text{if } \lambda < 0 \end{cases}$$

for any $\lambda \in \mathbb{A}$.

Example 2.2. *Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. Note that if $p > 2$, then $|2^n| = 1$ for all integers n.*

Definition 2.3. *Let A be a nonempty set and $d : A \times A \longrightarrow [0, \infty]$ fulfilling the ensuing properties:*

(i) $d(u, v) = 0$ *if and only if* $u = v$,

(ii) $d(u, v) = d(v, u)$ *(symmetry),*

(iii) $d(u, w) \leq max\{d(u, v), d(v, w)\}$ *(strong triangle inequality),*

for all $u, v, w \in A$. *Then* (A, d) *is called a generalized non-Archimedean metric space.* (A, d) *is called complete if every d-Cauchy sequence in A is d-convergent.*

Example 2.4. *For each nonempty set A, define*

$$d(u, u^\star) = \begin{cases} 0 & \text{if } u = u^\star \\ \infty & \text{if } u \neq u^\star. \end{cases}$$

Then d is a generalized non-Archimedean metric on A.

Example 2.5. Let A and B be two non-Archimedean spaces over a non-Archimedean field $\mathbb{K}$. If B has a complete non-Archimedean norm over $\mathbb{K}$ and $\phi : A \longrightarrow [0, \infty)$, for each $s, t : A \longrightarrow B$, define

$$d(s, t) = \inf\{\delta > 0 : |s(u) - t(u)| \leq \delta\phi(u),\ \forall u \in A\}.$$

Using Theorem 2.5 [6], Mirmostafaee [21] introduced non-Archimedean version of the alternative fixed point theorem as follows:

Theorem 2.6. [21] (Non-Archimedean Alternative Contraction Principle) *If* (A, d) *is a non-Archimedean generalized complete metric space and* $\sigma : A \to A$ *a strictly contractive mapping (that is* $d(\sigma(u), \sigma(v)) \leq Ld(v, u)$*), for all* $u, v \in A$ *and a Lipsctiz constant* $L < 1$*), then either*

(i) $d\left(J^n(u), J^{n+1}(u)\right) = \infty$, *for all* $n \geq 0$, *or*

(ii) there exists some $n_0 \geq 0$ *such that* $d\left(J^n(u), J^{n+1}(u)\right) < \infty$ *for all* $n \geq n_0$;

the sequence $\{J^n(u)\}$ *is convergent to a fixed point* $u^\star$ *of* J; $u^\star$ *is the unique fixed point of* J *in the set*

$$V = \{v \in A : d\left(J^{n_0}(u), v\right) < \infty\}$$

and $d(v, u^\star) \leq d(v, J(v))$ *for all v in this set.*

Throughout this paper, we consider that $\mathbb{E}$ and $\mathbb{F}$ is a non-Archimedean field and a complete non-Archimedean field, respectively. In the sequel, we denote $\mathbb{E}^* - \{0\}$ as a non-Archimedean field. In order to simplify manipulations, let us symbolize the difference operator $D_r : \underbrace{\mathbb{E}^* \times \mathbb{E}^* \times \cdots \times \mathbb{E}^*}_{\text{p times}} \to \mathbb{F}$ by

$$D_r(u_1, u_2, \ldots, u_p) = r\left(\sum_{j=1}^{p} \beta_j u_j\right) - \frac{\prod_{j=1}^{p} r\left(u_j\right)}{\left[\sum_{j=1}^{p} \beta_j \prod_{k=1, k\neq j}^{p} \sqrt{r\left(u_k\right)}\right]^2}$$

for all $u_i \in \mathbb{E}^*$, $i = 1, 2, \ldots, p$.

Definition 2.7. *A mapping* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{p\ times} \longrightarrow \mathbb{F}$ *is said to be as generalized reciprocal-quadratic mapping if r satisfies the equation (5).*

Assumptions on the above definition and equation (5): By assuming $r(u_i) \neq 0$, for all $u_i \in \mathbb{A}^*$, $i = 1, 2, \ldots, p$, the singular cases are eliminated.

