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On Mixed Quasi-Einstein Spacetimes

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Abstract. The object of the present paper is to study mixed quasi-Einstein spacetimes, briefly $M(QE)_4$ spacetimes. First we prove that every Z Ricci pseudosymmetric $M(QE)_4$ spacetimes is a Z Ricci semisymmetric spacetime. Then we study Z flat spacetimes. Also we consider Ricci symmetric $M(QE)_4$ spacetimes and among others we prove that the local cosmological structure of a Ricci symmetric $M(QE)_4$ perfect fluid spacetime can be identified as Petrov type I, D or O. We show that such a spacetime is the Robertson-Walker spacetime. Moreover we deal with mixed quasi-Einstein spacetimes with the associated generators U and V as concurrent vector fields. As a consequence we obtain some important theorems. Among others it is shown that a perfect fluid $M(QE)_4$ spacetime of non zero scalar curvature with the basic vector field of spacetime as velocity vector field of the fluid is of Segré characteristic $[(1,1,1),1]$. Also we prove that a $M(QE)_4$ spacetime can not admit heat flux provided the smooth function b is not equal to the cosmological constant k. This means that such a spacetime describe a universe which has already attained thermal equilibrium. Finally, we construct a non-trivial Lorentzian metric of $M(QE)_4$.

1. Introduction

General relativity is the flagship of Applied Mathematics. Almost from the inception, general relativity was regarded as an extra-ordinary difficult theory and triuamph of the human intellect, the most beautiful physical theory ever created. General relativity is an essential tool for the study of cosmology, the science of universe and a model of nature, especially of gravity that neglects quantum effects.

To day for the domain of macrophysics, general theory of relativity is the best available simple and elegant theory and is expected to be very exciting for many years to come. What Physicists / Mathematical Scientists can learn or acquire knowledge from the discovery of general relativity ? They generally learn/study the three stages in the evolution of Einstein's ideas from special to general relativity. (i) The first is the abandonment of the privileged states of inertial frame, i.e., Newtonian-Euclidean space mechanics for the study of nature. (ii) The second is acceptance of the dynamical role of the mertic (g), i.e., the study of non-linear behaviour of nature. (iii) The third is that a spacetime has to be considered as an equivalence class of pseudo-Riemannian geometry, i.e., by modern differential geometry.

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Modern differential geometry has become more and more important in theoretical physics which it has led to a greater simplicity in mathematics and a more fundamental understanding of physics.

The spacetime of general relativity and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that Lorentz manifold becomes a convenient choice for the study of general relativity. Indeed by basing its study on Lorentzian manifold the general theory of relativity opens the way to the study of global questions about it ([2], [7], [11], [12], [13]) and many others.

A non-flat semi-Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold [4] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1}$$

where a, b are scalars. Such an n-dimensional manifold is denoted by $(QE)_n$.
In 2010 Nagaraja [17] generalizes the quasi-Einstein manifold as follows:
A non-flat semi-Riemannian manifold (M^n, g) $(n \geq 3)$ is called mixed quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)B(Y) + cB(X)A(Y), \tag{2}$$

where a, b and c are smooth functions and A and B are non-zero 1-forms such that $g(X, U) = A(X)$ and $g(X, V) = B(X)$ for all vector fields X and U and V being the orthogonal unit vector fields called the generator of the manifold.
From (2), it follows that

$$S(Y, X) = ag(Y, X) + bA(Y)B(X) + cB(Y)A(X). \tag{3}$$

From (2) and (3), it follows that

$$(b - c)[A(X)B(Y) - A(Y)B(X)] = 0.$$

This shows that either $b = c$ or, $A(X)B(Y) = A(Y)B(X)$. Motivated by this result De and Mallick [14] have given the following definition:
A non-flat semi-Riemannian manifold (M^n, g) $(n \geq 3)$ is called mixed quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition:

$$S(X, Y) = ag(X, Y) + b[A(X)B(Y) + A(Y)B(X)], \tag{4}$$

where a, b are scalars of which $b \neq 0$ and A and B are non-zero 1-forms such that

$$g(X, U) = A(X), \quad g(X, V) = B(X), \quad g(U, V) = 0,$$

where U, V are unit vector fields. In such a case A, B are called associated 1-forms and U, V are called the generators of the manifold. Such an n-dimensional manifold is denoted by the symbol $M(QE)_n$.
If $b = 0$, then the manifold becomes an Einstein manifold. If $A = B$, then the manifold reduces to a quasi-Einstein manifold. This justifies the name mixed quasi-Einstein manifold.
In 2012, Mantica and Molinari [15] defined a generalized $(0, 2)$ symmetric Z tensor given by

