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Multivalued Fixed Point Theorem in b-Metric Spaces and its Application to Differential Inclusions

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Abstract. Our purpose in this paper is to present a fixed point result for multivalued mappings satisfying nonlinear quasi-contractive condition only on related points. Moreover, we provide a qualitative study of well-posedness, limit shadowing property and Ulam-Hyers stability of our fixed point problem. As application, we discuss the existence of a unique solution for a class of differential inclusions.

Introduction

The study of existence of fixed points for multivalued mappings is of increasing interest, and has numerous applications in optimal control, mathematical economics, mechanical systems. For instance, the seeking of optimal strategies, equilibrium costs or points of rest of dynamical systems may be derived from studying the existence of fixed points for some integral or differential inclusions (see, e.g., [1, 2]). In 1969, Nadler [3] established one of the most useful theorems in multivalued fixed point analysis. Because of its importance, Nadler's result has been developed in various directions and in more general spaces (see, e.g., [4–9]).

We intend in this paper to develop Nadler's result in the context of b-metric spaces of Czerwik [10]. More precisely, we present some sufficient conditions under which a multivalued mapping has fixed points, by using a nonlinear contractive condition of Ćirić-type [11], where the nonlinearity is controlled by an appropriate class of Matkowski functions [12], and the contraction is satisfied only on related points. Some results in the existing literature are obtained as special cases of our result. Moreover, we provide a qualitative study of well-posedness, limit shadowing property and Ulam-Hyers stability of our fixed point problem. For more details on this topics we refer the reader to [13–21]. Finally, as application, we discuss the existence of a unique solution for a class of first-order differential inclusions.

This paper is divided into four sections. Section 1, introduces the notations used throughout this paper and gather together some known results on b-metric spaces. In Section 2, we present the fixed point result followed by its consequences. Results on well-posedness, limit shadowing properties and Ulam-Hyers stability are presented in Section 3. Finally, Section 4 is devoted to discuss the existence of solutions for some differential inclusions.

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1. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, $\mathbb{R}$ the set of all real numbers, $\mathbb{R}^+$ the set of all nonnegative real numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, we denote by $\overline{\mathcal{P}}(X)$ (resp. $\mathcal{P}(X)$) the family of all subsets (resp. nonempty subsets) of a nonempty set X. A fixed point of a multivalued mapping $T : X \to \mathcal{P}(X)$ is an $x_* \in X$ satisfying the fixed point inclusion, that is,

$$x_* \in Tx_*. \tag{1}$$

The fixed points set containing all solutions of the fixed point problem (1) is denoted by $Fix(T) = \Big\{x \in X : x \in Tx\Big\}$. Next, let us recall some definitions and properties on b-metric spaces.

Definition 1.1. *Let X be a non empty set and let the real $b \geq 1$. A function d_b is called a b-metric when it belongs to the class of functions $d : X \times X \to \mathbb{R}^+$ satisfying:*

(i) *$d(x, y) = 0$ if and only if $x = y$;*

(ii) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

(iii) *$d(x, y) \leq b\, d(x, z) + b\, d(z, y)$ for all $x, y, z \in X$.*

The pair (X, d_b) is called a b-metric space.

Definition 1.2 (See [22])**.** *Let X be a topological space.*

(1) *A subset U of X is called sequentially open if each sequence $\{x_n\}$ in X converging to a point x in U is eventually in U, that is, there exists n_0 such that $x_n \in U$ for all $n \geq n_0$.*

(2) *A subset F of X is called sequentially closed if no sequence in F converges to a point not in F.*

(3) *X is called a sequential space if each sequentially open subset of X is open, equivalently, each sequentially closed subset of X is closed.*

Definition 1.3 (See [22])**.** *Let (X, d_b) be a b-metric space and $\Big\{x_n\Big\}_{n\in\mathbb{N}_0}$ be a sequence in X. Then*

(1) *$\Big\{x_n\Big\}_{n\in\mathbb{N}_0}$ is called convergent to an element $x \in X$ if and only if for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}_0$ such that for all $n \geq n_\varepsilon$ we have $d_b(x_n, x) < \varepsilon$.*

(2) *$\Big\{x_n\Big\}_{n\in\mathbb{N}_0}$ is called Cauchy if and only if for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}_0$ such that for each $n, m \geq n_\varepsilon$ we have $d_b(x_n, x_m) < \varepsilon$.*

(3) *(X, d_b) is called complete if every Cauchy sequence in (X, d_b) is a convergent sequence.*

Definition 1.4 (See [22])**.** *Let (X, d_b) be a b-metric space. A subset A of X is called open if for $a \in A$, there exists $\varepsilon > 0$ such that $B(a, r) \subset A$, where*

$$B(a, r) = \{y \in X : d_b(x, y) < r\}.$$

Hence, the family of all open subset in X denoted by $\mathfrak{T}_{d_b}$ form a topology on X.

Remark 1.5 ([22, Proposition 3.3])**.** *Every open (resp. closed) is an open (resp. closed) subset of $\mathfrak{T}_{d_b}$.*

Definition 1.6. *The set of closed, compact, and compact convex subsets of X are denoted by $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cp}(X)$ and $\mathcal{P}_{cp,cv}(X)$, that is, $\mathcal{P}_{cl}(X) = \Big\{Y \in \mathcal{P}(X) : Y \text{ is closed }\Big\}$, $\mathcal{P}_{cp}(X) = \Big\{Y \in \mathcal{P}(X) : Y \text{ is compact }\Big\}$ and $\mathcal{P}_{cp,cv}(X) = \Big\{Y \in \mathcal{P}_{cp}(X) : Y \text{ is convex}\Big\}$.*

Let (X, d_b) be a b-metric space. The gap functional will be denoted by $D_b : \overline{\mathcal{P}}(X) \times \overline{\mathcal{P}}(X) \to \mathbb{R} \cup \{+\infty\}$, and it is given by

$$D_b(A, B) = \begin{cases} \inf\left\{d_b(a, b) : a \in A, b \in B\right\}, & \text{if } A, B \in \mathcal{P}(X) \\ 0, & \text{if } A = B = \emptyset \\ +\infty, & \text{otherwise} \end{cases}$$

In particular, if $x_0 \in X$, the notation $D_b(x_0, A)$ will be used to represent $D_b(\{x_0\}, A)$. The generalized diameter functional will be denoted by $\delta_b : \overline{\mathcal{P}}(X) \times \overline{\mathcal{P}}(X) \to \mathbb{R} \cup \{+\infty\}$, and it is given by

$$\delta_b(A, B) = \begin{cases} \sup\left\{d_b(a, b) : a \in A, b \in B\right\}, & \text{if } A, B \in \mathcal{P}(X) \\ 0, & \text{if } A = B = \emptyset \\ +\infty, & \text{otherwise} \end{cases}$$

The notation $\delta_b(A)$ will be used to represent $\delta_b(A, A)$, and for any $x_0 \in X$ the notation $\delta_b(x_0, B)$ will be used to represent $\delta_b(\{x_0\}, B)$. Let's recall also the result of Czerwik [5].

