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Determination of a Jump by Conjugate Fourier-Jacobi Series

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Abstract. We prove the equiconvergence related to conjugate Fourier-Jacobi series and differentiated Fourier-Jacobi series for functions of harmonic bounded variation. A jump of a such function is determined by the partial sums of its conjugate Fourier-Jacobi series.

1. Introduction and Preliminaries

"Conjugacy" is an important concept in classical Fourier analysis which connects the study of the fundamental properties of harmonic functions to that of analytic functions [16]. Conjugate Fourier-Jacobi series was introduced by B. Muckenhoupt and E. M. Stein [11] when $\alpha = \beta$, and by Zh.-K. Li [9] for general α and β.
Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order $(\alpha,\beta), \alpha,\beta > -1$, normalized so that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. These polynomials are orthogonal on the interval $(-1,1)$ with respect to the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$.

Define $R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$, and denote by $L_p(\alpha,\beta), (1 \leqslant p < \infty)$ the space of functions $f(x)$ for which $\|f\|_{p(\alpha,\beta)} = \{\int_{-1}^{1} |f(x)|^p d\mu_{\alpha,\beta}(x)\}^{\frac{1}{p}}$ is finite.

For functions $f \in L_1(\alpha,\beta)$, its Fourier-Jacobi expansion is

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x), \tag{1}$$

where

$$\hat{f}(n) = \int_{-1}^{1} f(y) R_n^{(\alpha,\beta)}(y) d\mu_{\alpha,\beta}(y)$$

are the Fourier coefficients and

$$\omega_n^{(\alpha,\beta)} = \{\int_{-1}^{1} [R_n^{(\alpha,\beta)}(y)]^2 d\mu_{\alpha,\beta}(y)\}^{-1} \sim n^{2\alpha+1}.$$

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An alternative way is to define Fourier-Jacobi expansion of a function f on $(0,\pi)$ by (2).

$$f(\theta) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(cos\theta), \tag{2}$$

where

$$\hat{f}(n) = \int_0^{\pi} f(\varphi)R_n^{(\alpha,\beta)}(cos\varphi)d\mu_{\alpha,\beta}(\varphi),$$

$$\omega_n^{(\alpha,\beta)} = \{\int_0^{\pi}[R_n^{(\alpha,\beta)}(cos\varphi)]^2 d\mu_{\alpha,\beta}(\varphi)\}^{-1} \sim n^{2\alpha+1}, \tag{3}$$

and correspondingly $d\mu_{\alpha,\beta}(\theta) = 2^{\alpha+\beta+1}sin^{2\alpha+1}\frac{\theta}{2}cos^{2\beta+1}\frac{\theta}{2}d\theta$.

To the Fourier-Jacobi series of the form (2), its conjugate series is defined by

$$\tilde{f}(\theta) \sim \frac{1}{2\alpha+2}\sum_{n=1}^{\infty} n\hat{f}(n)\omega_n^{(\alpha,\beta)}R_{n-1}^{(\alpha+1,\beta+1)}(cos\theta)sin\theta, \tag{4}$$

(see [11]). If we start with (1), and $x = cos\theta$, this would correspond to

$$\tilde{f}(x) \sim \frac{-1}{2\alpha+2}\sum_{n=1}^{\infty} n\hat{f}(n)\omega_n^{(\alpha,\beta)}R_{n-1}^{(\alpha+1,\beta+1)}(x)\sqrt{1-x^2}. \tag{5}$$

Denote by $S_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1), and by $\tilde{S}_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (5). If $\alpha = \beta = -\dfrac{1}{2}$, the corresponding Fourier-Jacobi series becomes Fourier-Tchebycheff series, so by $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)$ we denote the n-th partial sum of the Fourier-Tchebycheff series of f (see [4]).