$$Z(X, Y) = S(X, Y) + \phi g(X, Y), \tag{5}$$

where ϕ is an arbitrary scalar function. In Refs. ([15], [16]) various properties of the Z tensor were pointed out. A spacetime is said to be Z flat if the Z tensor vanishes at each point of the spacetime.
A semi-Riemannian manifold $(M^n, g), n \geq 3$, is said to be Z Ricci pseudosymmetric [22] if and only if the relation

$$Z.L = f_L P(g, L). \tag{6}$$

holds on the set $U_L = \{x \in M : P(g, L) \neq 0 \quad at \quad x\}$, where L is the Ricci operator defined by $S(X, Y) = g(LX, Y)$ and f_L is some smooth function on U_L. Also $P(g, L)$ is defined by $P(g, L)(W; X, Y) = L((X \wedge_g Y)W)$

for all vector fields X, Y, W.
A Lorentzian manifold (M^4, g) is said to be mixed quasi Einstein spacetime if the Ricci tensor satisfies (4).
Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y). \tag{7}$$

The equation (7) of Einstein imply that matter determines the geometry of spacetime and conversely, the motion of matter is determined by the metric tensor of the space which is not flat.
In general relativity the matter content of the spacetime is described by the energy momentum tensor. The matter content is assumed to be a fluid having density and pressure and possing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion.
In a perfect fluid spacetime, the energy momentum tensor T of type $(0,2)$ is of the form ([18]):

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{8}$$

where σ and p are the energy density and the isotropic pressure respectively. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [12]. In addition, p and σ are related by an equation of state governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\sigma, T_0)$, where T_0 is the absolute temperature. However, we shall only be concerned with situations in which T_0 is effectively constant so that the equation of state reduces to $p = p(\sigma)$. In this case, the perfect fluid is called isentropic [12]. Moreover, if $p = \sigma$, then the perfect fluid is termed as stiff matter (see [21], page 66).

The curl of a vector field U [18] is given by

$$(curlU)(X,Y) = g(\nabla_X U, Y) - g(\nabla_Y U, X).$$

If we denote the projection tensor h, then $h(X) = X + A(X)U$. Since the vorticity tensor ω is the projection of curl of U, from above equation we get

$$\omega(X,Y) = g(\nabla_{hX} U, hY) - g(\nabla_{hY} U, hX) = 0.$$

Again the shear tensor σ [6] is given by

$$\sigma = \frac{1}{2}[g(\nabla_{hX} U, hY) - g(\nabla_{hY} U, hX)] - \frac{1}{3}(divU)g(hX,Y).$$

Several authors have studied spacetimes in different ways. Motivated by the studies of those authors in the present paper we characterize mixed quasi-Einstein spacetimes.
The present paper is organized as follows:
After introduction in Section 2, we study Z Ricci pseudosymmetric mixed quasi-Einstein spacetimes. Section 3 is devoted to study Z flat spacetimes. In Section 4, we study Ricci symmetric mixed quasi-Einstein spacetimes. Section 5 deals with mixed quasi-Einstein spacetimes with the associated vector fields U and V as concurrent vector fields. Section 6 deals with heat flux in a $M(QE)_4$ spacetime. Finally, we construct an example of a mixed quasi-Einstein spacetime.

2. *Z* Ricci pseudosymmetric mixed quasi-Einstein spacetimes

In this section we consider Z Ricci pseudosymmetric $M(QE)_4$. Therefore from (6) we get

$$(Z(X,Y).L)W = f_L P(g,L)(W;X,Y), \tag{9}$$

for any vector fields X, Y, W. If $f_L = 0$, then the manifold reduces to a Z Ricci semisymmetric manifold. Now

$$\begin{aligned}(Z(X,Y).L)(W) &= ((X \wedge_Z Y).L)W \\ &= (X \wedge_Z Y)LW - L((X \wedge_Z Y)) \\ &= Z(Y,LW)X - Z(X,LW)Y - Z(Y,W)LX \\ &\quad -Z(X,W)LY.\end{aligned} \tag{10}$$

Also

$$\begin{aligned} P(g,L)(W;X,Y) &= L((X\wedge_g Y)W) \\ &= L(g(Y,W)X - g(X,W)Y) \\ &= g(Y,W)LX - g(X,W)LY. \end{aligned} \tag{11}$$

From (10) and (11) we have

$$\begin{aligned} &Z(Y,LW)X - Z(X,LW)Y - Z(Y,W)LX - Z(X,W)LY \\ &\qquad = f_L\{g(Y,W)LX - g(X,W)LY\}. \end{aligned} \tag{12}$$

From (4) we infer that

$$LX = aX + b[A(X)V + B(X)U]. \tag{13}$$

Using (13) in (12) it follows that

$$\begin{aligned} & aZ(Y,W)X + b(\phi + a)\{A(W)B(Y) + B(W)A(Y)\}X \\ & +b^2\{A(W)A(Y) + B(W)B(Y)\}X \\ & -aZ(X,W)Y - b(\phi + a)\{A(W)B(X) + B(W)A(X)\}Y \\ & -b^2\{A(W)A(X) + B(W)B(X)\}Y \\ = \;& \{Z(Y,W) + f_L g(Y,W)\}LX + \{Z(X,W) - f_L g(X,W)\}LY. \end{aligned} \tag{14}$$