Proposition 1.7 ([5]). *Let (X, d_b) be a b-metric space and $A \in \mathcal{P}(X)$. Then,*

(1) $\delta_b(x, A) \leq b\, d_b(x, y) + b\, \delta_b(y, A)$, *for all $x, y \in X$;*

(2) $D_b(x, A) \leq b\, d_b(x, y) + b\, D_b(y, A)$, *for all x, $y \in X$ and $A \subset X$.*

(3) $D_b(x, A) = 0$ *if and only if $x \in \overline{A}$, where $\overline{A}$ is the closure of A.*

In the sequel, a binary relation $\mathcal{R}$ on a nonempty set X will mean a subset of $X \times X$. We say that a points $x, y \in X$ are *$\mathcal{R}$-related* if the pair $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. A partial order (resp. an equivalence relation) is a reflexive, transitive binary relation which is antisymmetric (resp. symmetric) on a set X. Next, we define a binary relation on $\mathcal{P}(X)$.

Definition 1.8. *Let X be a non empty set endowed with a binary relation $\mathcal{R}$. We define a binary relation $\mathcal{S}_{\mathcal{R}}$ on $\mathcal{P}(X)$ by*

$$(A, B) \in \mathcal{S}_{\mathcal{R}} \iff \text{ for all } x \in A, \text{ there exists } y \in B \text{ such that } (x, y) \in \mathcal{R}.$$

Remark 1.9. *It is worthwhile noting that if $\mathcal{R}$ is transitive (resp. reflexive) then $\mathcal{S}_{\mathcal{R}}$ is transitive (resp. reflexive). However, $\mathcal{S}_{\mathcal{R}}$ is not necessarily symmetric (resp. anti-symmetric) if $\mathcal{R}$ is symmetric (resp. anti-symmetric).*

Remark 1.10. *Let X be a non empty set and $\mathcal{S}$ be an arbitrary binary relation defined on $\mathcal{P}(X)$. For every $x_0 \in X$, the notations $(A, x_0) \in \mathcal{S}$ and $(x_0, B) \in \mathcal{S}$ will be used to represent $\left(A, \left\{x_0\right\}\right) \in \mathcal{S}$ and $\left(\left\{x_0\right\}, B\right) \in \mathcal{S}$, respectively.*

Next, we introduce two classes of multivalued mappings.

Definition 1.11. *Let X be a non empty set, $\mathcal{R}$ be a binary relation on X and $\mathcal{S}_{\mathcal{R}}$ be a binary relation on $\mathcal{P}(X)$. We say that a multivalued mapping $T : X \to \mathcal{P}(X)$ is $\mathcal{S}_{\mathcal{R}}$-preserving if*

$$x, y \in X : (x, y) \in \mathcal{R} \Longrightarrow (Tx, Ty) \in \mathcal{S}_{\mathcal{R}}.$$

We say that T is $\mathcal{S}_{\mathcal{R}}$-preserving from $x_0 \in X$ if T is $\mathcal{S}_{\mathcal{R}}$-preserving and $(x_0, Tx_0) \in \mathcal{S}_{\mathcal{R}}$.

Definition 1.12. *Let X be a non empty set and $\mathcal{S}$ be an arbitrary binary relation on $\mathcal{P}(X)$. We say that a multivalued mapping $T : X \to \mathcal{P}(X)$ is $\mathcal{S}$-proper if*

$$(x, Tx) \in \mathcal{S}, \text{ for all } x \in X.$$

Definition 1.13. *Let X be a non empty set, $\mathcal{R}$ be a binary relation and $T : X \to \mathcal{P}(X)$ be a multivalued mapping. For $x_0 \in X$, the set of all sequences $\{x_i\}_{i \in \mathbb{N}_0}$ satisfying $x_{i+1} \in Tx_i$ with $(x_i, x_{i+1}) \in \mathcal{R}$ for all $i \in \mathbb{N}_0$ will be denoted by $\mathcal{T}(x_0, \mathcal{R})$. Let $\xi = \{x_i\}_{i \in \mathbb{N}_0}$ be an element in $\mathcal{T}(x_0, \mathcal{R})$. For every $m, n \in \mathbb{N}_0$ with $m \leq n$, the set $O_{m,n}(\xi) := \{x_m, x_{m+1}, x_{m+2}, \cdots, x_n\}$ is called the ξ–orbit from m to n, and the set $O_{m,\infty}(\xi) := \{x_m, x_{m+1}, x_{m+2}, \cdots\}$ is called the ξ–orbit from m.*

Definition 1.14. *Let X be a non empty set, $\mathcal{R}$ be a binary relation and $T : X \to \mathcal{P}(X)$ be a multivalued mapping. A b-metric space (X, d_b) is called T-orbitally complete if and only if for all $x_0 \in X$ and $\xi \in \mathcal{T}(x_0, \mathcal{R})$ we have: if ξ is a Cauchy sequence, then ξ converges in X.*

Definition 1.15. *Let X be a non empty set and $\mathcal{R}$ be a binary relation on X. We say that (X, d_b) is $\mathcal{R}$-regular if for every convergent sequence $\{x_i\}$ to some point $x_* \in X$ such that x_i and x_{i+1} are $\mathcal{R}$-related for all $i \in \mathbb{N}_0$, we have $(x_i, x_*) \in \mathcal{R}$ for all $i \in \mathbb{N}_0$.*

The set of simulation functions as well as the nonlinear contraction are defined below.

Definition 1.16. *Let r be a positive real number. The set of simulation functions is denoted by Φ_r and contains all increasing functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:*

(i) $\lim\limits_{t \to +\infty} (t - r\,\varphi(t)) = +\infty$;

(ii) $\lim\limits_{n \to +\infty} \varphi^n(t) = 0$, *for all* $t > 0$;

(iii) $r\,\varphi(t) < t$, *for all* $t > 0$.