This paper consists of three main parts. In the first part we give a review of results about the determination of a jump of functions of generalized bounded variation by their Fourier series. Determination of a jump through generalized Fourier-Jacobi series is presented in the second part, where we also give a proof of the corresponding (C,α) summability result for the Wiener's classes $\mathcal{V}_p$ from the review paper [12] (by the author of this work Samra Sadiković - former surname Pirić and the coauthor Z. Šabanac). In the third part we prove the equiconvergence theorem related to conjugate Fourier-Jacobi series and the differentiated Fourier-Jacobi series and obtain a new way for determination of a jump by conjugate Fourier-Jacobi series.

2. Determination of a jump trough Fourier series

The problem of determination of jump discontinuities in piecewise smooth functions from their spectral data is relevant in signal processing [6]. Locating the discontinuities of a function by means of its truncated Fourier series, arises naturally from an attempt to overcome the Gibbs phenomenon, the poor approximative properties of the Fourier partial sums of a discontinuous function (i.e. the finite sum approximation of the discontinuous function overshoots the function itself, at a discontinuity by about 18 percent).

If a function f is integrable on $[-\pi,\pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, cos\, nx, sin\, nx\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of f by $S_n(x,f)$, i.e.

$$S_n(x,f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n}(a_k(f)cos\, kx + b_k(f)sin\, kx), \tag{6}$$

where $a_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\cos kt\,dt$ and $b_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\sin kt\,dt$
are the $k-$th Fourier coefficients of the function f.
By $\tilde{S}_n(x, f)$ we denote the $n-$th partial sum of the conjugate series, i.e.,

$$\tilde{S}_n(x, f) = \sum_{k=1}^{n}(a_k(f)\sin kx - b_k(f)\cos kx).$$

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time. Let $f(x)$ be a function of bounded variation with period 2π, and $S_n(x, f)$ be the partial sum of order n of its Fourier series. By the classical theorem of Fejér [16] the identity

$$\lim_{n\to\infty} \frac{S'_n(x, f)}{n} = \frac{1}{\pi}(f(x + 0) - f(x - 0)) \tag{7}$$

holds at any point x. To characterize continous periodic functions of BV in terms of their Fourier coefficients, Wiener [15] has introduced a concept of higher variation. A function f is said to be of bounded p-variation, $p \geq 1$, on the segment $[a, b]$ and to belong to the class $\mathcal{V}_p[a, b]$ if

$$V_{a\,p}^{b}(f) = \sup_{\Pi_{a,b}} \Big\{ \sum_i |f(x_i) - f(x_{i-1})|^p \Big\}^{\frac{1}{p}} < \infty,$$

where $\Pi_{a,b} = \{a = x_0 < x_1 < ... < x_n = b\}$ is an arbitrary partition of the segment $[a, b]$. $V_{a\,p}^{b}(f)$ is the p-variation of f on $[a, b]$.
B. I. Golubov [7] has shown that identity (7) is valid for classes $\mathcal{V}_p$, $(1 \leq p < \infty)$, i.e. for $r \in \mathbb{N}_0$

$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(x, f)}{n^{2r+1}} = \frac{(-1)^r}{(2r + 1)\pi}(f(x + 0) - f(x - 0)).$$

Problems of everywhere convergence of Fourier series for every change of variable have led D. Waterman [14] to another type of generalization.

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n}$ diverges and $\{I_n\}$ be a sequence of nonoverlapping segments $I_n = [a_n, b_n] \subset [a, b]$. A function f is said to be of Λ-bounded variation on $I = [a, b]$, i.e. $f \in \Lambda BV$ if $\sum \frac{|f(b_n) - f(a_n)|}{\lambda_n} < \infty$ for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f on I. In the case $\Lambda = \{n\}$, one speaks of harmonic bounded variation (HBV).

The class HBV contains all Wiener classes. M. Avdispahić has shown in [3] that equation (7) holds for any function $f \in HBV$ and that HBV is the limiting case for validity of the identity (7) in the scale of ΛBV spaces.

