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# Linear Combinations of a Class of Harmonic Univalent Mappings

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**Abstract.** A planar harmonic mapping is a complex-valued function $f : \mathbb{U} \to \mathbb{C}$ of the form $f(x + iy) = u(x, y) + iv(x, y)$, where $u$ and $v$ are both real harmonic. Such a function can be written as $f = h + \overline{g}$, where $h$ and $g$ are both analytic; the function $\omega = g'/h'$ is called the dilatation of $f$. We consider the linear combinations of planar harmonic mappings that are the vertical shears of the asymmetrical vertical strip mappings $\varphi_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$ with various dilatations, where $\alpha_j \in [\frac{\pi}{2}, \pi)$, $j = 1, 2$. We prove sufficient conditions for the linear combination of this class of harmonic univalent mappings to be univalent and convex in the direction of the imaginary axis.

## 1. Introduction

A continuous complex-valued function $f = \mu + i\nu$ defined in a simply connected domain $\Omega \subset \mathbb{C}$ is said to be harmonic in $\Omega$ if both $\mu$ and $\nu$ are real harmonic in $\Omega$. In any simply connected domain $\Omega$, we can write

$$f = h + \overline{g}, \tag{1.1}$$

where $h$ and $g$ are analytic in $\Omega$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\Omega$ is that $|h'| > |g'|$ in $\Omega$.

Denote by $S_H$ the class of functions $f$ of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f_z(0) - 1 = 0$. It is obvious that the normalization condition is equivalent to

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Furthermore, let $S_H^0$ be the subclass of $S_H$ consisting of $f$ with $b_1 = 0$. The classical family $S$ of analytic univalent, normalized functions on $\mathbb{U}$ is the subclass of $S_H$ in which $b_n = 0$ for all $n$. In 1984, Clunie and Sheil-Small [5] investigated the class $S_H$ for the first time. Since then, harmonic mappings have been an

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area of active research. Many remarkable results for harmonic mappings can be found in the literature [1-3, 5-11, 13-18, 20, 21, 24, 25].

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction $\gamma$, if for all $z_0 \in \mathbb{C}$, the set $\Omega \cap \{z_0 + te^{i\gamma} : t \in \mathbb{R}\}$ is either connected or empty. Particularly, a domain is convex in the direction of the real (resp. imaginary) axis if its intersection with each horizontal (resp. vertical) line is connected. A function is convex in the direction of real (resp. imaginary) axis if it maps $\mathbb{U}$ onto a domain convex in the direction of real (resp. imaginary) axis. The following result due to Hengarther and Schober [12] is very useful in checking if an analytic function is convex in the direction of the imaginary axis.

**Lemma 1.1.** *Suppose $f$ is analytic and non constant in $\mathbb{U}$. Then*

$$\Re[(1-z^2)f'(z)] \geq 0, \quad z \in \mathbb{U}$$

*if and only if*

*(i) it is univalent in $\mathbb{U}$;*

*(ii) it is convex in the direction of the imaginary axis;*

*(iii) there exist sequences $\{z'_n\}$ and $\{z''_n\}$ converging to $z = 1$ and $z = -1$, respectively, such that*

$$\begin{aligned} \lim_{n\to\infty} \Re(f(z'_n)) &= \sup_{|z|<1} \Re(f(z)), \\ \lim_{n\to\infty} \Re(f(z''_n)) &= \inf_{|z|<1} \Re(f(z)). \end{aligned} \tag{1.2}$$

Clunie and Sheil-Small [5] introduced the shear construction method that produces a harmonic univalent mapping with a given dilatation onto domains convex in one direction. This result is given in Lemma 1.2.

**Lemma 1.2.** *A locally univalent harmonic function $f = h + \overline{g}$ in $\mathbb{U}$ is a univalent harmonic mapping of $\mathbb{U}$ onto a domain convex in a direction of the imaginary (resp. real) axis if and only if $h + g$ (resp. $h - g$) is an analytic univalent mapping of $\mathbb{U}$ onto a domain convex in the direction of the imaginary (resp. real) axis.*

A linear combination is an important method to construct a new function. MacGregor [19] showed that the linear combination $tf + (1-t)g$ for $0 \leq t \leq 1$ of analytic functions need not to be univalent even if $f$ and $g$ are convex functions. Some results on the linear combinations of analytic functions are obtained in [4, 19, 23]. Recently, the linear combinations of harmonic mappings have started to be studied [6, 16, 24, 25].

It is interesting and meaningful to investigate the classes of harmonic functions that map $\mathbb{U}$ onto specific domains. One can refer to [1, 7, 8, 10, 11, 17, 18]. Specifically, the collection of functions $f = h + \overline{g} \in S_H^o$ that map $\mathbb{U}$ onto the right half-plane, $R = \{w : Rew > -1/2\}$, have the form

$$h(z) + g(z) = \frac{z}{1-z}$$

and those that map $\mathbb{U}$ onto the vertical strip, $\Omega_\alpha = \{w : \frac{\alpha-\pi}{2i\sin\alpha} < \Re(w) < \frac{\alpha}{2\sin\alpha}\}$, where $\frac{\pi}{2} \leq \alpha < \pi$, have the form

$$h(z) + g(z) = \frac{1}{2i\sin\alpha} \log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right). \tag{1.3}$$

In this paper, we derive several sufficient conditions for the combination $f = tf_1 + (1-t)f_2$ to be univalent and convex in the imaginary direction, where $f_1$ and $f_2$ are univalent harmonic mappings obtained by shearing of $h + g$ as given in (1.3).