Remark 1.17. *Note that condition* (ii) *implies that $\varphi(t) < t$ for all $t > 0$.*

Remark 1.18. *Observe that if $r = 1$, then* (ii) *implies* (iii)*, and if $r > 1$, then* (iii) *implies* (ii)*. Moreover, if $r_1 \leq r_2$, then $\Phi_{r_2} \subseteq \Phi_{r_1}$ (see [23] for further discussion on simulation functions).*

Definition 1.19. *Let (X, d_b) be a b-metric space and $\mathcal{R}$ be a binary relation on X. We say that $T : X \to \mathcal{P}(X)$ is a multivalued φ-quasi-contractive on $\mathcal{R}$-related points if there exists $\varphi \in \Phi_b$ such that*

$$\delta_b(Tx, Ty) \leq \varphi(M_b(x, y)), \text{ for all } (x, y) \in \mathcal{R}, \text{ where}$$
$$M_b(x, y) = \max\left\{d_b(x, y), D_b(x, Tx), D_b(y, Ty), D_b(x, Ty), D_b(y, Tx)\right\}. \tag{2}$$

2. Multivalued fixed point theorems

In this section, we state and prove our multivalued fixed point theorems.

Theorem 2.1. *Let (X, d_b) be a b-metric space, $\mathcal{R}$ be a transitive binary relation on X and $T : X \to \mathcal{P}_{cl}(X)$ be a multivalued mapping such that:*

(T1) *X is $\mathcal{R}$-regular and T-orbitally complete;*

(T2) *T is either $\mathcal{S}_{\mathcal{R}}$-preserving from some $x_0 \in X$ or T is $\mathcal{S}_{\mathcal{R}}$-proper;*

(T3) *T is a multivalued φ-quasi-contractive on $\mathcal{R}$-related points.*

Then T has at least one fixed point x_* in X.

Before proving the theorem, we need to establish three lemmas which will be presented only for the case of T is $\mathcal{S}_{\mathcal{R}}$-preserving from some $x_0 \in X$. However, if T is $\mathcal{S}_{\mathcal{R}}$-proper, the proof is similar, and hence omitted.

Lemma 2.2. *Let $\mathcal{R}$ be a binary relation on a non empty set X, and $T : X \to \mathcal{P}(X)$ be a multivalued mapping. Suppose that* (T1) *and* (T2) *hold, then $\mathcal{T}(x_0, \mathcal{R})$ is nonempty.*

Proof. To show that $\mathcal{T}(x_0, \mathcal{R})$ is not empty, we have to prove the existence of some sequences $\left\{x_i\right\}_{i\in\mathbb{N}_0}$ satisfying $x_{i+1} \in Tx_i$ and $(x_i, x_{i+1}) \in \mathcal{R}$ for all $i \in \mathbb{N}_0$. We construct such sequence by induction on i. The basis step is obtained immediately from (T1). Indeed, $(x_0, Tx_0) \in \mathcal{S}_{\mathcal{R}}$, so by definition of $\mathcal{S}_{\mathcal{R}}$ there exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathcal{R}$. Now for the induction step, suppose that we have $x_i \in Tx_{i-1}$ and $(x_{i-1}, x_i) \in \mathcal{R}$ for some $i \in \mathbb{N}$, hence using the fact that T is $\mathcal{S}_{\mathcal{R}}$-preserving, it follows that $(Tx_{i-1}, Tx_i) \in \mathcal{S}_{\mathcal{R}}$, then by definition of $\mathcal{S}_{\mathcal{R}}$, there exists $x_{i+1} \in Tx_i$ such that $(x_i, x_{i+1}) \in \mathcal{R}$ and this achieves the proof. $\square$

Lemma 2.3. *Under hypotheses of Theorem 2.1. For every $m, n \in \mathbb{N}_0$ with $m < n$, and for every $\xi = \left\{x_i\right\}_{i\in\mathbb{N}_0} \in \mathcal{T}(x_0, \mathcal{R})$ there exists an integer k satisfying $m < k \leq n$ such that*

$$d_b(x_m, x_k) = \delta_b(O_{m,n}(\xi)).$$

Moreover, every sequence $\xi = \left\{x_i\right\}_{i\in\mathbb{N}_0} \in \mathcal{T}(x_0, \mathcal{R})$ is bounded.

Proof. From Lemma 2.2, it follows that $\mathcal{T}(x_0, \mathcal{R})$ is nonempty.
Let $\xi = \left\{x_i\right\}_{i\in\mathbb{N}_0}$ be a sequence in $\mathcal{T}(x_0, \mathcal{R})$ and $m, n \in \mathbb{N}_0$ with $m < n$, then we have

$$x_{i+1} \in Tx_i \text{ and } (x_i, x_{i+1}) \in \mathcal{R}.$$

As $\mathcal{R}$ is transitive, then for all $i, j \in \mathbb{N}$ with $i < j$, we deduce

$$(x_{i-1}, x_{j-1}) \in \mathcal{R}.$$

So, by condition (T2) it follows that

$$\delta_b(Tx_{i-1}, Tx_{j-1}) \leq \varphi(M_b(x_{i-1}, x_{j-1})). \tag{3}$$

Now, from Remark 1.17, we obtain

$$\varphi(M_b(x_{i-1}, x_{j-1})) < M_b(x_{i-1}, x_{j-1}). \tag{4}$$

Observe now that $x_i \in Tx_{i-1}$ and $x_j \in Tx_{j-1}$, so we have

$$d_b(x_i, x_j) \leq \delta_b(Tx_{i-1}, Tx_{j-1}) \tag{5}$$

and

$$M_b(x_{i-1}, x_{j-1}) \leq \delta_b(O_{m,n}(\xi)), \tag{6}$$

where $m < i < j \leq n$. Therefore, combining 3, 4, 5 and 6, we deduce that,

$$d_b(x_i, x_j) < \delta_b(O_{m,n}(\xi)),$$

which implies the existence of some integer k satisfying $m < k \leq n$ such that

$$d_b(x_m, x_k) = \delta_b(O_{m,n}(\xi)),$$

and this prove the first assertion of the lemma. We shall next show that the sequence ξ is bounded. Observe first that

$$O_{0,1}(\xi) \subseteq O_{0,2}(\xi) \subseteq \cdots$$

which implies that

$$\delta_b(O_{0,1}(\xi)) \leq \delta_b(O_{0,2}(\xi)) \leq \cdots$$

So, the sequence $\left\{\delta_b(O_{0,n}(\xi))\right\}_{n\in\mathbb{N}_0}$ is increasing. To prove our result, it suffice to show that $\left\{\delta_b(O_{0,n}(\xi))\right\}_{n\in\mathbb{N}_0}$ is bounded. From the first assertion of this lemma there is a $k \in \mathbb{N}$ satisfying $k \leq n$ such that $d_b(x_0, x_k) = \delta(O_{0,n}(\xi))$. If $k = 1$ for all $n \geq 1$, then $\delta_b(O_{0,n}(\xi)) = d_b(x_0, x_1)$ for all n. Otherwise, there exist $N > 0$ such that $k \neq 1$ for all $n \geq N$. Thus, by property (iii) of the b-metric d_b, we have