3. Non-Archimedean Stability of Equation (5) by Direct Method

In this section, we accompolish various stabilities of the equation (5) in non-Archimedean fields related with Ulam, Hyers, T. M. Rassias, J. M. Rassias and Gavruta by direct method.

Theorem 3.1. *Let $\ell \in \{-1, 1\}$. Let $\psi : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} |\beta|^{2\ell(m+1)} \psi\left(\beta^{\ell(i+1)} u_1, \beta^{\ell(i+1)} u_2, \ldots, \beta^{\ell(i+1)} u_p\right) = 0 \tag{6}$$

for all $u_i, \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. Suppose that $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ is a mapping satisfying the inequality

$$\left|D_r(u_1, u_2, \ldots, u_p)\right| \leq \psi(u_1, u_2, \ldots, u_p) \tag{7}$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. Then there exists a unique generalized reciprocal-quadratic mapping R_q : $\underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \ldots \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ such that

$$\left|r(u) - R_q(u)\right| \leq \max\left\{|\beta|^{2\ell i} \psi\left(\beta^{\ell(i+1)} u, \beta^{\ell(i+1)} u, \ldots, \beta^{\ell(i+1)} u\right) : i \in \mathbb{N} \cup \{0\}\right\} \tag{8}$$

for all $u \in \mathbb{E}^\star$.

Proof. Let $\ell = -1$. Substituting $u_i = u$ for $i = 1, 2, \ldots, p$ in (7), we acquire

$$\left|r(\beta u) - \frac{1}{\beta^2} r(u)\right| \leq \psi(u, u, \ldots, u) \tag{9}$$

for all $u \in \mathbb{E}^\star$. Now, replacing u by $\frac{u}{\beta}$ in (9), we attain

$$\left|r(u) - \frac{1}{\beta^2} r\left(\frac{u}{\beta}\right)\right| \leq \psi\left(\frac{u}{\beta}, \frac{u}{\beta}, \ldots, \frac{u}{\beta}\right) \tag{10}$$

for all $u \in \mathbb{E}^\star$. Plugging u into $\frac{u}{\beta^m}$ in (10) and multiplying by $\left|\frac{1}{\beta}\right|^{2m}$, we find

$$\left|\frac{1}{\beta^{2m}} r\left(\frac{u}{\beta^m}\right) - \frac{1}{\beta^{2(m+1)}} r\left(\frac{u}{\beta^{m+1}}\right)\right| \leq \left|\frac{1}{\beta}\right|^{2m} \psi\left(\frac{u}{\beta^{m+1}}, \frac{u}{\beta^{m+1}}, \ldots, \frac{u}{\beta^{m+1}}\right) \tag{11}$$

for all $u \in \mathbb{E}^{\sim\approx\partial\setminus}$. Therefore it is easy to see that the sequence $\left\{\frac{1}{\beta^{2m}} r\left(\frac{u}{\beta^m}\right)\right\}$ is a Cauchy sequence by (6) and (11). Since $\mathbb{F}$ is complete, we can define a mapping R_q given by

$$R_q(u) = \lim_{m\to\infty} \frac{1}{\beta^{2m}} r\left(\frac{u}{\beta^m}\right). \tag{12}$$

