Filomat 32:8 (2018), 2867–2874
https://doi.org/10.2298/FIL1808867J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Multiordered Fuzzy Difference Sequence Spaces

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Abstract. In this paper we introduce some new multi ordered difference operator on sequence spaces of fuzzy real numbers by using ideal convergence and modulus function and study their some algebraic and topological properties.

1. Introduction

The concept of ideal convergence as a generalization of statistical convergence, and any concept involving statistical convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information system, population modelling and motion planning in robotics.

Zadeh [1] introduced the concept of fuzzy sets and fuzzy set operations. Subsequently, several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy ordering, fuzzy measures of fuzzy events, and fuzzy mathematical programming. The concept of fuzzy topology has very important applications in quantum particle physics, especially in connection with both string and ε^{∞} theory, which were given and studied by El Naschie [2]. The theory of sequences of fuzzy numbers was first introduced by Matloka [3]. Matloka introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties, and showed that every convergent sequence of fuzzy numbers is bounded. In [4], Nanda studied sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Different classes of sequences of fuzzy real numbers have been discussed by Nuray and Savas [5], Savas and Mursaleen [6] and many others.

Kostyrko et al. [7] introduced the notion of I -convergence based on the structure of admissible ideal I of subset of natural numbers . Later on it was studied by Salat et al.[8-9] and Demirci [10]. Recently it was further studied by Tripathy and Hazarika [11-12], Jalal [13-16] and several others.

Let X be a non empty set. A set $I \subseteq 2^X$(2^X denoting the power set of X) is said to be an ideal if I is additive i.e.,$A \subseteq B \in I \Rightarrow A \cup B \in I$ and hereditary i.e., $A \in I, B \subseteq A \Rightarrow B \in I$. An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $\mathfrak{J}(I)$ corresponding to I i.e $\mathfrak{J}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$ (where $\mathbb{N}$ denotes the set of natural numbers).

2010 *Mathematics Subject Classification*. Primary 46B20; 46B45; ; Secondary 46A45; 46A80; 46E30.
Keywords. Fuzzy real numbers, difference sequence space, ideal convergence, modulus function
Received: 22 June 2017; Revised: 18 January 2018; Accepted: 19 January 2019
Communicated by Eberhard Malkowsky
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Kizmaz [17] defined the difference Sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_\infty(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$$

for $Z = \ell_\infty, c$ and c_0 and $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. the idea of difference sequence was generalized by Et and Colak [18] and Et and Esi [19]. The operator $\Delta^s : \omega^F \to \omega^F$ is defined by

$$(\Delta^0 x_k) = x_k, (\Delta^1 x_k) = \Delta x_k = x_k - x_{k+1}, (\Delta^s x_k) = \Delta^{s-1} x_k - \Delta^{s-1} x_{k+1}, s \geq 2$$

and for all $k \in \mathbb{N}$.The generalized difference operator has the following representation,

$$(\Delta^s x_k) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} x_{k+i}, \text{ for all } k \in \mathbb{N}.$$

Recently Dutta [20] further generalized this notion and introduced the following. Let $r \geq 1, s \geq 1$ and $v = (v_k)$ be a sequence of non-zero real's

$$Z(\Delta^s_{(vr)}) = \{x = (x_k) : (\Delta^s_{(vr)} x_k) \in Z\}$$

for $Z = \ell_\infty, c$ and c_0 The generalized difference operator has the following binomial representation:

$$(\Delta^s_{(vr)} x_k) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} v_{k+ri} x_{k+ri}.$$

For $s = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $\ell_\infty(\Delta_r), c(\Delta_r)$ and $c_\infty(\Delta_r)$ Tripathy and Esi [21]. For $r = 1$and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $\ell_\infty(\Delta^s), c(\Delta^s)$ and $c_\infty(\Delta^s)$ Et and Colak [18]. For $r = 11, s = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, $r = s = 1$ we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_\infty(\Delta)$ [17].

Nakano [22] introduced the concept of modulus function.
A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if
(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t + u) = f(t) + f(u)$ for all $t, u \geq 0$,
(iii) f is non decreasing, and
(iv) f is continuous from the right at zero.
It follows that f must be continuous everywhere on $[0, \infty)$ and a modulus function may be bounded or unbounded.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm if
(i) $p(x) \geq 0$ for all $x \in X$,
(ii) $p(-x) = p(x)$ for all $x \in X$,
(iii) $p(x + y) \leq p(x) + p(y)$, is non decreasing, and
(iv) If (λ_n) be a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$ and (x_n) be a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.
A paranorm p for which $p(x) = 0 \Rightarrow x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space.