$$d_b(x_0, x_k) \leq b\, d_b(x_0, x_1) + b\, d_b(x_1, x_k),$$

and since $x_1 \in Tx_0$ and $x_k \in Tx_{k-1}$, it follows

$$d_b(x_0, x_k) \leq b\, d_b(x_0, x_1) + b\, \delta_b(Tx_0, Tx_{k-1}).$$

Since $(x_0, x_i) \in \mathcal{R}$ for all $i \in \mathbb{N}$, then we have

$$\delta_b(Tx_0, Tx_{k-1}) \leq \varphi(M_b(x_0, x_{k-1})).$$

From other hand observe that $M_b(x_0, x_{k-1}) \leq \delta_b(O_{0,n}(\xi))$, which implies $\varphi(M_b(x_0, x_{k-1})) \leq \varphi(\delta_b(O_{0,n}(\xi))$, and hence we deduce

$$d_b(x_0, x_k) \leq b\, d_b(x_0, x_1) + b\, \varphi(\delta_b(O_{0,n}(\xi))),$$

that is,

$$\delta_b(O_{0,n}(\xi)) - b\, \varphi(\delta_b(O_{0,n}(\xi))) \leq b\, d_b(x_0, x_1),$$

and this show that $\delta_b(O_{0,n}(\xi)) - b\, \varphi(\delta_b(O_{0,n}(\xi)))$ is bounded.
If $\left\{\delta_b(O_{0,n}(\xi))\right\}_{n\in\mathbb{N}_0}$ is unbounded, then we have $\lim\limits_{n\to+\infty} \delta_b(O_{0,n}(\xi)) = +\infty$, so it follows from Definition 1.16-(i) that,

$$\lim_{n\to\infty} \delta_b(O_{0,n}(\xi)) - b\, \varphi(\delta_b(O_{0,n}(\xi))) = +\infty,$$

which yields a contradiction. Consequently, $\left\{\delta_b(O_{0,n}(\xi))\right\}_{n\in\mathbb{N}_0}$ is bounded. □

Lemma 2.4. *Under hypotheses of Theorem 2.1, every $\xi = \left\{x_i\right\} \in \mathcal{T}(x_0, \mathcal{R})$ converges to some $x_* \in X$.*

Proof. Let $\xi = \left\{x_i\right\}_{i\in\mathbb{N}_0}$ in $\mathcal{T}(x_0, \mathcal{R})$, then we have

$$(x_m, x_n) \in \mathcal{R}, \quad \text{for all } m < n.$$

So, it follows from Lemma 2.3 that there exists an integer k satisfying $m < k \leq n$ such that $d_b(x_m, x_k) = \delta_b(O_{m,n}(\xi))$. Hence, we deduce from (T3) that

$$d_b(x_{m+1}, x_{n+1}) \leq \delta_b(Tx_m, Tx_n) \leq \varphi(\delta_b(O_{m,n}(\xi))) = \varphi(d_b(x_m, x_k)). \tag{7}$$

Again, as $(x_m, x_k) \in \mathcal{R}$ for $m < k$, it follows

$$d_b(x_m, x_k) \leq \delta_b(Tx_{m-1}, Tx_{k-1}) \leq \varphi(\delta_b(O_{m-1,k-1}(\xi))) \leq \varphi(\delta_b(O_{m-1,n}(\xi))). \tag{8}$$

Combining now (7) and (8), we get

$$d_b(x_{m+1}, x_{n+1}) \leq \varphi^2(\delta_b(O_{m-1,n}(\xi))).$$

Consequently, an induction yields

$$d_b(x_{m+1}, x_{n+1}) \leq \varphi^{m+1}(\delta_b(O_{0,n}(\xi))).$$

From other hand, we know from Lemma 2.3 that $O_{0,n}(\xi)$ is bounded, so there exists a real constant $C > 0$ such that $\delta_b(O_{0,n}(\xi)) \leq C$ for all $n \in \mathbb{N}$. Therefore, we obtain

$$\lim_{n,m\to\infty} d_b(x_{n+1}, x_{m+1}) \leq \lim_{m\to\infty} \varphi^{m+1}(C) = 0.$$

Now, since (X, d) is T-orbitally complete, we deduce that the sequence $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ is Cauchy, and hence there exists an $x_* \in X$ such that $\lim_{n\to\infty} d_b(x_n, x_*) = 0$. $\square$

Proof of Theorem 2.1. Let $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ in $\mathcal{T}(x_0, \mathcal{R})$, then by Lemma 2.4 the sequence ξ converge to some element $x_* \in X$. Next, we shall prove that x_* is a fixed point of T. Observe first that

$$D_b(x_{n+1}, Tx_*) \leq \delta_b(Tx_n, Tx_*),$$

then using Proposition 1.7, it follows

$$D_b(x_*, Tx_*) \leq b\, d_b(x_*, x_{n+1}) + b\, D_b(x_{n+1}, Tx_*) \leq b\, d_b(x_*, x_{n+1}) + b\, \delta_b(Tx_n, Tx_*).$$

As $(x_i, x_{i+1}) \in \mathcal{R}$ for all $i \in \mathbb{N}_0$ and ξ is convergent (by Lemma 2.4), then from the $\mathcal{R}$-regularity of (X, d_b), we deduce

$$(x_n, x_*) \in \mathcal{R}, \text{ for all } n \in \mathbb{N}_0. \tag{9}$$

Consequently, by using condition (T2), we have

$$D_b(x_*, Tx_*) \leq b\, d_b(x_*, x_{n+1}) + b\, \varphi(M_b(x_n, x_*)). \tag{10}$$

Now, observe that

$$\max\left\{d_b(x_n, x_*), D_b(x_n, Tx_n), D_b(x_*, Tx_n)\right\} \leq b\, d_b(x_n, x_*) + b\, d_b(x_*, x_{n+1}).$$

As ξ converges to x_*, we have $\lim_{n\to\infty} d_b(x_n, x_*) = \lim_{n\to\infty} d_b(x_*, x_{n+1}) = 0$, then

$$\lim_{n\to\infty} \max\left\{d_b(x_n, x_*), D_b(x_n, Tx_n), D_b(x_*, Tx_n)\right\} = 0.$$

Assume now that $D_b(x_*, Tx_*) \neq 0$. Let n_0 be a positive integer such that for all $n > n_0$, we have

$$\max\left\{d_b(x_n, x_*), D_b(x_n, Tx_n), D_b(x_*, Tx_n)\right\} < D_b(x_*, Tx_*).$$

Hence, for all $n > n_0$, we have

$$M_b(x_n, x_*) = D_b(x_*, Tx_*) \quad \text{or} \quad M_b(x_n, x_*) = D_b(x_n, Tx_*).$$