The third interesting generalization of the Jordan variation was given by Z. A. Chanturiya [5]. The modulus of variation of a bounded 2π periodic function f is the function ν_f with domain the positive integers, given by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(b_k) - f(a_k)|,$$

where $\Pi_n = \{[a_k, b_k]; k = 1, ..., n\}$ is an arbitrary partition of $[0, 2\pi]$ into n nonoverlapping segments.
By [1], there exist the following inclusion relations between Wiener's, Waterman's and Chanturiya's classes:

$$\{n^\alpha\}BV \subset \mathcal{V}_{\frac{1}{1-\alpha}} \subset V[n^\alpha] \subset \{n^\beta\}BV,$$

for $0 < \alpha < \beta < 1$.

Clearly, Fejér's identity (7) is a statement about Cesàro summability of the sequence $\{kb_k \cos kx - ka_k \sin kx\}$, $a_k = a_k(f)$ and $b_k = b_k(f)$ being the k-th cosine and sine coefficient, respectively. As it is well-known, a sequence s_n is Cesàro or $(C, 1)$ summable to s if the sequence σ_n of its arithmetical means converges to s, i.e. $\sigma_n = \dfrac{s_0 + s_1 + \ldots + s_n}{n+1} \to s, n \to \infty$.

Analogously, the sequence s_n is (C, α), $\alpha > -1$, summable to s if the sequence

$$\sigma_n^{(\alpha)} = \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} s_k, \tag{8}$$

converges to s.

Looking at Fejer's theorem in this way, several mathematicians have extended it to more general summability methods. We note two results [2] which represent the extension to (C, α) summability, $\alpha > 0$:

Theorem(A). If $f \in \mathcal{V}_p$, $p > 1$ the sequence $\{kb_k \cos kx - ka_k \sin kx\}$ is (C, α) summable to $\dfrac{1}{\pi}(f(x+0) - f(x-0))$ for any $\alpha > 1 - \dfrac{1}{p}$ and every x.

Corollary(B). *If $f \in V[n^\beta]$ ($\{n^\beta\}BV$) for some $0 < \beta < 1$, then the sequence $\{kb_k \cos kx - ka_k \sin kx\}$ is summable to $\dfrac{1}{\pi}(f(x+0) - f(x-0))$ by any Cesaro method of order $\alpha > \beta$.*

Theorem (A) and Corollary (B), are in some sense the most natural generalization of Fejér's theorem. Indicating the relationship between the order of Cesàro summability of the sequence $\{kb_k(f) \cos kx - ka_k(f) \sin kx\}$ and the "order of variation" of a function f, they complete the earlier picture whose elements were:
1) (C, α) summability for $\alpha > 0$ and the class BV;
2) (C, α) summability for $\alpha > 1$ and whole class of regulated functions W(i.e. functions possessing the one-sided limits at each point), which is the union of all ΛBV spaces;
3) $(C, 1)$ summability for the class HBV.

As an application of Theorem (A) we point to the next result [2].

Corollary(C). *If $f \in \mathcal{V}_p$, $p > 1$, then the conjugate series $\tilde{S}(f; x)$ is (C, α), $\alpha > -\dfrac{1}{p}$, summable to*

$$\tilde{f}(x) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2tg\frac{t}{2}} dt;$$

whenever this limit exists.

The following theorem was proved by Ferenc Lukács [10]. This well-known result determines the generalized jumps of a periodic, Lebesgue integrable function f in terms of the partial sum of the conjugate series to the Fourier series of f.

Theorem(D). Let $f \in L^1(T)$ and $x \in T$, where $T := [-\pi, \pi)$. If there exist a number $d_x(f)$ such that

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h |[f(x+t) - f(x-t)] - d_x(f)| dt = 0, \tag{9}$$

then

$$\lim_{n\to\infty}\frac{\tilde{S}_n(f,x)}{logn}=-\frac{1}{\pi}d_x(f). \tag{10}$$