## 2. Main Results

**Theorem 2.1.** *Let* $f_i = h_j + \overline{g_j} \in S_H$, *where* $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$, $\alpha_j \in [\frac{\pi}{2}, \pi)$ *for* $j = 1, 2$. *Then* $f = tf_1 + (1-t)f_2 \in S_H$ *is convex in the direction of the imaginary axis for* $0 \le t \le 1$, *if* $f$ *is locally univalent and sense-preserving.*

*Proof.* Define $F = tF_1 + (1-t)F_2$, where $F_j = h_j + g_j = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$ for $j = 1, 2$. Let

$$\varphi_j(z) := (1-z^2)F_j'(z) = \frac{1-z^2}{(1+ze^{i\alpha_j})(1+ze^{-i\alpha_j})}.$$

It is easy to verify that $\varphi_j(0) = 1$ and $\Re[\varphi_j(z)] = 0$ for $|z| = 1$. By the minimum principle for harmonic functions with $z \in \mathbb{U}$, we have

$$\Re[\varphi_j(z)] = \Re[(1-z^2)F_j'(z)] > 0.$$

Therefore,

$$\Re[(1-z^2)F'(z)] = t\Re[\varphi_1(z)] + (1-t)\Re[\varphi_2(z)] > 0,$$

for all $z \in \mathbb{U}$. It follows from Lemma 1.1 that $F = tF_1 + (1-t)F_2 = th_1 + (1-t)h_2 + tg_1 + (1-t)g_2$ is analytic and convex in the direction of the imaginary axis. Therefore if

$$f = tf_1 + (1-t)f_2 = th_1 + (1-t)h_2 + \overline{tg_1 + (1-t)g_2}$$

is locally univalent and sense-preserving, then by Lemma 1.2 we have $f \in S_H$ is convex in the direction of the imaginary axis. □

In view of Theorem 2.1, the main object we need to focus on is the condition *f is locally univalent and sense-preserving*. Actually, we just need $\omega$, the dilation of $f$, to satisfy $|\omega| < 1$. We begin by finding an expression for $\omega$.

**Lemma 2.2.** *Let* $f_i = h_j + \overline{g_j} \in S_H$, *where* $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$, $\alpha_j \in [\frac{\pi}{2}, \pi)$ *for* $j = 1, 2$. *If* $\omega_j = g_j'/h_j'$ *are dilatation functions of* $f_j$, $j = 1, 2$, *respectively, then the dilatation function* $\omega$ *of* $f = tf_1 + (1-t)f_2$, $0 \le t \le 1$, *is given by*

$$\omega = \frac{t(1+\omega_2)(1+2z\cos\alpha_2+z^2)\omega_1 + (1-t)(1+\omega_1)(1+2z\cos\alpha_1+z^2)\omega_2}{t(1+\omega_2)(1+2z\cos\alpha_2+z^2) + (1-t)(1+\omega_1)(1+2z\cos\alpha_1+z^2)}. \tag{2.1}$$

*Proof.* Since $f = tf_1 + (1-t)f_2 = th_1 + (1-t)h_2 + \overline{tg_1 + (1-t)g_2}$,

$$\omega = \frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'} = \frac{t\omega_1 h_1' + (1-t)\omega_2 h_2'}{th_1' + (1-t)h_2'}. \tag{2.2}$$

From $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$ and $\omega_j = g_j'/h_j'$, $j = 1, 2$, we have

$$h_j'(z) = \frac{1}{(1+\omega_j(z))(1+2z\cos\alpha_j+z^2)}.$$

Substituting these into (2.2), we get (2.1). □

If $\omega_1 = \omega_2$, then (2.1) reduces to $\omega = \omega_1 = \omega_2$. Considering Theorem 2.1, we get the following result.

**Theorem 2.3.** *Let $f_i = h_j + \overline{g_j} \in S_H$, where $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$, $\alpha_j \in [\frac{\pi}{2}, \pi)$ for $j = 1, 2$. If $\omega_1 = \omega_2$, then $f = tf_1 + (1-t)f_2 \in S_H$ is convex in the direction of the imaginary axis for $0 \le t \le 1$.*

If $\alpha_1 = \alpha_2$, we have the following theorem.