First, assume that there exists an infinite subsequence $(n_k)_{k\geq 0}$ such that $n_k > n_0$ and $M_b(x_{n_k}, x_*) = D_b(x_*, Tx_*)$. By (10), we deduce

$$D_b(x_*, Tx_*) \leq b\, d_b(x_*, x_{n_k}) + b\, \varphi(M_b(x_{n_k}, x_*)),$$

that is,

$$D_b(x_*, Tx_*) - b\, \varphi(D_b(x_*, Tx_*)) \leq b\, d_b(x_*, x_{n_k}).$$

As consequence of $\lim_{n\to\infty} d_b(x_n, x_*) = 0$, we obtain that $D_b(x_*, Tx_*) - b\,\varphi(D_b(x_*, Tx_*)) = 0$, which contradict Definition 1.16-(iii). Hence, there exist $N > 0$ such that $M_b(x_n, x_*) = D_b(x_n, Tx_*)$ for all $n > N$. So, by (9) and Remark 1.17, we deduce that for all $k > 0$,

$$\begin{aligned} D_b(x_*, Tx_*) \le D_b(x_{N+k}, Tx_*) \le \delta_b(Tx_{N+k-1}, Tx_*) &\le \varphi(M_b(x_{N+k-1}, x_*)) \\ &\le \varphi(D_b(x_{N+k-1}, Tx_*)) \\ &\vdots \\ &\le \varphi^k(D_b(x_N, Tx_*)). \end{aligned}$$

As $\lim_{k\to\infty} \varphi^k(D_b(x_N, Tx_*)) = 0 \ge D_b(x_*, Tx_*) \neq 0$, we obtain a contradiction.
Consequently, $D_b(x_*, Tx_*) = 0$ and the results follows from Proposition 1.7. □

Theorem 2.5. *Under hypotheses of Theorem 2.1. If $Fix(T) \times Fix(T) \subseteq \mathcal{R}$, then $Fix(T)$ is a singleton.*

Proof. Let $(x_*, y_*) \in Fix(T) \times Fix(T)$, then $(x_*, y_*) \in \mathcal{R}$. By applying the contraction, it follows

$$\delta_b(Tx_*, Ty_*) \le \varphi(M_b(x_*, y_*)).$$

So, by observing that $D_b(x_*, Tx_*) = D_b(y_*, Ty_*) = 0$, and that $\max\left\{D_b(x_*, Ty_*), D_b(y_*, Tx_*)\right\} \le d_b(x_*, y_*)$, we obtain

$$d_b(x_*, y_*) \le \delta_b(Tx_*, Ty_*) \le \varphi(d_b(x_*, y_*)) < d_b(x_*, y_*),$$

which is a contradiction whenever x_* and y_* are different. □

Definition 2.6. *Let X be a non empty set and $\mathcal{R}$ be a binary relation on X. We say that $A \in \mathcal{P}(X)$ is $\mathcal{R}$-directed, if for all $x, y \in A$ there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.*

Theorem 2.7. *In addition to the hypotheses of Theorem 2.1, suppose that $\mathcal{R}$ is an equivalence relation and $Fix(T)$ is $\mathcal{R}$-directed, then T has a unique fixed point.*

Proof. Let $x_*, y_* \in Fix(T)$. The set $Fix(T)$ is supposed $\mathcal{R}$-directed, then there exists $z \in X$ such that $(x_*, z) \in \mathcal{R}$ and $(y_*, z) \in \mathcal{R}$. Using the symmetry and the transitivity of $\mathcal{R}$, it follows that $(x_*, y_*) \in \mathcal{R}$, and this proves that $Fix(T) \times Fix(T) \subseteq \mathcal{R}$, so the result follows immediately from Theorem 2.5. □

Next, we derive some corollaries for both multivalued and single valued mappings.

Corollary 2.8. *Let (X, d_b) be a complete b-metric space and $T : X \to \mathcal{P}_{cl}(X)$ be a multivalued mapping. Suppose there exists $\varphi \in \Phi_b$, such that*

$$\delta_b(Tx, Ty) \le \varphi(M_b(x, y)), \text{ for all } x, y \in X.$$

Then T has a unique fixed point..

Proof. By choosing $\mathcal{R} = X \times X$, we deduce that all hypotheses of Theorem 2.1 are satisfied, and hence T has a fixed point x_* in X. Moreover, since $Fix(T) \times Fix(T) \subseteq \mathcal{R}$, then we deduce the uniqueness of the fixed point. □

Consider now a binary relation on $\mathcal{P}(X)$ induced by a partial ordered $\le$ on X, then by taking $\le$ as the binary relation $\mathcal{R}$ in Theorem 2.1, we obtain the following result.

Corollary 2.9. *Let $\le$ be a partially order on X and $\mathcal{S}_{\le}$ be a binary relation on $\mathcal{P}(X)$. Let (X, d_b) be a complete b-metric space and $\le$-regular. Suppose that the multivalued mapping $T : X \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) *T is either $\mathcal{S}_{\le}$-preserving from some $x_0 \in Tx_0$ or T is $\mathcal{S}_{\le}$–proper;*
(2) *there exists $\varphi \in \Phi_b$ such that $\delta_b(Tx, Ty) \le \varphi(M_b(x, y))$ for all $x \le y$.*

Then, there exists $x_* \in Tx_*$. In addition, if we suppose that every two fixed points of $Fix(T)$ are comparable w.r.t. the partial order, then T has a unique fixed point.

Corollary 2.10. *Let $\mathcal{R}$ be a transitive binary relation on a nonempty set X. Let $f : X \to X$ be a single-valued mapping and such that (X, d_b) is a b-metric space, $\mathcal{R}$-regular and f-orbitally complete. Suppose that f satisfies the following conditions:*

(1) *f is $\mathcal{R}_{\mathcal{R}}$-preserving from some $x_0 \in X$;*
(2) *there exists $\varphi \in \Phi_b$ such that*
$d_b(fx, fy) \le \varphi\left(\max\left\{d_b(x,y), d_b(x,fx), d_b(y,fy), d_b(x,fy), d_b(y,fx)\right\}\right)$, *for all $(x,y) \in \mathcal{R}$.*

Then, f has at least one fixed point x_* in X. In addition, if $Fix(f) \times Fix(f) \subseteq \mathcal{R}$, then x_* is the unique fixed point of f.