Throughout the article, ω^F denotes the class of all fuzzy real-valued sequence spaces. Also, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively.

2. Definitions and preliminaries

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$ we define $d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space [23]. Also, the relation $'' \leq ''$ is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$,i.e., a mapping $X : \mathbb{R} \to J(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X

is convex if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$. If there exist $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$, then the fuzzy number X is called normal. A fuzzy number X is said to be upper semi-continuous if for each $\varepsilon > 0, X^{-1}([0, a + \varepsilon))$ for all $a \in [0, 1]$ is open in the usual topology in $\mathbb{R}$.Let $\mathbb{R}(J)$ denote the set of all fuzzy numbers which are upper semi-continuous and have compact support, i.e.,if $X \in \mathbb{R}(J)$, then for any $\alpha \in [0, 1], [X]^\alpha$ is compact, where

$$[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha, \text{if } \alpha \in [0, 1]\}$$
$$[X]^0 = \overline{\{t \in \mathbb{R} : X(t) > \alpha, \text{if } \alpha = 0\}}.$$

The set $\mathbb{R}$ of real numbers can be embedded in $\mathbb{R}(J)$ if we define $\bar{r} \in \mathbb{R}(J)$ by

$$\bar{r} = \begin{cases} 1 & \text{if } t = r; \\ 0 & \text{if } t \neq r. \end{cases}$$

The additive identity and multiplicative identity of $\mathbb{R}(J)$ are defined by $\bar{0}$ and $\bar{1}$ respectively. The arithmetic operations on $\mathbb{R}(J)$ are defined as follows:

$$(X \oplus Y)(t) = \sup\{X(s) \wedge Y(t - s)\}, t \in \mathbb{R},$$

$$(X \Theta Y)(t) = \sup\{X(s) \wedge Y(s - t)\}, t \in \mathbb{R},$$

$$(X \otimes Y)(t) = \sup\left\{X(s) \wedge Y\left(\frac{t}{s}\right)\right\}, t \in \mathbb{R},$$

$$\left(\frac{X}{Y}\right)(t) = \sup\{X(st) \wedge Y(s)\}, t \in \mathbb{R}.$$

Let $X, Y \in \mathbb{R}(J)$ and the α-level sets be $[X]^\alpha = [x_1^\alpha, x_2^\alpha]$,$[Y]^\alpha = [y_1^\alpha, y_2^\alpha]$,$\alpha \in [0, 1]$. Then the above operations can be defined in terms of α-level sets as follows:

$$[X \oplus Y]^\alpha = [x_1^\alpha + y_1^\alpha, x_2^\alpha + y_2^\alpha],$$
$$[X \Theta Y]^\alpha = [x_1^\alpha - y_1^\alpha, x_2^\alpha - y_2^\alpha],$$
$$[X \otimes Y]^\alpha = \left[\min_{i \in \{1,2\}} x_i^\alpha y_i^\alpha, \max_{i \in \{1,2\}} x_i^\alpha y_i^\alpha\right],$$
$$[X^{-1}]^\alpha = \left[(x_2^\alpha)^{-1}, (x_1^\alpha)^{-1}\right], x_i^\alpha > 0 \text{ for each } 0 < \alpha \leq 1.$$

For $r \in \mathbb{R}$ and $X \in \mathbb{R}(J)$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1})t & \text{if } r \neq 0; \\ 0 & \text{if } r = 0. \end{cases}$$

The absolute value $|X|$ of $X\mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\} & \text{if } t \geq 0; \\ 0 & \text{if } t < 0. \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

A metric $\bar{d}$ on $\mathbb{R}(J)$ is said to be a translation invariant if $\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$ for $X, Y, Z \in \mathbb{R}(J)$.

Proposition 1. If $\bar{d}$ is a translation invariant metric on $\mathbb{R}(J)$,then
(i) $\bar{d}(\Delta^s_{(vr)}X_k + \Delta^s_{(vr)}Y_k, 0) \leq \bar{d}(\Delta^s_{(vr)}X_k, 0) + \bar{d}(\Delta^s_{(vr)}Y_k, 0)$
(ii)$\bar{d}(\alpha\Delta^s_{(vr)}X_k, 0) \leq |\alpha|\bar{d}(\Delta^s_{(vr)}X_k, 0)$.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded. A sequence $X = (X_k)$ of fuzzy numbers is said to be I-convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ such that

$$A = \{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_k) of fuzzy numbers, and we write $I - \lim X_k = X_0$. A sequence $X = (X_k)$ of fuzzy numbers is said to be I-bounded if there exists $M > 0$, such that