**Theorem 2.4.** *Let $f_j = h_j + \overline{g_j} \in S_H$, where $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$, $\alpha_j \in [\frac{\pi}{2}, \pi)$ for $j = 1, 2$. If $\alpha_1 = \alpha_2$, then $f = tf_1 + (1-t)f_2 \in S_H$ is convex in the direction of the imaginary axis for $0 \le t \le 1$.*

*Proof.* In view of Theorem 2.1, it suffices to show that $f$ is locally univalent and sense-preserving. Substituting $\alpha_1 = \alpha_2$ into (2.1), we have

$$\omega = \frac{t(1+\omega_2)\omega_1 + (1-t)(1+\omega_1)\omega_2}{t(1+\omega_2) + (1-t)(1+\omega_1)} = \frac{t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2}{1 + t\omega_2 + (1-t)\omega_1}. \tag{2.3}$$

Let

$$\begin{aligned} I &:= |1 + t\omega_2 + (1-t)\omega_1|^2 - |t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2|^2 \\ &= 1 + 2t(1-|\omega_1|^2)\Re\omega_2 + 2(1-t)(1-|\omega_2|^2)\Re\omega_1 \\ &\quad + (1-2t)(|\omega_1|^2 - |\omega_2|^2) - |\omega_1|^2|\omega_2|^2 \end{aligned}$$

and $\omega_j = \rho_j e^{i\theta_j}$, $j = 1, 2$. Then

$$\begin{aligned} I &= 1 + 2t(1-\rho_1^2)\rho_2\cos\theta_2 + 2(1-t)(1-\rho_2^2)\rho_1\cos\theta_1 + (1-2t)(\rho_1^2 - \rho_2^2) - \rho_1^2\rho_2^2 \\ &\ge 1 - 2t(1-\rho_1^2)\rho_2 - 2(1-t)(1-\rho_2^2)\rho_1 + (1-2t)(\rho_1^2 - \rho_2^2) - \rho_1^2\rho_2^2 \\ &= (1-\rho_2^2)(1-\rho_1)^2 + 2t(\rho_1 - \rho_2)[1 + \rho_1\rho_2 - (\rho_1 + \rho_2)] := II. \end{aligned}$$

It is easy to verify that $1 + \rho_1\rho_2 - (\rho_1 + \rho_2) > 0$ for $\rho_j \in [0, 1)$, $j = 1, 2$. Actually, $0 < 1 + \rho_1\rho_2 - (\rho_1 + \rho_2) \le 1$ for all $\rho_j \in [0, 1)$, $j = 1, 2$. If $\rho_1 - \rho_2 \ge 0$, then

$$I \ge II \ge (1-\rho_2^2)(1-\rho_1)^2 > 0.$$

If $\rho_1 - \rho_2 < 0$, then

$$\begin{aligned} I \ge II &\ge (1-\rho_2^2)(1-\rho_1)^2 + 2(\rho_1 - \rho_2)[1 + \rho_1\rho_2 - (\rho_1 + \rho_2)] \\ &= (1-\rho_1^2)(1-\rho_2)^2 > 0. \end{aligned}$$

Therefore, $I > 0$ which implies $|\omega| < 1$. Hence $f$ is locally univalent and sense-preserving. □

**Remark 2.5.** *In an earlier paper [24, Thm. 3], it was claimed that for $\omega$ given in (2.3), $|\omega| < 1$. The authors assumed that the function $I$ is monotonic in $t$ for $t \in [0, 1]$ and this is not the case. Our proof establishes the validity of the result.*

The following lemma is popularly known as Cohn's Rule.

**Lemma 2.6 (Cohn's Rule, see (22, p. 375)).** *Given a polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ of degree $n$, let*

$$p^*(z) = z^n\, \overline{p\left(\frac{1}{\bar{z}}\right)} = \bar{a}_n + \bar{a}_{n-1}z + \bar{a}_{n-2}z^2 + ... + \bar{a}_0 z^n.$$

*Denote by $r$ and $s$ the number of zeros of $p$ inside and on the circle $|z| = 1$, respectively. If $|a_0| < |a_n|$, then*

$$p_1 = \frac{\bar{a}_n p(z) - a_0 p^*(z)}{z}$$

*is of degree $n-1$ and has $r_1 = r - 1$ and $s_1 = s$ number of zeros inside and on the unit circle $|z| = 1$, respectively.*

In order to get our next result, we need Lemma 6.

**Lemma 2.7.** *Let* $a \in (-1, 0) \cup (0, 1)$, $t \in (0, 1)$ *and* $\alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi)$. *If* $a(\alpha_1 - \alpha_2) > 0$, *then*

$$\begin{aligned}
&(\mathbf{1}) \quad |a(2t-1)+1| > |a(2t-1)+1+2at(1-t)(\cos\alpha_1 - \cos\alpha_2)|; && (2.4)\\
&(\mathbf{2}) \quad |a(2t-1)+1+at(1-t)(\cos\alpha_1 - \cos\alpha_2)| &&\\
&\qquad\qquad > |(at+1)t\cos\alpha_2 + (1-t)(at+1-a)\cos\alpha_1|. && (2.5)
\end{aligned}$$