Proof. Since every singleton set $\left\{x\right\}$ is closed, then we can define a closed multivalued mapping $T : X \to \mathcal{P}_{cl}(X)$ by putting $Tx = \left\{fx\right\}$ for all $x \in X$. Let $x, y \in X$ such that $(x, y) \in \mathcal{R}$, then we have

$$\delta_b(Tx, Ty) = d_b(fx, fy) \le \varphi\left(M_b(x, y)\right).$$

Hence, T is φ-quasi-contractive on $\mathcal{R}$-related points. It is not difficult to see that all conditions of Theorem 2.1 are satisfied and hence T has a fixed point x_*, that is, $x_* \in Tx_* = \left\{fx_*\right\}$, which implies that $x_* = fx_*$. In addition, if $Fix(f) \times Fix(f) \subseteq \mathcal{R}$, then all the conditions of Theorem 2.5 are satisfied, and thus x_* becomes the unique fixed point of T and therefore the unique fixed point of f. □

Corollary 2.11. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exists $\varphi \in \Phi_1$ such that for all $x, y \in X$, we have*

$$d(fx, fy) \le \varphi\left(\max\left\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\right\}\right).$$

Then f has a unique fixed point x_* in X.

Proof. The result follows from Corollary 2.10 by using the binary relation $\mathcal{R} = X \times X$. □

Remark 2.12. *From Corollary 2.11, we can derive a similar result to [11, Theorem 1.(a)] for a different class of multivalued mappings, and this by taking $\varphi(t) = qt$ for all $t > 0$. However, we can derive [24, Corollary 4] from our Corollary 2.11 for a different class of simulation functions.*

Remark 2.13. *More consequences may be derived by specifying various binary relations.*

3. Well-posedness, limit shadowing property and Ulam-Hyers stability

In the previous section, some sufficient conditions for existence and uniqueness of fixed points have been established. In the current section, we present a qualitative study of our fixed point inclusion (1) such as the well-posedness, the limit shadowing properties and the Ulam-Hyers stability. The well-posedness is a concept introduced by De Blasi and Myjak [27] in order to show whether every sequence will be convergent to the unique fixed point. However, the limit shadowing property has been introduced by Eirola et al. [26] and used to study the existence of trajectory approaches the real trajectory of certain dynamical system. The Ulam-Hyers stability was originated by Ulam when he asked about the existence of a linear mapping near an approximately additive mapping, which has been affirmatively answered by Hyers in [25].

Definition 3.1. *Let (X, d_b) be a b-metric space and $T : X \to \mathcal{P}_{cl}(X)$ be a multivalued mappings. We say that the fixed point inclusion* (1) *is well-posed w.r.t. δ_b if T has a unique fixed point x_* and every sequence $\left\{x_n\right\}$ satisfying $\lim_{n\to\infty} \delta_b(x_n, Tx_n) = 0$ converges to x_*.*

Theorem 3.2. *Let (X, d_b) be a b-metric space and $\mathcal{R}$ be a binary relation on X. Suppose there exists $\varphi \in \Phi_b$ such that $T : X \to \mathcal{P}_{cl}(X)$ is a multivalued φ-quasi-contractive on $\mathcal{R}$-related points and satisfies:*

(W1) *T has a unique fixed point x_*;*

(W2) *for every sequence $\left\{x_n\right\}_{n\in\mathbb{N}_0}$ satisfying $\lim_{n\to\infty} \delta_b(x_n, Tx_n) = 0$, there exists $n_0 \in \mathbb{N}_0$ such that $(x_n, x_*) \in \mathcal{R}$ for all $n \geq n_0$.*

Then, the fixed point inclusion (1) *is well posed w.r.t. δ_b.*

In order to prove this theorem, we need the following lemma:

Lemma 3.3. *In addition to the hypotheses of Theorem 3.2, suppose there exists a subsequence $\left\{x_{n(k)}\right\}_{k\in\mathbb{N}_0}$ such that $M_b(x_{n(k)}, x_*) = D_b(x_{n(k)}, Tx_{n(k)})$. Let $\varepsilon > 0$, so if there exists $N \in \mathbb{N}_0$ such that $D_b(x_n, Tx_n) < \varepsilon$ for all $n \geq N$, then there exists $N_1 \geq N$ such that $d_b(x_n, x_*) < \varepsilon$ for all $n \geq N_1$.*

Proof. We proceed by induction. Let $\varepsilon > 0$ and $k_0 \in \mathbb{N}_0$ such that $n(k_0) \geq \max\{N, n_0\}$, then we have

$$d_b(x_{n(k_0)+1}, x_*) \leq \delta_b(Tx_{n(k_0)}, Tx_*) \leq \varphi(M_b(x_{n(k_0)}, x_*)) = \varphi(D_b(x_{n(k_0)}, Tx_{n(k_0)})) < \varepsilon.$$

Suppose next that $d_b(x_{n(k_0)+p}, x_*) < \varepsilon$ and prove that $d_b(x_{n(k_0)+p+1}, x_*) < \varepsilon$. We have,

$$d_b(x_{n(k_0)+p+1}, x_*) \leq \delta_b(Tx_{n(k_0)+p}, Tx_*) \leq \varphi\left(M_b(x_{n(k_0)+p}, x_*)\right).$$

Observe that $D_b(Tx_{n(k_0)+p}, x_*) \leq d_b(x_{n(k_0)+p+1}, x_*)$, so if $M_b(x_{n(k_0)+p}, x_*) = D_b(Tx_{n(k_0)+p}, x_*)$, we obtain

$$\begin{aligned} d_b(x_{n(k_0)+p+1}, x_*) &\leq \delta_b(Tx_{n(k_0)+p}, Tx_*) \\ &\leq \varphi\left(M_b(x_{n(k_0)+p}, x_*)\right) = \varphi\left(D_b(Tx_{n(k_0)+p}, x_*)\right) \\ &\leq \varphi\left(d_b(x_{n(k_0)+p+1}, x_*)\right) \\ &< d_b(x_{n(k_0)+p+1}, x_*), \end{aligned}$$

which is a contradiction. Next, observe that $D_b(x_*, Tx_*) = 0$ and that $D_b(x_{n(k_0)+p}, Tx_*) \leq d_b(x_{n(k_0)+p}, x_*)$. Hence, we have

$$M_b(x_{n(k_0)+p}, x_*) = \max\left\{d_b(x_{n(k_0)+p}, x_*), D_b(Tx_{n(k_0)+p}, x_{n(k_0)+p})\right\}.$$

Therefore,

$$d_b(x_{n(k_0)+p+1}, x_*) \leq \varphi\left(M_b(x_{n(k_0)+p}, x_*)\right) < \max\left\{d_b(x_{n(k_0)+p}, x_*), D_b(Tx_{n(k_0)+p}, x_{n(k_0)+p})\right\},$$

which implies that for all $n \geq n(k_0) + 1 = N_1$, we have $d(x_n, x_*) < \varepsilon$. □