By "log" we mean the natural logarithm.
It is clear that if the finite limit

$$d_x(f):=\lim_{t\to 0+}[f(x+t)-f(x-t)]$$

exists, then condition (9) is also satisfied with the same $d_x(f)$. In particular, if a periodic function f is of bounded variation over $[-\pi,\pi]$, then (10) is satisfied at every point $x\in T$ with

$$d_x(f):=f(x+0)-f(x-0).$$

3. Cesàro summability and determination of a jump through generalized Fourier-Jacobi series

We say that a function w is a generalized Jacobi weight and write $w\in GJ$, if

$$w(t)=h(t)(1-t)^{\alpha}(1+t)^{\beta}|t-x_1|^{\delta_1}...|t-x_M|^{\delta_M},$$

$$h\in C[-1,1],\ h(t)>0\ (|t|\le 1),\ \ \omega(h;t;[-1,1])t^{-1}\in L[0,1],$$

$$-1<x_1<...<x_M<1,\ \ \alpha,\beta,\delta_1,...,\delta_M>-1.$$

By $\sigma(w)=(P_n(w;x))_{n=0}^{\infty}$ we denote the system of algebraic polynomials $P_n(w;x)=\gamma(w)x^n+$ lower degree terms with positive leading coefficients $\gamma_n(w)$, which are orthonormal on $[-1,1]$ with respect to the weight $w\in GJ$, i.e.,

$$\int_{-1}^{1}P_n(w;t)P_m(w;t)w(t)dt=\delta_{nm}.$$

Such polynomials are called the generalized Jacobi polynomials.
If $fw\in L[-1,1],\ w\in GJ$, then the n-th partial sum of the Fourier series of f with respect to the system $\sigma(w)$ is given by $S_n(w;f;x)=\displaystyle\sum_{k=0}^{n-1}a_k(w;f)P_k(w;x)=\int_{-1}^{1}f(t)K_n(w;x;t)w(t)dt$, where $a_k(w;f)=\displaystyle\int_{-1}^{1}f(t)P_k(w;t)w(t)dt$ is the k-th Fourier coefficient of the function f and

$$K_n(w;x;t)=\sum_{k=0}^{n-1}P_k(w;x)P_k(w;t)$$

is the Dirichlet kernel of the system $\sigma(w)$.
For a given weight $w\in GJ$ it is assumed that $x_0=-1$ and $x_{M+1}=1$. In addition,

$$\Delta(\nu;\varepsilon)=[x_\nu+\varepsilon;x_{\nu+1}-\varepsilon]$$

for a fixed $\varepsilon\in(0,\frac{x_{\nu+1}-x_\nu}{2}),\ \ \nu=1,2,...,M.$

When $h(t)\equiv 1$, $|t|\le 1$, and $M=0$ (i.e. a weight does not have singularities strictly inside of the segment $(-1,1)$), $w\in GJ$ is called a Jacobi weight, and in this case we use the commonly accepted notation "(α,β)" instead of "w" throughout. For example, we write $S_n^{(\alpha,\beta)}(f,x)$ instead of $S_n(w;f;x)$, and so on.

The following theorem establishes an identity which determines the jumps of a bounded function by means of its differentiated partial sums of generalized Fourier-Jacobi series [8].

Theorem(E). Let $r \in \mathbb{N}_0$, $w \in GJ$, and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then the identity

$$\lim_{n\to\infty} \frac{(S_n(w;f;x))^{2r+1}}{n^{2r+1}} = \frac{(-1)^r(1-x^2)^{-r-\frac{1}{2}}}{(2r+1)\pi}(f(x+0)-f(x-0)) \tag{11}$$

is valid for every $f \in \Lambda BV$ and each fixed $x \in (-1,1)$, $x \neq x_1, ..., x_M$, if $\Lambda BV \subseteq HBV$. If, in addition, the weight $w \in GJ$ satisfies the following conditions:

$$\alpha \geq -\frac{1}{2},\ \ \beta \geq -\frac{1}{2},\ \delta_1 \geq 0, ..., \delta_M \geq 0,\ \ \omega(h;t)t^{-1}\ln t \in L^1([0,1]), \tag{12}$$

then condition $\Lambda BV \subseteq HBV$ is necessary for the validity of identity (11) for every $f \in \Lambda BV$ and each fixed $x \in (-1,1)$, $x \neq x_1, ..., x_M$ as well.