*Proof.* (**1**) It is obvious that $a(2t-1)+1 > 0$ holds for $a \in (-1, 0) \cup (0, 1)$ and $t \in (0, 1)$. Thus we just need to prove the following double inequality

$$a(2t-1)+1 > a(2t-1)+1+2at(1-t)(\cos\alpha_1 - \cos\alpha_2) > -[a(2t-1)+1].$$

That is,

$$0 > at(1-t)(\cos\alpha_1 - \cos\alpha_2) > -[a(2t-1)+1]. \tag{2.6}$$

First, because $\alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi)$, then $a(\alpha_1 - \alpha_2) > 0$ is equivalent to $a(\cos\alpha_1 - \cos\alpha_2) < 0$. Therefore, for $t \in (0, 1)$ we have

$$0 > at(1-t)(\cos\alpha_1 - \cos\alpha_2). \tag{2.7}$$

Next, to prove the second inequality of (2.6), we consider two subcases.
*Subcase 1*: if $a \in (0, 1)$ and $\alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi)$, then $a(\alpha_1 - \alpha_2) > 0$ implies $-1 < \cos\alpha_1 - \cos\alpha_2 < 0$. Thus,

$$at(1-t)(\cos\alpha_1 - \cos\alpha_2) > -at(1-t) > -[a(2t-1)+1] \tag{2.8}$$

holds for $t \in (0, 1)$. The last inequality holds because of $a(t^2+t-1) > -1$ for $t \in (0, 1)$ and $a \in (0, 1)$.
*Subcase 2*: if $a \in (-1, 0)$ and $\alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi)$, then $a(\alpha_1 - \alpha_2) > 0$ implies $0 < \cos\alpha_1 - \cos\alpha_2 < 1$. Thus,

$$at(1-t)(\cos\alpha_1 - \cos\alpha_2) > at(1-t) > -[a(2t-1)+1] \tag{2.9}$$

holds for $t \in (0, 1)$. The last inequality holds because of $a(-t^2+3t-1) > -1$ for $t \in (0, 1)$ and $a \in (-1, 0)$.

Therefore, the second inequality of the double inequality (2.6) follows from inequalities (2.8) and (2.9).

**(2)** If $a(\alpha_1 - \alpha_2) > 0$, then in view of inequality (2.6) we know that $a(2t-1)+1+at(1-t)(\cos\alpha_1 - \cos\alpha_2) > 0$ for $a \in (-1, 0) \cup (0, 1)$, $t \in (0, 1)$, and $\alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi)$. So inequality (2.5) is equivalent to the double inequality

$$\begin{aligned}
&a(2t-1)+1+at(1-t)(\cos\alpha_1 - \cos\alpha_2)\\
&\quad > (at+1)t\cos\alpha_2 + (1-t)(at+1-a)\cos\alpha_1\\
&\qquad > -[a(2t-1)+1+at(1-t)(\cos\alpha_1 - \cos\alpha_2)].
\end{aligned} \tag{2.10}$$

Now, let

$$\begin{aligned}
f(a,t) :=& a(2t-1)+1+at(1-t)(\cos\alpha_1 - \cos\alpha_2)\\
&- [(at+1)t\cos\alpha_2 + (1-t)(at+1-a)\cos\alpha_1]\\
=&(1-a)(1-\cos\alpha_1)\\
&+ t[(1+a)(\cos\alpha_1 - \cos\alpha_2) + 2a(1-\cos\alpha_1)].
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial f(a,t)}{\partial a} &= -(1-\cos\alpha_1) + t[\cos\alpha_1 - \cos\alpha_2 + 2(1-\cos\alpha_1)];\\
\frac{\partial f(a,t)}{\partial t} &= (1+a)(\cos\alpha_1 - \cos\alpha_2) + 2a(1-\cos\alpha_1).
\end{aligned}$$

Let

$$\frac{\partial f(a,t)}{\partial a} = 0 \quad \text{and} \quad \frac{\partial f(a,t)}{\partial t} = 0.$$

Then we have

$$a = a_0 = \frac{\cos\alpha_2 - \cos\alpha_1}{2 - (\cos\alpha_1 + \cos\alpha_2)} \quad \text{and} \quad t = t_0 = \frac{1 - \cos\alpha_1}{2 - (\cos\alpha_1 + \cos\alpha_2)}.$$

Therefore, it is obvious that

$$f(a,t) \geq f(a_0,t_0) = \frac{2(1-\cos\alpha_1)(1-\cos\alpha_2)}{2 - (\cos\alpha_1 + \cos\alpha_2)} > 0. \tag{2.11}$$

Inequality (2.11) implies that the first inequality of the double inequality (2.10) holds.