Proof of Theorem 3.2. Assume that $Fix(T) = \left\{x_*\right\}$ and let $\left\{x_n\right\}_{n\in\mathbb{N}_0}$ be a sequence satisfying

$$\lim_{n\to\infty} \delta_b(x_n, Tx_n) = 0. \tag{11}$$

Then from (W2), there exists $n_0 \in \mathbb{N}_0$ such that

$$(x_n, x_*) \in \mathcal{R}, \text{ for all } n \geq n_0. \tag{12}$$

In order to prove that T is well posed w.r.t. δ_b, we shall show that $\lim_{n\to\infty} d_b(x_n, x_*) = 0$.
Using T is φ-quasi-contractive on $\mathcal{R}$-related points together with (12), then for all $n \geq n_0$ and every $z_n \in Tx_n$, we have

$$\begin{aligned} d_b(x_n, x_*) &\leq b\, d_b(x_n, z_n) + b\, d_b(z_n, x_*) \\ &\leq b\, \delta_b(x_n, Tx_n) + b\, \delta_b(Tx_n, Tx_*) \\ &\leq b\, \delta_b(x_n, Tx_n) + b\, \varphi\left(M_b(x_n, x_*)\right). \end{aligned}$$

Now, since $D_b(x_*, Tx_n) \leq d_b(x_*, x_{n+1})$, then if $M_b(x_n, x_*) = D_b(x_*, Tx_n)$, we obtain

$$d_b(x_{n+1}, x_*) \leq \delta_b(Tx_n, Tx_*) \leq \varphi(M_b(x_n, x_*)) \leq \varphi(d_b(x_{n+1}, x_*)) < d_b(x_{n+1}, x_*),$$

which is absurd. Next, from (W1) observe that $D_b(x_*, Tx_*) = 0$ and $D_b(x_n, Tx_*) \leq d_b(x_n, x_*)$, so $M_b(x_n, x_*) = \max\left\{d_b(x_n, x_*), D_b(x_n, Tx_n)\right\}$. Suppose now that there exists a subsequence $\left\{n(k)\right\}_{k\in\mathbb{N}_0}$ such that $M_b(x_{n(k)}, x_*) = D_b(x_{n(k)}, Tx_{n(k)})$. Then, by (11) and Lemma 3.3, we obtain $\lim_{n\to\infty} d_b(x_n, x_*) = 0$. Otherwise, if such a subsequence does not exist, then there exists m_0 such that $M_b(x_n, x_*) = d_b(x_n, x_*)$ for all $n \geq m_0 \geq n_0$ and this implies

$$d_b(x_{m_0+k}, x_*) \leq \varphi^k\left(d_b(x_{m_0}, x_*)\right), \quad \text{for all } k \geq 0.$$

Therefore, we have always $d_b(x_n, x_*)$ tends to 0 when n tends to infinity. □

Remark 3.4. *Under hypotheses of Corollary 2.8, the fixed point inclusion* (1) *is well-posed.*

Next, we introduce an appropriate limit shadowing property corresponding to the fixed point inclusion (1).

Definition 3.5. *We say that the fixed point inclusion* (1) *has the limit shadowing property for a multivalued mapping $T : X \to \mathcal{P}_{cl}(X)$ if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} \delta_b(x_n, Tx_n) = 0$, there exists $z_0 \in X$ and a sequence $\zeta = \left\{z_n\right\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}(z_0, \mathcal{R})$ such that $\lim_{n\to\infty} d_b(x_n, z_n) = 0$.*

Theorem 3.6. *If the fixed point inclusion* (1) *is well posed (in the sense of Theorem 3.2), then* (1) *possess the limit shadowing property.*

Proof. From theorems 2.1 and 2.5, we have that for any z_0 in X, every sequence $\zeta := \{z_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}(z_0, \mathcal{R})$ satisfies

$$\lim_{n\to\infty} d_b(z_n, x_*) = 0. \tag{13}$$

Let $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence such that $\lim_{n\to\infty} \delta_b(x_n, Tx_n) = 0$. Since we have the following inequality

$$d_b(x_n, z_n) \leq b\, d_b(x_n, x_*) + b\, d_b(x_*, z_n). \tag{14}$$

Using the fact that the fixed inclusion is well-posed w.r.t δ_b, then we deduce that

$$\lim_{n\to\infty} d_b(x_n, x_*) = 0. \tag{15}$$

Consequently, combining (13), (14) and (15), we obtain $\lim_{n\to\infty} d_b(x_n, z_n) = 0$. □

We study next, the Ulam-Hyers stability of the fixed point inclusion (1).

Definition 3.7 (see [28])**.** *Let (X, d_b) be a b-metric space and $T : X \to \mathcal{P}(X)$ be a multivalued operator. The fixed point inclusion* (1) *is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which is increasing and continuous at 0 and $\psi(0) = 0$, such that for every $\varepsilon > 0$ and for each $y_* \in X$ solution of the inequality*

$$\delta_b(y, Ty) \leq \varepsilon, \tag{16}$$

there exists an $x_* \in Fix(T)$ such that

$$d_b(y_*, x_*) \leq \psi(\varepsilon).$$

Remark 3.8. *If $\psi(t) = c\,t$ where c is a positive constant and $t \geq 0$, then the fixed point inclusion* (1) *is said to be Ulam-Hyers stable.*

Theorem 3.9. *In addition to the hypotheses of Theorem 2.1, suppose that $T : X \to \mathcal{P}_{cl}(X)$ satisfies*

(U1) *for any $y_* \in X$ satisfying* (16) *and $x_* \in Fix(T)$, we have $(x_*, y_*) \in \mathcal{R}$;*

(U2) *the function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\beta(t) := t - b^2\varphi(t)$ is increasing, bijective and continuous.*

Then, the fixed point inclusion (1) *is generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$, $x_* \in Fix(T)$ and $y_* \in X$ satisfying (16). From condition (U1), we have $(x_*, y_*) \in \mathcal{R}$, so from (T1) we deduce that $(Tx_*, Ty_*) \in \mathcal{S}_{\mathcal{R}}$. Therefore, for any $z_* \in Ty_*$, we have

$$\begin{aligned} d_b(y_*, x_*) &\leq b\, d_b(y_*, z_*) + b\, d_b(z_*, x_*) \\ &\leq b\, \delta_b(y_*, Ty_*) + b\, \delta_b(Ty_*, Tx_*) \\ &\leq b\, \varepsilon + b\, \varphi(M_b(y_*, x_*)). \end{aligned}$$

