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A Hilbert's Type Inequality With Two Parameters

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Abstract. In this paper, by introducing a parameter α and λ, using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality of a weight coefficient. Using this inequality, we derive generalizations of a Hilbert's type inequality.

1. Introduction

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1) is Hardy-Hilbert's inequality. Inequality (2) is a Hilbert's type inequality [1].

In [4], [8] and [7], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1). In [2], Kuang and Debnath gave a reinforcement of inequality (2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty}[pq - G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[pq - G(q,n)]b_n^q\right\}^{\frac{1}{q}} \tag{3}$$

where $G(r,n) = \frac{r + \frac{1}{3r} - \frac{4}{3}}{(2n+1)^{\frac{1}{r}}} > 0$ $(r = p, q)$.

In [5] and [6], Xi gave a generalizations and reinforcements of inequalities (2) and (3):

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$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^{\lambda},n^{\lambda})} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{4}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda}+A,n^{\lambda}+B\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{5}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$, $2-\min\{p,q\}<\lambda\le 2$, $0\le A\le B\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$.

In this paper, by introducing a parameter α and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality for a weight coefficient. Using this inequality, we derive a generalization of inequalities (4).

2. A Lemma

First, we need the following formula of the Riemann-ζ function (see [3], [10] and [9]):

$$\zeta(\sigma) = \sum_{k=1}^{n}\frac{1}{k^{\sigma}}-\frac{n^{1-\sigma}}{1-\sigma}-\frac{1}{2n^{\sigma}}-\sum_{k=1}^{l-1}\frac{B_{2k}}{2k}\binom{-\sigma}{2k-1}\frac{1}{n^{\sigma+2k-1}}-\frac{B_{2l}}{2l}\binom{-\sigma}{2l-1}\frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{6}$$

where $\sigma>0$, $\sigma\neq 1$, $n,\ l\ge 1$, $n,\ l\in N$, $0<\varepsilon=\varepsilon(\sigma,l,n)<1$. The numbers $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $\cdots$ are Bernoulli numbers. In particular, $\zeta(\sigma)=\sum_{k=1}^{\infty}\frac{1}{k^{\sigma}}$ $(\sigma>1)$.

Since $\zeta(0)=-1/2$, then the formula of the Riemann-ζ function (6) is also true for $\sigma=0$.

Lemma 2.1. *If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\}<\lambda\le 2$, $0\le\alpha\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, $n\ge 1$ and $n\in N$, then*

$$\omega(n,\lambda,p,\alpha)=\sum_{k=1}^{\infty}\frac{1}{\max\{k^{\lambda},n^{\lambda}\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} < n^{1-\lambda}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{\alpha}{1+\alpha}\right)\right], \tag{7}$$

and

$$\omega(n,\lambda,q,\alpha)=\sum_{k=1}^{\infty}\frac{1}{\max\{k^{\lambda},n^{\lambda}\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} < n^{1-\lambda}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{\alpha}{1+\alpha}\right)\right], \tag{8}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$. When $\lambda=1$, we have following the stronger inequality:

$$\omega(n,1,p,\alpha)=\sum_{k=1}^{\infty}\frac{1}{\max\{k,n\}+\alpha}\left(\frac{n}{k}\right)^{\frac{1}{p}} < \left[pq-\frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2+3q+5p}{12pq}-\frac{\alpha}{1+\alpha}\right)\right], \tag{9}$$

and

$$\omega(n,1,q,\alpha)=\sum_{k=1}^{\infty}\frac{1}{\max\{k,n\}+\alpha}\left(\frac{n}{k}\right)^{\frac{1}{q}} < \left[pq-\frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2+3p+5q}{12pq}-\frac{\alpha}{1+\alpha}\right)\right]. \tag{10}$$

Proof. Equalities (7) and (8) define the weight coefficient. When $2-min\{p,q\} < \lambda \le 2$, taking $\sigma = \frac{2-\lambda}{p} \ge 0$, $l = 1$, in (6), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\varepsilon_1, \tag{11}$$

where $0 < \varepsilon_1 < 1$.

Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\varepsilon_2, \tag{12}$$

where $0 < \varepsilon_2 < 1$.