For each $u \in \mathbb{E}^\star$ and non-negative integers m, we obtain

$$\begin{aligned}
\left|\frac{1}{\beta^{2m}} r\left(\frac{u}{\beta^m}\right) - r(u)\right| &= \left|\sum_{j=0}^{m-1} \left\{\frac{1}{\beta^{2(j+1)}} r\left(\frac{u}{\beta^{j+1}}\right) - \frac{1}{\beta^{2j}} r\left(\frac{u}{\beta^j}\right)\right\}\right| \\
&\leq \max\left\{\left|\frac{1}{\beta^{2(j+1)}} r\left(\frac{u}{\beta^{j+1}}\right) - \frac{1}{\beta^{2j}} r\left(\frac{u}{\beta^j}\right)\right| : 0 \leq i < m\right\} \\
&\leq \max\left\{\left|\frac{1}{\beta}\right|^{2(j+1)} \psi\left(\frac{u}{\beta^{j+1}}, \frac{u}{\beta^{j+1}}, \ldots, \frac{u}{\beta^{j+1}}\right) : 0 \leq j < m\right\}.
\end{aligned} \tag{13}$$

Using (12) and allowing $m \to \infty$ in the above inequality (13), we find that the inequality (8) is valid . Using (6), (7) and (12), we have for all $u_1, u_2, \ldots, u_p \in \mathbb{E}^*$

$$\begin{aligned}\left|D_{R_q}(u_1, u_2, \ldots, u_p)\right| &= \lim_{m\to\infty} \left|\frac{1}{\beta}\right|^{2m} \left|D_r\left(\frac{u_1}{\beta^m}, \frac{u_2}{\beta^m}, \ldots, \frac{u_p}{\beta^m}\right)\right| \\ &\le \lim_{m\to\infty} \left|\frac{1}{\beta}\right|^{2m} \psi\left(\frac{u_1}{\beta^m}, \frac{u_2}{\beta^m}, \ldots, \frac{u_p}{\beta^m}\right) = 0.\end{aligned}$$

Therefore the mapping R_q satisfies (5), which implies that it is reciprocal-quadratic mapping. In order to demonstrate the distinctivity of R_q, let us presume $R_Q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \ldots \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ be another reciprocal-quadratic mapping satisfying (8). Then we have

$$\begin{aligned}\left|R_q(u) - R_Q(u)\right| &= \lim_{n\to\infty} \left|\frac{1}{\beta}\right|^{2n} \left|R_q\left(\frac{u}{\beta^n}\right) - R_Q\left(\frac{u}{\beta^n}\right)\right| \\ &\le \lim_{n\to\infty} \left|\frac{1}{\beta}\right|^{2n} \max\left\{\left|R_q\left(\frac{u}{\beta^n}\right) - r\left(\frac{u}{\beta^n}\right)\right|, \left|r\left(\frac{u}{\beta^n}\right) - R_Q\left(\frac{u}{\beta^n}\right)\right|\right\} \\ &\le \lim_{n\to\infty}\lim_{m\to\infty} \max\Big\{\max\Big\{\left|\frac{1}{\beta}\right|^{2(j+n)} \psi\left(\frac{u}{\beta^{j+n+1}}, \frac{u}{\beta^{j+n+1}}, \ldots, \frac{u}{\beta^{j+n+1}}\right) : n \le j \le m+n\Big\}\Big\} \\ &= 0\end{aligned}$$

for all $u \in \mathbb{E}^\star$. Similar proof follows for the case $\ell = 1$. This implies that R_q is unique which finishes the proof. □

In the following corollaries, we assume that $|2| < 1$ for a non-Archimdean field $\mathbb{E}$. We obtain the stability results of equation (5) associated with Hyers, T. M. Rassias and J. M. Rassias by Theorem 3.1.

Corollary 3.2. *Let* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *be a mapping satisfying the following inequality*

$$\left|D_r(u_1, u_2, \ldots, u_p)\right| \le \epsilon$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$, *where* $\epsilon > 0$ *is a constant. Then there exists a unique generalized reciprocal-quadratic mapping* $R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfying* (5) *and*

$$\left|r(u) - R_q(u)\right| \le \epsilon$$

for every $u \in \mathbb{E}^\star$.