$$\{k \in \mathbb{N} : \bar{d}(X_k, \bar{0}) > M\} \in I.$$

A sequence space E_F of fuzzy numbers is said to be normal (or solid) if $(\alpha_k X_k) \in E_F$ whenever$(X_k) \in E_F$ and for all sequences (α_k)a of scalars $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ and $|Y_k| \leq |X_k|$ for all $k \in \mathbb{N}$. A sequence space E_F of fuzzy numbers is said to be symmetric if implies $(X_{\pi(k)}) \in E_F$ where π is the permutation on $\mathbb{N}$. A sequence space E_F is said to be monotone if E_F contains the canonical pre-image off all its step spaces.

Example 1. If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$, then I_f is a non-trivial admissible ideal of $\mathbb{N}$, and the corresponding convergence coincides with the usual convergence.

Example 2. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A) = 0$ denotes the asymptotic density of the set A, then I_δ is a non-trivial admissible ideal of $\mathbb{N}$, and the corresponding convergence coincides with the statistical convergence.

Lemma 1. (Kostyrko, Salat,and Wilczynski [7],Lemma 5.1.) If $I \subset 2^{\mathbb{N}}$ is amaximal ideal, then for each $A \subset \mathbb{N}$, we have either $A \in I$ or $\mathbb{N} - A \in I$.

Lemma 2. Every normal space is monotone (Kampthan and Gupta [24]).

The following well known inequality will be used throughout the article. Let $p = (p_k)$ be a sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = G, H = \max\{1, 2^{G-1}\}$, then

$$|\alpha_k + \beta_k|^{p_k} \leq H(|\alpha_k|^{p_k} + |\beta_k|^{p_k})$$

for all $k \in \mathbb{N}$ and $a_k, b_k \in \mathbb{C}$. Also $|\alpha_k|^{p_k} \leq \max\{1, |\alpha|^G\}$ for all $\alpha \in \mathbb{C}$.

3. Some new sequence spaces of fuzzy numbers

The main object of the paper is to introduce the following sequence spaces and examine algebraic and topological properties of the resulting sequence spaces.
Let f be a modulus function. Let r and s be two non-negative integers and $v = (v_k)$ be a sequence of non-zero reals. Then for a sequence $p = (p_K)$ of strictly positive real numbers, we define the following classes of sequences:

$$\omega^F[\Delta^s_{(vr)}, f, p]^I = \left\{X \in \omega^F : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, X_0))\right]^{p_k} \geq \varepsilon\right\} \in I,$$

$$\omega^F[\Delta^s_{(vr)}, f, p]^I_0 = \left\{X \in \omega^F : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, \bar{0}))\right]^{p_k} \geq \varepsilon\right\} \in I,$$

$$\omega^F[\Delta^s_{(vr)}, f, p]^I_\infty = \left\{X \in \omega^F : \exists K > 0 \; s.t \; \frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, \bar{0}))\right]^{p_k} \geq K\right\} \in I,$$

$$\omega^F[\Delta^s_{(vr)}, f, p]_\infty = \left\{ X \in \omega^F : \sup_n \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} X_k, \bar{0})) \right]^{p_k} < \infty \right\}.$$

Theorem 1. Let f be a modulus function and $p = (p_k)$ be a sequence of strictly positive real numbers, then the classes of sequences $\omega^F[\Delta^s_{(vr)}, f, p]^I$,$\omega^F[\Delta^s_{(vr)}, f, p]^I_0$ and $\omega^F[\Delta^s_{(vr)}, f, p]^I_\infty$ are closed under addition and scalar multiplication.

Proof. We shall give the proof for $\omega^F[\Delta^s_{(vr)}, f, p]^I_0$. The others can be proved similarly. Let $X = (X_k), Y = (Y_k) \in \omega^F[\Delta^s_{(vr)}, f, p]^I_0$. For scalars γ and μ there exists integers M_γ and N_μ such that $\gamma < |M_\gamma|$ and $\mu < |N_\mu|$. Since f is sub-additive and the operator $\Delta^s_{(vr)}$ is linear, we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{\infty} & \left[f(\bar{d}(\Delta^s_{(vr)}(\gamma X_k + \mu Y_k), \bar{0})) \right]^{p_k} \\ & \leq \frac{1}{n} \sum_{k=1}^{\infty} \left[f(|\gamma| \bar{d}(\Delta^s_{(vr)}(X_k), \bar{0})) + f(|\mu| \bar{d}(\Delta^s_{(vr)}(Y_k), \bar{0})) \right]^{p_k} \\ & \leq C(M_\gamma)^H \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} X_k, \bar{0})) + C(N_\mu)^H \frac{1}{n} \sum_{k=1}^{\infty} f(\bar{d}(\Delta^s_{(vr)} X_k, \bar{0})) \right]^{p_k} \to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof.