In [12], we established the corresponding (C,α) summability result for the Wiener's classes $\mathcal{V}_p$ in the case of Fourier-Jacobi series.

Theorem(F). Let f be a function of bounded p-variation, i.e. $f \in \mathcal{V}_p$, $p > 1$, such that $fw \in L^1([-1,1])$, $w \in GJ$. Then the sequence $(a_n(w;f)P_n'(w;x))$ is (C,α), $\alpha > 1 - \dfrac{1}{p}$ summable to $\dfrac{(1-x^2)^{-\frac{1}{2}}}{\pi}(f(x+0)-f(x-0))$ for every $x \in (-1,1)$, $x \neq x_1, ..., x_M$, where $a_n(w;f)P_n'(w;x)$ is the n-th term of the differentiated Fourier-Jacobi series of f.

Proof. We use the uniform equiconvergence of Fourier-Tchebycheff series and Fourier series with respect to the system of generalized Jacobi polynomials for an arbitrary function $f \in HBV$ and a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\nu = 0, 1, 2, ..., M$,

$$||S_n(w;f;x) - S_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)||_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1), \tag{13}$$

proved by G. Kvernadze [8, (63)]. By the triangle inequality we get

$$||a_n(w;f)P_n(w;x) - a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)||_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1). \tag{14}$$

So, it is enough to show that the sequence $(a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})'}(x))$, $f \in \mathcal{V}_p$ is (C,α) summable, $\alpha > 1 - \dfrac{1}{p}$, to $\dfrac{(1-x^2)^{-\frac{1}{2}}}{\pi}(f(x+0)-f(x-0))$ for each fixed $x \in (-1,1)$, $x \neq x_1, ..., x_M$.

From an obvious identity [13]

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x) = S_n(g,\theta),$$

where $x = \cos\theta$, one has

$$a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = a_n(g)\cos n\theta + b_n(g)\sin n\theta. \tag{15}$$

Differentiating the identity (15) we obtain

$$a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})'}(x) = -(1-x^2)^{-\frac{1}{2}}(nb_n(g)\cos n\theta - na_n(g)\sin n\theta).$$

By Theorem (A) and taking into account that $f(x\pm) = g(\theta\mp)$, $\theta \in [0,\pi]$, we derive (C,α) summability of the sequence $(a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})'}(x))$, $\alpha > 1 - \dfrac{1}{p}$.

Finally, the result follows from the equiconvergence formula (14). □

Consesequently, the corresponding (C,α) summability result holds for the Waterman's classes $\{n^\beta\}BV$ and the Chanturiya's classes $V[n^\beta]$ if $\alpha > \beta$, $0 < \beta < 1$.

Corollary(G). If f belongs to $\{n^\beta\}BV$ or $V[n^\beta]$, $0 < \beta < 1$, then the claim of Theorem (F) is valid for (C,α), $\alpha > \beta$.

Using the equiconvergence formula [8, (63)] i.e.

$$\|S_n(w;f;x) - S_n^{(\alpha,\beta)}(f,x)\|_{C[\Delta(\nu;\varepsilon)]} = o(1),$$

for every fixed $\varepsilon \in (0,(x_{\nu+1}-x_\nu)/2)$, $\nu = 0,1,\ldots,M$, G. Kvernadze in [8] obtained an identity determining the jumps of a piecewise continuous function of bounded harmonic variation by means of its Fourier-Jacobi partial sums. In particular,

Theorem(H). Let $r \in \mathbb{N}_0$, $\alpha > -1, \beta > -1$, and $f \in HBV$. Then the identity

$$\lim_{n\to\infty} \frac{(S_n^{(\alpha,\beta)}(f,x))^{2r+1}}{n^{2r+1}} = \frac{(-1)^r(1-x^2)^{-r-\frac{1}{2}}}{(2r+1)\pi}(f(x+0)-f(x-0))$$

is valid for each fixed $x \in (-1,1)$.