Observe now that $D_b(x_*, Tx_*) = 0$, $D_b(y_*, Ty_*) \leq \varepsilon$ and that $D_b(y_*, Tx_*) \leq d_b(y_*, x_*)$, then we obtain

$$d_b(y_*, x_*) \leq b\, \varepsilon + \varphi(b\, d_b(y_*, x_*) + b\, \varepsilon)$$

or equivalently

$$b\, d_b(y_*, x_*) \leq b^2\, \varepsilon + b^2\, \varphi(b\, d_b(y_*, x_*) + b\, \varepsilon).$$

Now, using (U2), it follows

$$\beta(b\, d_b(y_*, x_*) + b\, \varepsilon) \leq b\,(b + 1)\, \varepsilon,$$

and this yields,

$$d_b(y_*, x_*) \leq b^{-1}\beta^{-1}(b\,(b + 1)\, \varepsilon),$$

which proves that (1) is generalized Ulam-Hyers stable, where $\psi(t) = b^{-1}\beta^{-1}(b\,(b + 1)\, t)$ for all $t > 0$. □

4. Application

This section is devoted to the study of the Cauchy differential inclusion:

$$\begin{cases} y'(t) \in F(t, y(t)), \quad a.e. \quad t \in J := [t_0, t_1] \\ y(t_0) = y_0 \end{cases}, \tag{17}$$

with $y_0 \in E$ and $F : J \times E \to \mathcal{P}_{cp,cv}(E)$ is a multivalued mapping, where E is a separable Banach space endowed with a norm $\|\cdot\|$. We intend here to prove that (17) has a unique solution in $C(J, E)$. Consider the Banach space $X = C(J, E)$ endowed with a norm $\|\cdot\|_*$, and equipped with the distance:

$$d_2(x, y) = \sup_{t \in J} \left\| x(t) - y(t) \right\|^2.$$

It is well known that (X, d_2) is a complete b-metric where $b = 2$.

Let $T_F : C(J, E) \to \mathcal{P}(C(J, E))$ be the multivalued Carathéodory operator associated to F:

$$T_F(y) = \left\{ h \in C(J, E) : \exists\, g \in S_F(y) \text{ such that } h(t) = y_0 + \int_{t_0}^{t} g(r)\, dr,\ \forall t \in J \right\},$$

with $S_F : C(J, E) \to \mathcal{P}(L^1(J, E))$ be the Niemytzki operator associated to F:

$$S_F(y) = \left\{ g \in L^1(J, E) : g(r) \in F(r, y(r)),\ \text{for a.e. } r \in J \right\},$$

where $L^1(J, E)$ is the Bochner space of integrable functions on J with values in the Banach space E. Consider now the following assumptions:

(A_1) $t \mapsto F(t, y)$ is measurable for each $y \in C(J, E)$;

(A_2) $y \mapsto F(t, y)$ is continuous for a.e $t \in J$;

(A_3) for all $\rho > 0$, ther exists $h_\rho \in L^1(J)$ such that

$$\left|F(t, y)\right| := \sup\left\{\left\|v\right\|_* : v \in F(t, y)\right\} \leq h_\rho(t) \ \text{ a.e. } t \in J \text{ and for all } y \in \overline{B(0, \rho)};$$

(A_4) there exists $\lambda \in (0, 1)$ and a positive function $\ell \in L^1(J \times \mathbb{R}^+)$ and $\varphi \in \Phi_2$ such that

$$\int_{t_0}^{t_1} \ell(t, r)\, dt \leq \sqrt{\varphi(r)} \text{ for all } r \geq 0 \quad \text{with}$$

$$\delta_2\Big(F(t, u(t)), F(t, v(t))\Big) \leq \ell\Big(t,\, M_2(u, v)\Big), \ \text{ for all } t \in J,\ u, v \in C(J, E),$$

where $M_2(u, v) = \max\Big\{d_2(u, v), d_2(u, T(u)), d_2(v, T(v)), d_2(u, T(v)), d_2(v, T(u))\Big\}$.

Theorem 4.1. *Suppose that the assumptions* (A_1) *to* (A_4) *are satisfied. Then the problem* (17) *has a unique solution in* $C(J, E)$.

First, we need the following lemma to prove the theorem.

Lemma 4.2 (See [29, Lemma 2.3]). *Let* $F : J \times E \to \mathcal{P}_{cp,cv}(E)$ *be a multivalued mapping satisfying* (A_1) *to* (A_3), *then* $T_F(y)$ *is nonempty and closed for all* $y \in E$.

Proof of Theorem 4.1. From Lemma 4.2 we have that $T(y) \in \mathcal{P}_{cl}(C(J, E))$ for each $y \in C(J, E)$. It remains to prove that T is multivalued φ-quasi-contractive for some $\varphi \in \Phi_2$. Let $y_1, y_2 \in C(J, E)$, then it follows from (A_4) that,

$$\delta_2(F(t, y_1(t)), F(t, y_2(t))) \leq \ell\Big(t, M_2(y_1, y_2)\Big), \ \text{ for all } t \in J. \tag{18}$$

Now, let $h_i \in T(y_i)$ for $i = 1, 2$, then there exists $g_i \in S_F(y_i)$, such that

$$h_i(t) = y_0 + \int_{t_0}^{t} g_i(r) dr, \ \text{ for } i = 1, 2 \text{ and } t \in J.$$

Since $g_1(r) \in F(r, y_1(r))$ and $g_2(r) \in F(r, y_2(r))$ for all $r \in J$, then it follows from (18) that,

$$d_2(g_1(r), g_2(r)) \leq \ell\Big(t,\, M_2(y_1, y_2)\Big), \ \text{ for all } r \in J,$$

which implies,

$$\left\|g_1(r) - g_2(r)\right\| \leq \ell\Big(t,\, M_2(y_1, y_2)\Big)^{\frac{1}{2}}.$$

Hence, we have

$$\begin{aligned}
d_2(h_1(t), h_2(t)) &= \left\|h_1(t) - h_2(t)\right\|^2 \\
&= \left\|\int_{t_0}^{t} g_1(r) - g_2(r) dr\right\|^2 \\
&\leq \left(\int_{t_0}^{t} \left\|g_1(r) - g_2(r)\right\| dr\right)^2 \\
&\leq \left(\int_{t_0}^{t} \ell(t, \Big(M_2(y_1, y_2)\Big)^{\frac{1}{2}} dr\right)^2 \\
&\leq \varphi(M_2(y_1, y_2)),
\end{aligned}$$

which implies,

$$d_2(h_1, h_2) \leq \varphi(M_2(y_1, y_2)).$$

Now, since h_1 and h_2 are arbitrary in $T(y_1)$ and $T(y_2)$, respectively, it follows,

$$\delta_2(T(y_1), T(y_2)) \leq \varphi(M_2(y_1, y_2)), \text{ for all } y_1, y_2 \in C(J, E).$$

Consequently, the theorem follows from Corollary 2.8. □

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