Proof. The proof is achieved by assuming $\psi(u_1, u_2, \ldots, u_p) = \epsilon$ in Theorem 3.1 when $\ell = -1$. □

Corollary 3.3. *Assume* $k_1 \ge 0$ *and* $\alpha \ne -2$, *as fixed constants. Suppose* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfies*

$$\left|D_r(u_1, u_2, \ldots, u_p)\right| \le k_1 \left(\sum_{i=1}^{p} |u_i|^\alpha\right)$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. *Then there exists a unique generalized reciprocal-quadratic mapping* $R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{p\ times} \to \mathbb{F}$ *satisfying* (5) *and*

$$\left| r(u) - R_q(u) \right| \leq \begin{cases} \frac{|p|k_1}{|\beta|^\alpha} |u|^\alpha, & \text{for } \alpha > -2 \\ k_1 |p| |\beta|^2 |u|^\alpha, & \text{for } \alpha < -2 \end{cases}$$

for every $u \in \mathbb{E}^*$.

Proof. The proof is established by considering $\psi(u_1, u_2, \ldots, u_p) = k_1 \left(\sum_{i=1}^{p} |u_i|^\alpha \right)$ in Theorem 3.1 with $\alpha > -2$ and $\ell = -1$ and in Theorem 3.1 with $\alpha < -2$ and $\ell = 1$. $\square$

Corollary 3.4. *Let* $k_2 \geq 0$ *be fixed constant. Suppose* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *is a mapping satisfies*

$$\left| D_r(u_1, u_2, \ldots, u_p) \right| \leq k_2 \prod_{i=1}^{p} |u_i|^\alpha$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. *Then there exists a unique generalized reciprocal-quadratic mapping* $R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfying* (5) *and*

$$\left| r(u) - R_q(u) \right| \leq \begin{cases} \frac{k_2}{|\beta|^\alpha} |u|^\alpha, & \text{for } \alpha > -2 \\ k_2 |\beta|^2 |u|^\alpha, & \text{for } \alpha < -2 \end{cases}$$

for every $u \in \mathbb{E}^\star$.

Proof. The proof is complete by opting $\psi(x, y) = k_2 \prod_{i=1}^{p} |u_i|^\alpha$ in Theorem 3.1 with $\alpha > -2$ and $\ell = -1$ and in Theorem 3.1 with $\alpha < -2$ and $\ell = 1$. $\square$

4. Non-Archimedean Stability of Equation (5) by Fixed Point Method

Using fixed point alternative, we investigate various stabilities of the equation (5) in non-Archimedean fields by fixed point method.

Theorem 4.1. *Suppose that the mapping* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfies the inequality*

$$\left| D_r(u_1, u_2, \ldots, u_p) \right| \leq \phi(u_1, u_2, \ldots, u_p) \tag{14}$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$, *where* $\phi : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to [0, \infty)$ *is a given function. If* $0 < L < 1$,

$$|\beta|^{-2} \phi \left(\beta^{-1} u_1, \beta^{-1} u_2, \ldots, \beta^{-1} u_p \right) \leq L \phi(u_1, u_2, \ldots, u_p) \tag{15}$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$, *then there exists a unique generalized reciprocal-quadratic mapping*

$$R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$$

such that

$$|r(u) - R_q(u)| \leq L |\beta|^2 \phi(u, u, \ldots, u) \tag{16}$$

for all $u \in \mathbb{E}^\star$.

Proof. Plugging u_i by $\frac{u}{\beta}$ for $i = 1, 2, \dots, p$ in (14), we obtain

$$\left| r(u) - \beta^{-2} r\left(\beta^{-1}u\right) \right| \leq \phi\left(\beta^{-1}u, \beta^{-1}u, \dots, \beta^{-1}u\right) \tag{17}$$

for all $u \in \mathbb{E}^\star$.

For every $g, h \in \mathbb{E}^\star \to \mathbb{F}$ define

$$d(g,h) = inf\{\delta > 0 : |g(u) - h(u)| \leq \delta\phi(u, u, \dots, u)\}, \text{ for all } u \in \mathbb{E}^\star.$$

By Example 2.5, d defines a complete generalized non-Archimedean metric on $S = \{g | g : \mathbb{E}^\star \to \mathbb{R}\}$.