Next, let

$$\begin{aligned} I := &(at+1)t\cos\alpha_2 + (1-t)(at+1-a)\cos\alpha_1 \\ &+ a(2t-1) + 1 + at(1-t)(\cos\alpha_1 - \cos\alpha_2) \\ = &[a(2t-1)+1][1 + \cos\alpha_1 - t(\cos\alpha_1 - \cos\alpha_2)]. \end{aligned} \tag{2.12}$$

If $\cos\alpha_1 - \cos\alpha_2 > 0$, then

$$\begin{aligned} I &> [a(2t-1)+1][1 + \cos\alpha_1 - (\cos\alpha_1 - \cos\alpha_2)] \\ &= [a(2t-1)+1](1 + \cos\alpha_2) > 0. \end{aligned} \tag{2.13}$$

If $\cos\alpha_1 - \cos\alpha_2 < 0$, then

$$I > [a(2t-1)+1](1 + \cos\alpha_1) > 0. \tag{2.14}$$

Therefore, $I > 0$ follows from inequalities (2.13) and (2.14) and the second inequality of the double inequality (2.10) is proved. □

**Theorem 2.8.** *Let $f_j = h_j + \overline{g_j} \in S_H$, where $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$, $\alpha_j \in [\frac{\pi}{2}, \pi)$ for $j = 1, 2$. If $\omega_1(z) = z$, $\omega_2(z) = \frac{z+a}{1+az}$, $a \in (-1,1)$, then $f = tf_1 + (1-t)f_2 \in S_H$ $(0 < t < 1)$ is convex in the direction of the imaginary axis provided $a(\alpha_1 - \alpha_2) \geq 0$.*

*Proof.* By Theorem 2.1, we just need to show that $|\omega| < 1$ in $\mathbb{U}$. If $a = 0$, then $\omega_2(z) = \omega_1(z) = z$ and this case can follows from Theorem 2.3. If $\alpha_1 = \alpha_2$, then the case follows from Theorem 2.4. Therefore, we shall only consider the case when $a(\alpha_1 - \alpha_2) > 0$.

Setting $\omega_1(z) = z$ and $\omega_2(z) = \frac{z+a}{1+az}$ in (2.1), we get

$$\begin{aligned} \omega(z) = &\frac{t(1 + \frac{z+a}{1+az})(1 + 2z\cos\alpha_2 + z^2)z + (1-t)(1+z)(1 + 2z\cos\alpha_1 + z^2)\frac{z+a}{1+az}}{t(1 + \frac{z+a}{1+az})(1 + 2z\cos\alpha_2 + z^2) + (1-t)(1+z)(1 + 2z\cos\alpha_1 + z^2)} \\ = &\frac{t(1+a)(1 + 2z\cos\alpha_2 + z^2)z + (1-t)(1 + 2z\cos\alpha_1 + z^2)(z+a)}{t(1+a)(1 + 2z\cos\alpha_2 + z^2) + (1-t)(1 + 2z\cos\alpha_1 + z^2)(1+az)} \\ = &\frac{p(z)}{p^*(z)}, \end{aligned}$$

where

$$\begin{aligned} p(z) = &(at+1)z^3 + [2t(1+a)\cos\alpha_2 + 2(1-t)\cos\alpha_1 + a(1-t)]z^2 \\ &+ [1 + at + 2a(1-t)\cos\alpha_1]z + a(1-t) \\ := &a_3z^3 + a_2z^2 + a_1z + a_0 \end{aligned}$$

and

$$\begin{aligned} p^*(z) =& a(1-t)z^3 + [1 + at + 2a(1-t)\cos\alpha_1]z^2 \\ &+ [2t(1+a)\cos\alpha_2 + 2(1-t)\cos\alpha_1 + a(1-t)]z + (at+1) \\ =& z^3\overline{p(\tfrac{1}{\bar{z}})}. \end{aligned}$$

Thus if $z_0$ is a zero of $p$ and $z_0 \neq 0$, then $1/\overline{z_0}$ is a zero of $p^*$, and we can rewrite

$$\omega(z) = \frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)}.$$

For $|\beta| \leq 1$, the function $\phi(z) = \frac{z+\beta}{1+\bar{\beta}z}$ maps $\overline{\mathbb{U}} = \{z : |z| \leq 1\}$ onto $\overline{\mathbb{U}}$. Hence, to prove that $|\omega| < 1$ in $\mathbb{U}$, it suffices to show that $|A| \leq 1, |B| \leq 1, |C| \leq 1$ with at least one of these having modulus strictly less than one. As $|a_3| = at + 1 > |a_0| = |a(1-t)|$ holds for all $a \in (-1,0) \cup (0,1)$ and $t \in (0,1)$, we can apply Cohn' Rule to $p$, and thus it is sufficient to show that all the zeros of $p_1$ lie inside or on $|z| = 1$, where

$$p_1(z) = \frac{a_3 p(z) - a_0 p^*(z)}{z} = (1+a)\widetilde{p_1}(z)$$

and

$$\begin{aligned} \widetilde{p_1}(z) =& [a(2t-1)+1]z^2 + [2(1-t)(at+1-a)\cos\alpha_1 + 2t(at+1)\cos\alpha_2]z \\ &+ a(2t-1) + 1 + 2at(1-t)(\cos\alpha_1 - \cos\alpha_2) \\ :=& b_2 z^2 + b_1 z + b_0. \end{aligned}$$