In addition,

$$\begin{aligned}
\omega(n,\lambda,p,\alpha) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\} + \alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\} + \alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda+\alpha} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\} + \alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \sum_{k=1}^{n} \frac{1}{n^\lambda+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda+\alpha} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&\le \sum_{k=1}^{n} \frac{1}{n^\lambda}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda+\alpha} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda+\alpha} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.
\end{aligned}$$

Further, by (11) and (12) we have

$$\begin{aligned}
\omega(n,\lambda,p,\alpha) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2-\lambda}{p}}}\right] - \frac{1}{n^\lambda+\alpha} \\
&\quad + n^{\frac{2-\lambda}{p}}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12p\,qn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right] \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{\,1-\lambda}}{p+\lambda-2} + \frac{1}{2n^\lambda} - \frac{1}{n^\lambda+\alpha} + \frac{qn^{\,1-\lambda}}{q+\lambda-2} \\
&\quad + \frac{1}{2n^\lambda} + \frac{p\lambda+2q}{12p\,qn^{1+\lambda}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pq\,\lambda\ n^{\,1-\lambda}}{(p+\lambda-2)(q+\lambda-2)} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} + \frac{\alpha}{n^\lambda(n^\lambda+\alpha)} \\
&= n^{1-\lambda}\left\{\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12p\,qn^{\frac{p-\lambda+2}{p}}} - \frac{\alpha}{n^{\frac{2-\lambda}{p}}(n^\lambda+\alpha)}\right]\right\}.
\end{aligned} \tag{13}$$

In (11), taking $n = 1$, by $2 - \min\{p,q\} < \lambda \le 2$, we obtain

$$\begin{aligned}\zeta\left(\frac{2-\lambda}{p}\right) &= 1-\frac{p}{p+\lambda-2}-\frac{1}{2}+\frac{(2-\lambda)\varepsilon_1}{12p}\\ &< \frac{1}{2}-\frac{p}{p+\lambda-2}+\frac{2-\lambda}{12p}\\ &= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}\\ &< 0.\end{aligned}$$

So for $n \geq 1, n \in N, 2-\min\{p,q\} < \lambda \leq 2, 0 \leq \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, we have

$$\begin{aligned}&-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}}-\frac{\alpha}{n^{\frac{2-\lambda}{p}}(n^\lambda+\alpha)}\\ &> \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}-\frac{p\lambda+2q}{12pq}-\frac{\alpha}{1+\alpha}\\ &= \frac{q(\lambda-2-3p)(\lambda-2-2p)-(p\lambda+2q)(p+\lambda-2)}{12pq(p+\lambda-2)}-\frac{\alpha}{1+\alpha}\\ &> \frac{-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)}-\frac{\alpha}{1+\alpha}\\ &\geq \frac{-(2p+2q)+6pq}{12q(p+\lambda-2)}-\frac{\alpha}{1+\alpha}\\ &> \frac{1}{3(p+\lambda-2)}-\frac{\alpha}{1+\alpha}\\ &> \frac{1}{3p}-\frac{\alpha}{1+\alpha}\\ &\geq 0.\end{aligned} \tag{14}$$

Using the last result (14) and the inequality (13) for $\omega(n,\lambda,p,\alpha)$, we obtain (7).

When $\lambda = 1$, we have

$$\begin{aligned}&-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}}-\frac{\alpha}{n^{\frac{2-\lambda}{p}}(n^\lambda+\alpha)}\\ &> \frac{q(\lambda-2)^2+(p\lambda+5pq+2q)(2-\lambda)-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)}-\frac{\alpha}{1+\alpha}\\ &= \frac{p+3q-p^2+3pq+6p^2q}{12pq(p-1)}-\frac{\alpha}{1+\alpha}\\ &= \frac{5p^2+10p+12q}{12pq(p-1)}-\frac{\alpha}{1+\alpha}\\ &= \frac{(5p^2+10p+12q)(q-1)}{12pq}-\frac{\alpha}{1+\alpha}\\ &= \frac{12q^2+3q+5p}{12pq}-\frac{\alpha}{1+\alpha}.\end{aligned}$$

Using the last result and the inequality (13) for $\omega(n,\lambda,p,\alpha)$, we obtain (9).