Let $\sigma : S \to S$ be defined by $\sigma(g)(u) : \beta^{-2} g(\beta^{-1}u)$ for all $u \in \mathbb{E}^\star$ and $g \in S$. If for some $g, h \in S$ and $\delta > 0$,

$$|g(u) - h(u)| \leq \delta\phi(u, u, \dots, u) \quad (u \in \mathbb{E}^\star),$$

then

$$\begin{aligned} |\sigma(g)(u) - \sigma(h)(u)| &= |\beta|^{-2} \left| g(\beta^{-1}u) - h(\beta^{-1}u) \right| \\ &\leq \delta|\beta|^{-2}\phi(\beta^{-1}u, \beta^{-1}u, \dots, \beta^{-1}u) \\ &\leq \delta L\phi(u, u, \dots, u) \quad (u \in \mathbb{E}^\star). \end{aligned}$$

Therefore, $d(\sigma(g), \sigma(h)) \leq Ld(g, h)$. Hence d is a strictly contractive mapping on S with Lipschitz constant L.

Let $X = \{g \in S : d(r, g) < \infty\}$, since

$$\begin{aligned} |\sigma(r)(u) - \sigma^0 r(u)| &= \left| \beta^{-2} r(\beta^{-1}u) - r(u) \right| \\ &\leq \phi(\beta^{-1}u, \beta^{-1}u, \dots, \beta^{-1}u) \\ &\leq L|\beta|^2\phi(u, u, \dots, u) \quad (u \in \mathbb{E}^\star). \end{aligned}$$

This means that $d(\sigma(r), r) \leq L|\beta|^2$. By Theorem 2.6 (ii), σ has a unique fixed point $R_q : \mathbb{E}^\star \to \mathbb{F}$ which is defined by

$$R_q(u) = \lim_{n\to\infty} \sigma^n(r)(u) = \lim_{n\to\infty} \beta^{-2n} r(\beta^{-n}u) \quad (u \in \mathbb{E}^\star).$$

The inequality

$$\begin{aligned} \left| D_{R_q}(u_1, u_2, \dots, u_p) \right| &= \lim_{m\to\infty} |\beta|^{-2m} \left| D_r(\beta^{-m}u_1,)\beta^{-m}u_2, \dots, \beta^{-m}u_p \right| \\ &\leq \lim_{m\to\infty} |\beta|^{-2m}\phi(\beta^{-m}u_1, \beta^{-m}u_2, \dots, \beta^{-m}u_p) \\ &\leq \lim_{m\to\infty} L^m\phi(u_1, u_2, \dots, u_p) = 0 \end{aligned}$$

for all $u_i \in \mathbb{E}^\star, i = 1, 2, \dots, p$, implies that R_q is generalized reciprocal-quadratic. By Theorem 2.6 (ii), $d(r, R_q) \leq d(\sigma(r), r)$, that is,

$$|r(u) - R_q(u)| \leq L|\beta|^2\phi(u, u, \dots, u) \quad (u \in \mathbb{E}^\star).$$

If $R_Q : \mathbb{E}^\star \to \mathbb{F}$ is another generalized reciprocal-quadratic mapping which satisfies (16), then R_Q is a fixed point of σ in X. The uniqueness of the fixed point of σ in X implies that $R_q = R_Q$. □

The following theorem is dual of Theorem 4.1 and its proof follows directly from Theorem 4.1. Hence we omit the proof.

Theorem 4.2. *Suppose that the mapping* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfies* (14). *If* $0 < L < 1$,

$$|\beta|^2\phi\left(\beta u_1, \beta u_2, \dots, \beta u_p\right) \leq L\phi(u_1, u_2, \dots, u_p) \tag{18}$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$, *then there exists a unique generalized reciprocal-quadratic mapping*

$$R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$$

such that

$$|r(u) - R_q(u)| \leq L\phi(\beta^{-1}u, \beta^{-1}u, \ldots, \beta^{-1}u) \tag{19}$$

for all $u \in \mathbb{E}^\star$.