4. Main results

Theorem 4.1. *Let $\alpha > -1, \beta > -1$, and $f \in HBV$. If $S_n^{(\alpha,\beta)}(f,x)$ denotes the n-th partial sum of the Fourier-Jacobi series and $\tilde{S}_n^{(\alpha,\beta)}(f,x)$ denotes the n-th partial sum of conjugate Fourier-Jacobi series, then for $-1 < x < 1$,*

$$\lim_{n\to\infty}\left(\frac{\left(S_n^{(\alpha,\beta)}(f,x)\right)'}{n} + \frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\log n\sqrt{1-x^2}}\right) = 0.$$

Proof. Differentiating the expression for the n-th partial sum

$$S_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n} \hat{f}(k)\omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(x),$$

and using [13, 4.21.7.] i.e.

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

we get $\left[S_n^{(\alpha,\beta)}(f,x)\right]' = \sum_{k=0}^n \hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{d}{dx}(R_k^{(\alpha,\beta)}(x))$
$= \sum_{k=0}^n \hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{d}{dx}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} =$
$= \sum_{k=0}^n \hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{d}{dx}(P_k^{(\alpha,\beta)}(x))\frac{1}{P_k^{(\alpha,\beta)}(1)} =$
$= \sum_{k=0}^n \hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{(k+\alpha+\beta+1)}{2}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_k^{(\alpha,\beta)}(1)} =$
$= -\frac{1}{2}\sum_{k=0}^n (k+\alpha+\beta+1)\hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_k^{(\alpha,\beta)}(1)}.$

As $P_{k-1}^{(\alpha+1,\beta+1)}(1) = \begin{pmatrix} k+\alpha \\ k-1 \end{pmatrix}$, we have

$$\frac{P_{k-1}^{(\alpha+1,\beta+1)}(1)}{P_k^{(\alpha,\beta)}(1)} = \frac{\begin{pmatrix} k+\alpha \\ k-1 \end{pmatrix}}{\begin{pmatrix} k+\alpha \\ k \end{pmatrix}} = \frac{k}{\alpha+1}.$$

So $P_k^{(\alpha,\beta)}(1) = \dfrac{P_{k-1}^{(\alpha+1,\beta+1)}(1)}{\frac{k}{\alpha+1}}$, and we get

$$\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_k^{(\alpha,\beta)}(1)} = \frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_{k-1}^{(\alpha+1,\beta+1)}(1)} \cdot \frac{k}{\alpha+1} = R_{k-1}^{(\alpha+1,\beta+1)}(x) \cdot \frac{k}{\alpha+1}.$$

Now, we have

$$\left[S_n^{(\alpha,\beta)}(f,x)\right]' = \sum_{k=1}^{n} \frac{k\cdot(k+\alpha+\beta+1)}{2\alpha+2} \hat{f}(k)\omega_k^{(\alpha,\beta)} \cdot R_{k-1}^{(\alpha+1,\beta+1)}(x).$$

According to (4)

$$-\tilde{S}_n^{(\alpha,\beta)}(f,x) = \frac{1}{2\alpha+2}\sum_{k=1}^{n} k\cdot \hat{f}(k)\omega_k^{(\alpha,\beta)} \cdot R_{k-1}^{(\alpha+1,\beta+1)}(x)\sqrt{1-x^2} =$$

$$= \frac{1}{2\alpha+2}\sum_{k=1}^{n} \frac{1}{k+\alpha+\beta+1}(k+\alpha+\beta+1)k\cdot \hat{f}(k)\omega_k^{(\alpha,\beta)} \cdot R_{k-1}^{(\alpha+1,\beta+1)}(x)\sqrt{1-x^2}.$$

Using Abel's summation formula we get

$$\begin{aligned}
-\frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\sqrt{1-x^2}} &= \frac{1}{n+\alpha+\beta+1}\left[S_n^{(\alpha,\beta)}(f,x)\right]' + \\
&+ \sum_{k=1}^{n-1}\left[\frac{1}{k+\alpha+\beta+1} - \frac{1}{k+\alpha+\beta+2}\right]\left[S_k^{(\alpha,\beta)}(f,x)\right]' = \\
&= \frac{1}{n+\alpha+\beta+1}\left[S_n^{(\alpha,\beta)}(f,x)\right]' + \\
&+ \sum_{k=1}^{n-1}\frac{1}{(k+\alpha+\beta+1)(k+\alpha+\beta+2)}\left[S_k^{(\alpha,\beta)}(f,x)\right]'.
\end{aligned}$$