Theorem 2 The space $\omega^F[\Delta^s_{(vr)}, f, p]^I_0$ is a paranormed space with respect to the paranorm defined by

$$g(X) = \sup_n \left\{ \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} X_k, \bar{0})) \right]^{p_k} \right\}^{\frac{1}{M}} \leq 1,$$

where $\sup_k p_k < \infty$ and $M = \max(1, H)$.

Proof. Obviously $g(X) = g(-X)$ for all $X \in \omega^F[\Delta^s_{(vr)}, f, p]^I_0$. It is trivial that $v_k X_k = \bar{0}$ for $X_k = \bar{0}$. Since $\frac{p_k}{M} \leq 1$ and $M \geq 1$ by using Minkowskis inequality, we have,

$$\begin{aligned} \left\{ \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} X_k + \Delta^s_{(vr)} Y_k, \bar{0})) \right]^{p_k} \right\}^{\frac{1}{M}} & \leq \left\{ \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} X_k, \bar{0})) + f(\bar{d}(\Delta^s_{(vr)} Y_k, \bar{0})) \right]^{p_k} \right\}^{\frac{1}{M}} \\ & \leq \left\{ \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} X_k, \bar{0})) \right]^{p_k} \right\}^{\frac{1}{M}} \\ & \quad + \left\{ \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\Delta^s_{(vr)} Y_k, \bar{0})) \right]^{p_k} \right\}^{\frac{1}{M}}. \end{aligned}$$

It follows that $g(X + Y) \leq g(X) + g(Y)$.

Finally to check the continuity of scalar multiplication, let γ be any scaler, by definition we have,

$$g(\gamma X) = \sup_n \left\{ \frac{1}{n} \sum_{k=1}^{\infty} \left[f(\bar{d}(\gamma \Delta^s_{(vr)} X_k, \bar{0})) \right]^{p_k} \right\}^{\frac{1}{M}} \leq N_\gamma^{\frac{H}{M}} g(X),$$

where N_γ is an integer such that $|\gamma| \le N_\gamma$. Now let $\gamma \to 0$ for fixed X with $g(X) \ne 0$. By properties of f for $|\gamma| < 1$, we have,

$$\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\gamma\Delta^s_{(vr)}X_k,\bar{0}))\right]^{p_k} < \varepsilon \text{ for } n > N(\varepsilon). \tag{1}$$

Also for $1 \le n \le N$, taking γ small enough, since f is continuous we have

$$\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\gamma\Delta^s_{(vr)}X_k,\bar{0}))\right]^{p_k} < \varepsilon. \tag{2}$$

Equations (1) and (2) together imply that $g(\gamma X) \to 0$ as $\gamma \to 0$.
This completes the proof.

Theorem 3. Let I be an admissible ideal and f be a modulus function. Then the following hold:

$$\omega^F[\Delta^{s-1}_{(vr)}, f, p]^I_0 \subseteq \omega^F[\Delta^{s}_{(vr)}, f, p]^I_0;$$
$$\omega^F[\Delta^{s-1}_{(vr)}, f, p]^I \subseteq \omega^F[\Delta^{s}_{(vr)}, f, p]^I;$$
$$\omega^F[\Delta^{s-1}_{(vr)}, f, p]^I_\infty \subseteq \omega^F[\Delta^{s}_{(vr)}, f, p]^I_\infty$$

for $s \ge 1$ and the inclusions are strict. In general, for all $j = 1, 2, \cdots, s-1$ the following hold:

$$\omega^F[\Delta^{j}_{(vr)}, f, p]^I_0 \subseteq \omega^F[\Delta^{s}_{(vr)}, f, p]^I_0;$$
$$\omega^F[\Delta^{j}_{(vr)}, f, p]^I \subseteq \omega^F[\Delta^{s}_{(vr)}, f, p]^I;$$
$$\omega^F[\Delta^{j}_{(vr)}, f, p]^I_\infty \subseteq \omega^F[\Delta^{s}_{(vr)}, f, p]^I_\infty$$

and the inclusions are strict.