In the following corollaries, we investigate the stability results of equation (5) pertinent to Hyers, T. M. Rassias and J. M. Rassias using Theorems 4.1 and 4.2.

Corollary 4.3. *Let* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *be a mapping for which there exists a constant* ϵ (*independent of* $u_1, u_2, \ldots, u_p$)≥ 0 *such that the functional inequality*

$$\left|D_r(u_1, u_2, \ldots, u_p)\right| \leq \epsilon$$

holds for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. *Then there exists a unique generalized reciprocal-quadratic mapping* $r_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfying the functional equation* (5) *and*

$$|r(u) - R_q(u)| \leq \epsilon$$

for all $u \in \mathbb{E}^\star$.

Proof. Taking $\phi(u_1, u_2, \ldots, u_p) = \epsilon$ and choosing $L = |\beta|^{-2}$ in Theorem 4.1, the proof follows immediately. □

Corollary 4.4. *Let* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *be a mapping and let there exist real numbers* $\alpha \neq -2$ *and* $c_1 \geq 0$ *such that*

$$\left|D_r(u_1, u_2, \ldots, u_p)\right| \leq c_1 \left(\sum_{i=1}^{p} |u_i|^\alpha\right)$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. *Then there exists a unique generalized reciprocal-quadratic mapping* $R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfying the functional equation* (5) *and*

$$|r(u) - R_q(u)| \leq \begin{cases} \frac{|p|c_1}{|\beta|^\alpha}|u|^\alpha, & \text{for } \alpha > -2 \\ |p|c_1|\beta|^2|u|^\alpha, & \text{for } \alpha < -2 \end{cases}$$

for all $u \in \mathbb{E}^\star$.

Proof. Assume $\phi(u_1.u_2., \ldots, u_p) = c_1\left(\sum_{i=1}^{p} |u_i|^\alpha\right)$ and then select $L = |\beta|^{-\alpha-2}$, $\alpha > -2$ and $L = |\beta|^{\alpha+2}$, $\alpha < -2$, respectively in Theorem 4.1 and Theorem 4.2 to get the desired result. □

Corollary 4.5. *Let* $c_2 \geq 0$ *and* $\alpha \neq -2$ *be real numbers, and* $r : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *be a mapping satisfying the functional inequality*

$$\left|D_r(u_1, u_2, \ldots, u_p)\right| \leq c_3 \left(\prod_{i=1}^{p} |u_i|^{\alpha/p}\right)$$

for all $u_i \in \mathbb{E}^\star$, $i = 1, 2, \ldots, p$. *Then there exists a unique generalized reciprocal-quadratic mapping* $R_q : \underbrace{\mathbb{E}^\star \times \mathbb{E}^\star \times \cdots \times \mathbb{E}^\star}_{\text{p times}} \to \mathbb{F}$ *satisfying the functional equation* (5) *and*

$$|r(u) - R_q(u)| \leq \begin{cases} \frac{c_2}{|\beta|^\alpha}|u|^\alpha, & \textit{for } \alpha > -2 \\ c_2|\beta|^2|u|^\alpha, & \textit{for } \alpha < -2 \end{cases}$$

for all $u \in \mathbb{E}^\star$.

Proof. The proof goes through the same way as in Theorem 4.1 and Theorem 4.2 by considering

$$\phi(u_1, u_2, \ldots, u_p) = c_3 \left(\prod_{i=1}^{p} |u_i|^{\alpha/p} \right)$$

and then choosing $\beta = L^{-\alpha-2}$, $\alpha > -2$ and $L = \beta^{\alpha+2}$, $\alpha < -2$, respectively. □

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