So, we have

$$\begin{aligned}
\frac{-\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\sqrt{1-x^2}} &= \frac{1}{n+\alpha+\beta+1}\left[S_n^{(\alpha,\beta)}(f,x)\right]' + \\
&+ \sum_{k=1}^{n-1}\frac{1}{k+\alpha+\beta+2}\left(\frac{\left(S_k^{(\alpha,\beta)}(f,x)\right)'}{k+\alpha+\beta+1} - \frac{d}{\pi\sqrt{1-x^2}}\right] + \\
&+ \frac{d}{\pi\sqrt{1-x^2}}\sum_{k=1}^{n-1}\frac{1}{k+\alpha+\beta+2}, \qquad (16)
\end{aligned}$$

where $d = f(x+0) - f(x-0)$, as from Theorem (H) for $r = 0$ we have

$$\lim_{n\to\infty} \frac{\left(S_n^{(\alpha,\beta)}(f,x)\right)'}{n} = \frac{d}{\pi\sqrt{1-x^2}}. \tag{17}$$

Dividing (16) by $\log n$ (by "log" we mean the natural logarithm) we get

$$\begin{aligned}
-\frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\log n\sqrt{1-x^2}} &= \frac{1}{\log n}\frac{1}{n+\alpha+\beta+1}\left[S_n^{(\alpha,\beta)}(f,x)\right]' + \\
&+ \frac{1}{\log n}\sum_{k=1}^{n-1}\frac{1}{k+\alpha+\beta+2}\left[\frac{\left(S_k^{(\alpha,\beta)}(f,x)\right)'}{k+\alpha+\beta+1} - \frac{d}{\pi\sqrt{1-x^2}}\right] + \\
&+ \frac{1}{\log n}\frac{d}{\pi\sqrt{1-x^2}}\sum_{k=1}^{n-1}\frac{1}{k+\alpha+\beta+2}.
\end{aligned}$$

Now, subtracting $\dfrac{d}{\pi\sqrt{1-x^2}}$ from both sides of the last equality, we have

$$\begin{aligned}
-\frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\log n\sqrt{1-x^2}} - \frac{d}{\pi\sqrt{1-x^2}} &= \frac{1}{\log n}\frac{1}{n+\alpha+\beta+1}\left[S_n^{(\alpha,\beta)}(f,x)\right]' + \\
&+ \frac{1}{\log n}\sum_{k=1}^{n-1}\frac{1}{k+\alpha+\beta+2}\left[\frac{\left(S_k^{(\alpha,\beta)}(f,x)\right)'}{k+\alpha+\beta+1} - \frac{d}{\pi\sqrt{1-x^2}}\right] + \\
&+ \frac{d}{\pi\sqrt{1-x^2}}\left[\frac{1}{\log n}\sum_{k=1}^{n-1}\frac{1}{k+\alpha+\beta+2} - 1\right].
\end{aligned}$$

Letting $n\to\infty$, having in mind (17) and $\displaystyle\sum_{k=1}^{n-1}\frac{1}{k+1} \sim \log n$, we get

$$\lim_{n\to\infty}\left(-\frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\log n\sqrt{1-x^2}} - \frac{d}{\pi\sqrt{1-x^2}}\right) = 0. \tag{18}$$

Finally, adding (17) and (18) proves the result. □

Corollary 4.2. *Let $\alpha > -1$, $\beta > -1$ and $f \in HBV$. Then the identity*

$$\lim_{n\to\infty}\frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\log n} = -\frac{f(x+0)-f(x-0)}{\pi}$$

is valid for each fixed $x \in (-1,1)$.

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