Proof. Let $X = (X_k) \in \omega^F[\Delta^{s-1}_{(vr)}, f, p]^I_0$. Then for $\varepsilon > 0$, we have

$$\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k,\bar{0}))\right]^{p_k} \ge \varepsilon\right\} \in I.$$

By definition of modulus function f ,then the follows that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k,\bar{0}))\right]^{p_k} &\le \frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^{s-1}_{(vr)}X_k - \Delta^{s-1}_{(vr)}X_{k+1},\bar{0}))\right]^{p_k}\\ &\le C\left\{\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^{s-1}_{(vr)}X_k,\bar{0}))\right]^{p_k} + \frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^{s-1}_{(vr)}X_{k+1},\bar{0}))\right]^{p_k}\right\}.\end{aligned}$$

This completes the proof.
The following example shows that, in general, equality does not hold.

Example 3 Let $f(x) = (x)$ for all $x \in [0, \infty), s = 3, r, v = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (X_k) of fuzzy numbers as follows

$$X_k(t) = \begin{cases} -\frac{t}{k^3-1} + 1 & \text{if } k^3 - 1 \le t \le 0; \\ -\frac{t}{k^3+1} + 1 & \text{if } 0 < t \le k^3 + 1; \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha \in (0,1]$, the α-level sets of $X_k, \Delta X_k, \Delta^2 X_k$ and $\Delta^3 X_k$ are

$$\begin{aligned}
[X_k]^\alpha &= [(1-\alpha)(k^3-1),(1-\alpha)(k^3+1)]\\
[\Delta X_k]^\alpha &= [(1-\alpha)(-3k^2-3k-3),(1-\alpha)(-3k^2-3k-3)]\\
[\Delta^2 X_k]^\alpha &= [(1-\alpha)(6k+2),(1-\alpha)(6k+10)]\\
[\Delta^3 X_k]^\alpha &= [-14(1-\alpha),2(1-\alpha)]
\end{aligned}$$

respectively. It is easy to see that the sequence $[\Delta^2 X_k]^\alpha$ is not I-bounded, but $[\Delta^3 X_k]^\alpha$ is I-bounded.

Theorem 4. Let f be a modulus function. Then

$$\omega^F[\Delta^s_{(vr)}, f, p]^I_0 \subset \omega^F[\Delta^s_{(vr)}, f, p]^I \subset \omega^F[\Delta^s_{(vr)}, f, p]^I_\infty$$

and the inclusions are proper.

Proof. Let $(X_k) \in \omega^F[\Delta^s_{(vr)}, f, p]^I$. Let $\varepsilon > 0$ be given , then

$$\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, X_0))\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Since

$$\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, X_0))\right]^{p_k} \leq \frac{1}{2}\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, X_0))\right]^{p_k} + \frac{C}{2}\frac{1}{n}\sum_{k=1}^{\infty}\left[f(\bar{d}(\Delta^s_{(vr)}X_k, \bar{0}))\right]^{p_k}.$$

Taking supremum over k on both sides, we get $\omega^F[\Delta^s_{(vr)}, f, p]^I_\infty$. The inclusion $\omega^F[\Delta^s_{(vr)}, f, p]^I_0 \subset \omega^F[\Delta^s_{(vr)}, f, p]^I$ is obvious.

The following example shows that, in general, equality does not hold.

Example 4 Let $f(x) = x^2$ for all $x \in [0,\infty), s = 1, r, v = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (X_k) of fuzzy numbers as follows

Let $k = 2^i$, $i = 1, 2, 3, \cdots$

$$X_k(t) = \begin{cases} \frac{4}{k}+1 & \text{if } \frac{-k}{4} \leq t \leq 0;\\ -\frac{4}{k}+1 & \text{if } 0 < t \leq k^3+1;\\ 0 & \text{otherwise.}\end{cases}$$

For $\alpha \in (0,1]$, the α-level sets of $X_k, \Delta X_k$, are

$$[X_k]^\alpha = \begin{cases} \left[\frac{k}{4}(\alpha-1), \frac{k}{4}(\alpha-1)\right] & \text{if } k = 2^i\ i = 1,2,3,...;\\ [0,0] & \text{otherwise}\end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[\frac{k}{4}(\alpha-1), \frac{k}{4}(1-\alpha)\right] & \text{for } k = 2^i\ i = 1,2,3,...;\\ \left[\frac{k}{4}(\alpha-1), \frac{k}{4}(1-\alpha)\right] & \text{for } k+1 = 2^i\ (i > 2);\\ [0,0] & \text{otherwise.}\end{cases}$$

It is easy to prove that X_k and (ΔX_k) are I-bounded, but are not I-convergent.

Acknowledgment The author is thankful to the reviewers for their fruitful suggestions which improved the presentation of the paper.

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