By Lemma 2.7(1), we have $|b_2| > |b_0|$. So we can use Cohn's Rule again on $\widetilde{p_1}$. Now, we need to show that all the zeros of $p_2$ lie inside or on $|z| = 1$, where

$$p_2(z) = \frac{b_2\widetilde{p_1}(z) - b_0\widetilde{p}_1^*(z)}{z} = -4at(1-t)(\cos\alpha_1 - \cos\alpha_2)\widetilde{p_2}(z)$$

and

$$\begin{aligned} \widetilde{p_2}(z) =& [a(2t-1) + 1 + at(1-t)(\cos\alpha_1 - \cos\alpha_2)]z \\ &+ (at+1)t\cos\alpha_2 + (1-t)(at+1-a)\cos\alpha_1 \\ :=& c_1 z + c_0. \end{aligned}$$

By Lemma 2.7(2), we have $|c_1| > |c_0|$. Hence, the zeros of $\widetilde{p_2}, p_2, \widetilde{p_1}$, and $p_1$ lie in $|z| < 1$. Therefore, $|\omega| < 1$. □

**Theorem 2.9.** *Let $f_j = h_j + \overline{g_j} \in S_H$ with $\omega_j = e^{i\theta_j}z$ , where $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}})$, $\alpha_j \in [\frac{\pi}{2}, \pi)$ and $\theta_j \in [0, 2\pi)$ for $j = 1, 2$. For each case, if the stated conditions are satisfied, then $f = tf_1 + (1-t)f_2 \in S_H$ is convex in the direction of the imaginary axis for $0 < t < 1$.*

*Case (1): $\theta_1 = \theta_2$ or $\alpha_1 = \alpha_2$;*

*Case (2):*

$$\begin{aligned} &(a) \quad (\cos\alpha_1 - \cos\alpha_2)(\cos\alpha_1 - \cos\alpha_2 + \cos\theta_1 - \cos\theta_2) < 0, \textit{and} \\ &(b) \quad |\cos\alpha_1 - \cos\alpha_2 + \cos\theta_1 - \cos\theta_2| \\ &\qquad > |\cos\alpha_1 e^{-i\theta_1} - \cos\alpha_2 e^{-i\theta_2} + i\sin(\theta_1 - \theta_2)|. \end{aligned}$$

*Proof.* By Theorem 2.1, we just need to show that dilatation $\omega$ of $f$ satisfies $|\omega| < 1$ in $\mathbb{U}$. By Theorem 2.3 and 2.4, respectively, Case (1) is true.

If $\theta_1 \neq \theta_2$ and $\alpha_1 \neq \alpha_2$, by substituting $\omega_j = e^{i\theta_j}z$, $j = 1, 2$ into the equation (2.1), we derive

$$\omega(z) = ze^{i(\theta_1+\theta_2)}\ \frac{p(z)}{p^*(z)},$$

where

$$\begin{aligned} p(z) =& z^3 + [(1-t)e^{-i\theta_1} + te^{-i\theta_2} + 2(t\cos\alpha_2 + (1-t)\cos\alpha_1)]z^2 \\ &+ [2t\cos\alpha_2 e^{-i\theta_2} + 2(1-t)\cos\alpha_1 e^{-i\theta_1} + 1]z + (1-t)e^{-i\theta_1} + te^{-i\theta_2} \\ :=& a_3z^3 + a_2z^2 + a_1z + a_0 \end{aligned}$$

and

$$p^*(z) = z^3\overline{p(\tfrac{1}{\bar{z}})} = \bar{a}_0z^3 + \bar{a}_1z^2 + \bar{a}_2z + \bar{a}_3.$$

Because of $t \in (0, 1)$ and $\theta_1 \neq \theta_2$, we have $|(1-t)e^{-i\theta_1} + te^{-i\theta_2}| < 1$. Hence, $|a_0| < |a_3|$. By Cohn's Rule, we have

$$p_1(z) = \frac{\bar{a}_3p(z) - a_0p^*(z)}{z} = 2t(1-t)(e^{-i\theta_1} - e^{-i\theta_2})\widetilde{p_1}(z),$$

where

$$\begin{aligned} \widetilde{p_1}(z) =& \frac{1}{2}(e^{i\theta_1} - e^{i\theta_2})z^2 + (\cos\alpha_1 e^{i\theta_1} - \cos\alpha_2 e^{i\theta_2})z \\ &+ \cos\alpha_1 - \cos\alpha_2 + \frac{1}{2}(e^{i\theta_1} - e^{i\theta_2}) \\ :=& b_2z^2 + b_1z + b_0. \end{aligned}$$

In view of condition (a) in Case (2), we have

$$\begin{aligned} |b_0|^2 - |b_2|^2 =& |\cos\alpha_1 - \cos\alpha_2 + \frac{1}{2}(e^{i\theta_1} - e^{i\theta_2})|^2 - \frac{1}{4}|e^{i\theta_1} - e^{i\theta_2}|^2 \\ =& (\cos\alpha_1 - \cos\alpha_2)(\cos\alpha_1 - \cos\alpha_2 + \cos\theta_1 - \cos\theta_2) < 0. \end{aligned}$$

