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Notes on the Results of Lower Bounds for a Class of Harmonic Functions in the Half Space

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Abstract. In this note, we point out several gaps in the paper "On the lower bound for a class of harmonic functions in the half space" by Zhang, Deng and Kou (Acta Math. Sci. Ser. B Engl. Ed., 32(4), 2012) and give the main results under weaker conditions.

The origin of our work lies in Zhang, Deng and Kou [5]. In [5] Lemmas 1 and 2 and therefore also Theorem 1 are erroneous. We give now the correction of these statements. The present notation and terminology in the same as used in [5].

To this end, we start with an auxiliary proposition. Actually, this proposition is a direct corollary of [2, p. 3296], in which harmonic majorization Theorems with respect to a half-space and their applications were introduced. But it plays an important role in our discussions.

Proposition 1. *Let H be an admissible domain with boundary ∂H in $\mathbf{R}^n$. If u and v are two harmonic functions in $\overline{H}$, then we have*

$$\int_{\partial H}\left(u(x)\frac{\partial v(x)}{\partial n}-v(x)\frac{\partial u(x)}{\partial n}\right)d\sigma(x)=0,$$

where $d\sigma(x)$ is the surface element of sphere in H and $\partial/\partial n$ denotes differentiation along the inward normal into H.

We now return to [5, Lemma 1] and give a corrected proof of it. This result does not seem easy to be proved, hence we refer to utilize a slightly different approach. For more details about this procedure we refer to [1], where a different problem is studied by a similar argument.

Lemma 1. *Let $u(x)$ be a harmonic function in the upper half space $\mathbf{R}^n_+$ and continuous on $\partial\mathbf{R}^n_+$. Then*

$$\int_{\{x\in\mathbf{R}^n_+:|x|=R\}}u(x)\frac{nx_n}{R^{n+1}}d\sigma(x)+\int_{\{x\in\mathbf{R}^n_+:r<|x'|<R\}}u(x')\left(\frac{1}{|x'|^n}-\frac{1}{R^n}\right)dx'=c_1(r)+\frac{c_2(r)}{R^n} \tag{1}$$

for $0<r<R$, where

$$c_1(r)=\int_{\{x\in\mathbf{R}^n_+:|x|=r\}}\left(\frac{(n-1)x_n}{r^{n+1}}u(x)+\frac{x_n}{r^n}\frac{\partial u(x)}{\partial n}\right)d\sigma(x)$$

and

$$c_2(r)=\int_{\{x\in\mathbf{R}^n_+:|x|=r\}}\left(\frac{x_n}{r}u(x)-x_n\frac{\partial u(x)}{\partial n}\right)d\sigma(x).$$

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Remark 1. *In [5, Lemma 1] the definition of $\partial u/\partial n$ is inaccurate, the expressions of $c_1(r)$ and $c_2(r)$ are incorrect.*

Proof. Put

$$v(x) = \frac{x_n}{|x|^n} - \frac{x_n}{R^n}$$

in

$$B^+(r,R) = \{x \in \mathbf{R}^n_+ : r < |x| < R\}.$$

It is easy to see that $v(x)$ is a harmonic function in $B^+(r,R)$. It follows that

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{nx_n}{R^{n+1}} \tag{2}$$

on the half sphere $\{x \in \mathbf{R}^n_+ : |x| = R\}$,

$$\frac{\partial v(x)}{\partial n} = -\frac{x_n}{r}\left(\frac{n-1}{r^n} + \frac{1}{R^n}\right) \tag{3}$$

on the half sphere $\{x \in \mathbf{R}^n_+ : |x| = r\}$ and

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{1}{|x|^n} - \frac{1}{R^n} \tag{4}$$

on the set $\{x \in \mathbf{R}^n_+ : r < |x'| < R\}$.

By applying Proposition 1 to two harmonic functions $u(x)$ and $v(x)$ in $B^+(r,R)$, we obtain that

$$U_1 + U_2 + U_3 = 0, \tag{5}$$

where

$$U_1 = \int_{\{x\in\mathbf{R}^n_+:|x|=R\}} \left(u(x)\frac{\partial v(x)}{\partial n} - v(x)\frac{\partial u(x)}{\partial n}\right) d\sigma(x),$$

$$U_2 = \int_{\{x\in\mathbf{R}^n_+:|x|=r\}} \left(u(x)\frac{\partial v(x)}{\partial n} - v(x)\frac{\partial u(x)}{\partial n}\right) d\sigma(x)$$

and

$$U_3 = \int_{\{x\in\mathbf{R}^n_+:r<|x|<R\}} \left(u(x)\frac{\partial v(x)}{\partial n} - v(x)\frac{\partial u(x)}{\partial n}\right) d\sigma(x).$$

It follows that

$$U_1 = \int_{\{x\in\mathbf{R}^n_+:|x|=R\}} u(x)\frac{nx_n}{R^{n+1}} d\sigma(x), \quad U_2 = -c_1(r) - \frac{c_2(r)}{R^n}$$

and

$$U_3 = \int_{\{x\in\mathbf{R}^n_+:r<|x|<R\}} u(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right) dx',$$

from (2), (3) and (4), respectively, which together with (5) give that (1) holds.

This lemma is proved. □

The proof of [5, Lemma 2] fails at Line 3, p. 1491. The formula

$$G^+_R(x,y) = G^+_R(x,y) - G^+_R(x^*,y)$$

should read

$$G^+_R(x,y) = G^+_R(x^*,y) - G^+_R(x,y^*).$$

More importantly, the definition of the set B^+_R is incorrect. Moreover, the hypothesis $n > 2$ should be added in Lemma 2.

A correction of Lemma 2 reads as follows, which improve the corresponding one established by Kuran in [2].

Lemma 2. *Let $n > 2$ and $u(x)$ be defined as in Lemma 1. Then*

$$\begin{aligned} u(x) \quad = \quad & \int_{\{y \in \mathbf{R}^n_+ :\, |y|=R\}} \frac{R^2 - |x|^2}{\omega_n R} \Big(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \Big) u(y) d\sigma(y) \\ & + \frac{2x_n}{\omega_n} \int_{\{y \in \overline{\mathbf{R}^n_+} : |y'|<R\}} \Big(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y' - \widetilde{x}|^n} \Big) u(y') dy' \end{aligned}$$

for any

$$x \in \{x \in \overline{\mathbf{R}^n_+} : |x| \leq R\},$$

where $\widetilde{x} = R^2 x/|x|^2$ *and* $x^* = (x', -x_n)$.

Finally, what we get instead of [5, Theroem 1] is the following. The proof of it is carried out in the same way as for Theorem 1 in [5], except that instead of the erroneous Lemmas 1 and 2 their corrected versions above are used.

Theorem 1. *Let $u(x)$ be a harmonic function in $\mathbf{R}^n_+$ and continuous on ∂R^n_+. Suppose that*

$$u(x) \leq K r^{\rho}, \quad x \in \mathbf{R}^n_+, \quad r = |x| > 1, \quad \rho > 1 \tag{6}$$

and

$$u(x) \geq -K, \quad |x| \leq 1, \quad x_n \geq 0. \tag{7}$$

Then the result in [5, Theorem 1] holds.

Remark 2. *Conditions (6) and (7) are weaker than conditions (1) and (2) in [5, Theorem 1]. For the conical version of Theorem 1, we refer the reader to the paper by Armitage [1] and Li & Vetro [3].*

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