In a similar way, one can prove (8) and (10). □

3. Main Results

Theorem 3.1. *If* $p,q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\} < \lambda \le 2$, $0 \le \alpha \le \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $a_n \ge 0$, $b_n \ge 0$, *for* $n \ge 1, n \in N$ *and* $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}+\alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{15}$$

and

$$\sum_{m=1}^{\infty}m^{(p-1)(\lambda-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda,n^\lambda\}+\alpha}\right)^p < \kappa(\lambda)^{p-1}\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}a_n^p, \tag{16}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$. *When* $\lambda = 1$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}+\alpha} < \left\{\sum_{n=1}^{\infty}\left[pq-\frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2+3p+5q}{12pq}-\frac{\alpha}{1+\alpha}\right)\right]a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[pq-\frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2+3q+5p}{12pq}-\frac{\alpha}{1+\alpha}\right)\right]b_n^q\right\}^{\frac{1}{q}}. \tag{17}$$

Proof. By Hölder inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}+\alpha} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{(\max\{m^\lambda,n^\lambda\}+\alpha)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]\left[\frac{b_n}{(\max\{m^\lambda,n^\lambda\}+\alpha)^{\frac{1}{q}}}(\frac{n}{m})^{\frac{2-\lambda}{pq}}\right]\\ &\le \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m^p}{\max\{m^\lambda,n^\lambda\}+\alpha}(\frac{m}{n})^{\frac{2-\lambda}{q}}\right]\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\max\{m^\lambda,n^\lambda\}+\alpha}(\frac{n}{m})^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}}\\ &= \left\{\sum_{m=1}^{\infty}\omega(m,\lambda,q,\alpha)a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(n,\lambda,p,\alpha)b_n^q\right\}^{\frac{1}{q}}.\end{aligned}$$

By (7), (8), (9) and (10), we obtain (15) and (17).
By Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda,n^\lambda\}+\alpha} &= \sum_{n=1}^{\infty}\left[\frac{1}{(\max\{m^\lambda,n^\lambda\}+\alpha)^{\frac{1}{p}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}a_n\frac{1}{(\max\{m^\lambda,n^\lambda\}+\alpha)^{\frac{1}{q}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]\\ &\le \left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda,n^\lambda\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda,n^\lambda\}+\alpha}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}}\\ &= \left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda,n^\lambda\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}[\omega(m,\lambda,p,\alpha)]^{\frac{1}{q}}\\ &< \left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda,n^\lambda\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}[m^{1-\lambda}\kappa(\lambda)]^{\frac{1}{q}}.\end{aligned}$$

So

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda, n^\lambda\} + \alpha} \right)^p < \kappa(\lambda)^{p-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\} + \alpha} \left(\frac{n}{m} \right)^{\frac{2-\lambda}{q}} a_n^p \right]$$
$$< \kappa(\lambda)^{p-1} \sum_{n=1}^{\infty} \omega(n, \lambda, q, \alpha) a_n^p.$$

By Lemma 2.1, the proof of the theorem is completed. □

In inequality (17), taking $p = q = 2$, we have:

Corollary 3.2. *Let* $a_n \geq 0$, $b_n \geq 0$, $0 \leq \alpha \leq \frac{1}{5}$, *and* $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, *then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\} + \alpha} < 4 \left\{ \sum_{n=1}^{\infty} \left[1 - \frac{1}{3\sqrt{n}} \left(1 - \frac{3\alpha}{4 + 4\alpha} \right) \right] a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} \left[1 - \frac{1}{3\sqrt{n}} \left(1 - \frac{3\alpha}{4 + 4\alpha} \right) \right] b_n^2 \right\}^{\frac{1}{2}}. \quad (18)$$

In inequality (15), taking $\alpha = 0$, we obtain:

Corollary 3.3. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, $2 - \min\{p, q\} < \lambda \leq 2$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3qn^{\frac{q+\lambda-2}{q}}} \right] n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{3pn^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}. \quad (19)$$

Apparently, inequality (15) is a generalization of inequality (4).

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