It follows $|b_0| < |b_2|$. Therefore, we can make use of Cohn's Rule again. Let

$$\widetilde{p_1^*}(z) = z^2\overline{\widetilde{p_1}(\tfrac{1}{\bar{z}})} = \bar{b}_0z^2 + \bar{b}_1z + \bar{b}_2.$$

By direct computation, we have

$$\begin{aligned} p_2(z) :=& \frac{\bar{b}_2\widetilde{p_1}(z) - b_0\widetilde{p_1^*}(z)}{z} \\ :=& -(\cos\alpha_1 - \cos\alpha_2)\widetilde{p_2}(z) = -(\cos\alpha_1 - \cos\alpha_2)(c_1z + c_0), \end{aligned}$$

where

$$\begin{aligned} c_1 =& \cos\alpha_1 - \cos\alpha_2 + \cos\theta_1 - \cos\theta_2, \\ c_0 =& \cos\alpha_1 e^{-i\theta_1} - \cos\alpha_2 e^{-i\theta_2} + i\sin(\theta_1 - \theta_2). \end{aligned}$$

Because of condition (a) in Case (2), we have $|c_1| > |c_0|$. So the zero(s) of $\widetilde{p_2}$ and $\widetilde{p_1}$ both lie inside the unit circle $|z| = 1$. Therefore, it follows $|\omega| < 1$. $\square$

If we take specific values for $\theta_1$ and $\theta_2$, we have the following corollary.

**Corollary 2.10.** *Let* $f_j = h_j + \overline{g_j} \in S_H$*, where* $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$*,* $\alpha_j \in [\frac{\pi}{2}, \pi)$ *for* $j = 1, 2$*. For each case, if the stated conditions are satisfied, then* $f = tf_1 + (1-t)f_2 \in S_H$ *is convex in the direction of the imaginary axis for* $0 < t < 1$*.*

*Case (1): For* $\omega_1(z) = ze^{i(\theta+\frac{\pi}{2})}, \omega_2(z) = ze^{i\theta}$*.*

$(a)$ $(\cos\alpha_1 - \cos\alpha_2)(\cos\alpha_1 - \cos\alpha_2 - \sin\theta - \cos\theta) < 0,$ *and*

$(b)$ $(\cos\alpha_1 + \sin\theta)(\cos\theta - \cos\alpha_2) > (\sin\theta + \cos\theta)(\cos\alpha_1 - \cos\alpha_2).$

*Case (2): For* $\omega_1(z) = ze^{i(\theta+\pi)}, \omega_2(z) = ze^{i\theta}$*.*

$(a)$ $(\cos\alpha_1 - \cos\alpha_2)(\cos\alpha_1 - \cos\alpha_2 - 2\cos\theta) < 0,$ *and*

$(b)$ $(\cos\alpha_1 - \cos\theta)(\cos\alpha_2 + \cos\theta) < 0.$

*Case (3): For* $\omega_1(z) = z, \omega_2(z) = -z$*.*

$(a)$ $\alpha_1 > \alpha_2$.

**Theorem 2.11.** *Let* $f_j = h_j + \overline{g_j} \in S_H$*, where* $h_j(z) + g_j(z) = \frac{1}{2i\sin\alpha_j}\log\left(\frac{1+ze^{i\alpha_j}}{1+ze^{-i\alpha_j}}\right)$*,* $\alpha_j \in [\frac{\pi}{2}, \pi)$ *for* $j = 1, 2$*. For each case, if the stated conditions are satisfied, then* $f = tf_1 + (1-t)f_2 \in S_H$ *is convex in the direction of the imaginary axis for* $0 < t < 1$*.*

*Case (1): For* $\omega_1(z) = z, \omega_2(z) = z^2,$ *and* $\alpha_1 < \alpha_2$*.*

*Case (2): For* $\omega_1(z) = z, \omega_2(z) = -z^2,$ *and* $\alpha_1 > \alpha_2$*.*

*Proof.* As the proof is similar to the proof of Theorem 2.8 and Theorem 2.9, it is omitted. □

## 3. Example

In this section, we give two examples to illustrate our results.

**Example 3.1.** *If* $\alpha_1 = \alpha_2 = \frac{\pi}{2}$*, then* $h_j(z) + g_j(z) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$ *for* $j = 1, 2$*. Taking* $\omega_1(z) = z$*, we get*

$$\begin{aligned} h_1(z) =& \frac{1}{4}\log\frac{(1+z)^2}{1+z^2} + \frac{1}{2}\arctan z, \\ g_1(z) =& -\frac{1}{4}\log\frac{(1+z)^2}{1+z^2} + \frac{1}{2}\arctan z. \end{aligned}$$

*Taking* $\omega_2(z) = -z^2$*, we obtain*

$$\begin{aligned} h_2(z) =& \frac{1}{4}\log\frac{1+z}{1-z} + \frac{1}{2}\arctan z, \\ g_2(z) =& -\frac{1}{4}\log\frac{1+z}{1-z} + \frac{1}{2}\arctan z. \end{aligned}$$

*Let* $f_j = h_j + \overline{g_j}$ *for* $j = 1, 2$ *and* $f = tf_1 + (1-t)f_2$*. Then by Theorem 2.4, we know that* $f$ *is in* $S_H$ *and is convex in the direction of the imaginary axis. The images of* $\mathbb{U}$ *under* $f$ *with* $t = 0, \frac{1}{2}$ *and* $1$*, respectively, are shown in Figure 1.*

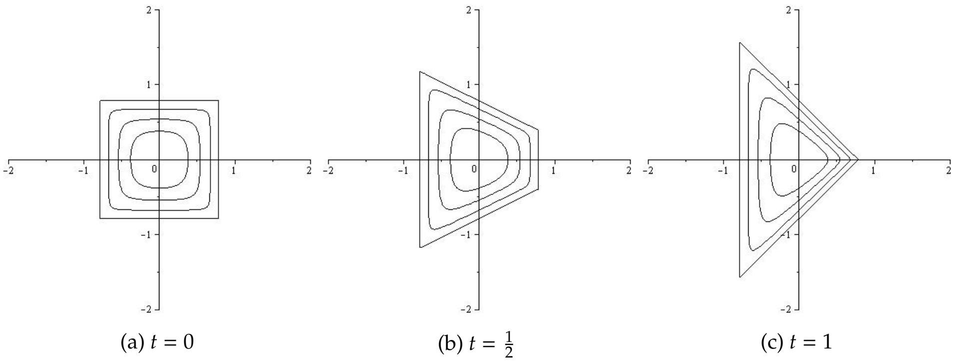


Figure 1: Images of concentric circles under $f$ of Example 3.1

**Example 3.2.** *Letting $\alpha_1 = \frac{3\pi}{4}$, we have*

$$h_1(z) + g_1(z) = \frac{1}{\sqrt{2}i} \log\left( \frac{\sqrt{2} + (-1+i)z}{\sqrt{2} - (1+i)z} \right).$$

*With $\omega_1(z) = z$, we have*

$$\begin{aligned} h_1(z) =& \tfrac{2-\sqrt{2}}{4} \log \tfrac{(1+z)^2}{1-\sqrt{2}z+z^2} + \tfrac{(1+\sqrt{2})\arctan(\sqrt{2}z-1)}{2+\sqrt{2}} + \tfrac{(1+\sqrt{2})\pi}{4(2+\sqrt{2})}, \\ g_1(z) =& -\tfrac{2-\sqrt{2}}{4} \log \tfrac{(1+z)^2}{1-\sqrt{2}z+z^2} + \tfrac{(1+\sqrt{2})\arctan(\sqrt{2}z-1)}{2+\sqrt{2}} + \tfrac{(1+\sqrt{2})\pi}{4(2+\sqrt{2})}. \end{aligned}$$

*With $\alpha_2 = \frac{\pi}{2}$, then $h_2(z) + g_2(z) = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$. Taking $\omega_2(z) = -z$, we get*

$$\begin{aligned} h_2(z) =& \tfrac{1}{4} \log \tfrac{1+z^2}{(1-z)^2} + \tfrac{1}{2} \arctan z, \\ g_2(z) =& -\tfrac{1}{4} \log \tfrac{1+z^2}{(1-z)^2} + \tfrac{1}{2} \arctan z. \end{aligned}$$

*Let $f_j = h_j + \overline{g_j}$ for $j = 1, 2$ and $f = tf_1 + (1-t)f_2$, Then Corollary 2.10 gives us that $f$ is in $S_H$ and is convex in the direction of the imaginary axis. The images of $\mathbb{U}$ under $f$ with $t = 0, \frac{1}{2}$ and $1$, respectively, are shown in Figure 2.*

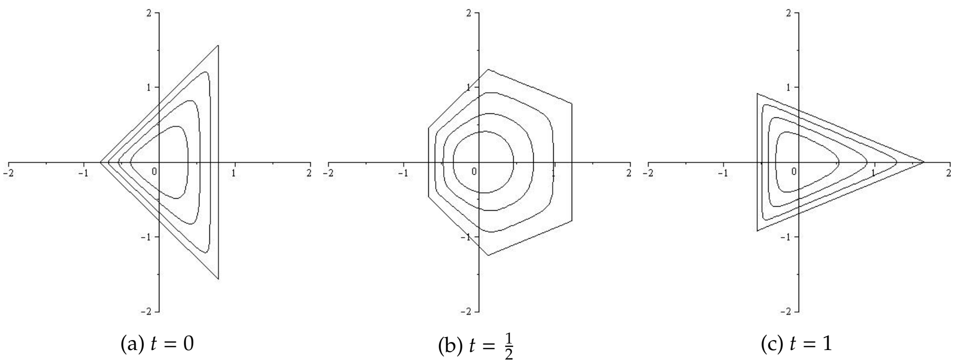


Figure 2: Images of concentric circles under $f$ of Example 3.2

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