Putting $X = U$ in (14) yields

$$\begin{aligned} & aZ(Y,W)U + b(\phi + a)\{A(W)B(Y) + B(W)A(Y)\}U \\ & +b^2\{A(W)A(Y) + B(W)B(Y)\}U \\ & -aZ(U,W)Y - b(\phi + a)\{A(W)B(U) + B(W)A(U)\}Y \\ & -b^2\{A(W)A(U) + B(W)B(U)\}Y \\ = \;& \{Z(Y,W) + f_L g(Y,W)\}LU + \{Z(U,W) - f_L g(U,W)\}LY. \end{aligned} \tag{15}$$

Again substituting $Y = V$ in (15) we have

$$\begin{aligned} & aZ(V,W)U + b(a+\phi)A(W)U + b^2B(W)U \\ & -a\{(a+\phi)A(W) + bB(W)\}V - b(a+\phi)B(W)V - b^2A(W)V \\ = \;& \{S(V,W) + \phi g(V,W) + f_L g(V,W)\}(aU + bV) \\ & +\{(a+\phi)A(W) + bB(W) - f_L A(W)\}LV. \end{aligned} \tag{16}$$

Taking inner product of (16) with U, we get

$$f_L(aB(W) - bA(W)) = 0. \tag{17}$$

Therefore either $f_L = 0$, or $aB(W) - bA(W) = 0$.
If $aB(W) - bA(W) = 0$, then $aV - bU = 0$. Thus $a = 0 = b$. Therefore from (4), we have $S(X,Y) = 0$. This contradicts the definition of $M(QE)_4$. Therefore in view of the above we can state the following:

Theorem 2.1. *A Z Ricci pseudosymmetric $M(QE)_4$ is Z Ricci semisymmetric spacetime.*

3. *Z* flat spacetimes

The Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{18}$$

where κ is the gravitational constant and T is the energy momentum tensor of type $(0,2)$.
Again for Z flat spacetimes we have from the definition

$$S(X,Y) = -\frac{r}{4}g(X,Y), \tag{19}$$

which implies that the manifold is an Einstein spacetime. Hence r is constant. Using (19) in (18) we get

$$-\frac{r}{4}g(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y) \tag{20}$$

Taking covariant derivative of (20), we obtain

$$\kappa(\nabla_W T)(X,Y) = 0. \tag{21}$$

Thus in view of (21) we conclude the following:

Theorem 3.1. *In a Z flat spacetime the energy momentum tensor is covariant constant.*

Remark 3.2. *In [5] Chaki and Ray characterize general relativistic spacetimes with covariant constant energy momentum tensor.*

4. Ricci symmetric mixed quasi-Einstein spacetimes

A Lorentzian manifold is said to be Ricci symmetric if the Ricci tensor satisfies the condition $\nabla S = 0$, where ∇ is the semi-Riemannian connection. In this section we consider Ricci symmetric $M(QE)_4$. Then from (4) we obtain

$$\begin{aligned}(\nabla_W S)(X,Y) = (Wa)g(X,Y) &+ (Wb)[A(X)B(Y) + A(Y)B(X)]\\ &+b[(\nabla_W A)(X)B(Y) + A(X)(\nabla_W B)(Y)\\ &+(\nabla_W B)(X)A(Y) + B(X)(\nabla_W A)(Y)].\end{aligned} \tag{22}$$

Since the manifold is Ricci symmetric, we have

$$(\nabla_W S)(X,Y) = 0. \tag{23}$$

From (22) and (23) we obtain that

$$\begin{aligned}&(Wa)g(X,Y) + (Wb)[A(X)B(Y) + A(Y)B(X)]\\ &\quad +b[(\nabla_W A)(X)B(Y) + A(X)(\nabla_W B)(Y)\\ &\quad +(\nabla_W B)(X)A(Y) + B(X)(\nabla_W A)(Y)] = 0.\end{aligned} \tag{24}$$

Taking a frame field and after contraction over X, Y we get from (24)

$$4(Wa) + 2[(\nabla_W A)(V) + (\nabla_W B)(V)] = 0. \tag{25}$$

Since $g(U,V) = 0$, we get $g(\nabla_W U, V) + g(U, \nabla_W V) = 0$ and hence

$$(\nabla_W A)(V) + (\nabla_W B)(V) = 0. \tag{26}$$

Therefore (25) and (26) yields

$$(Wa) = 0. \tag{27}$$

Thus a is constant.
Again putting $X = U$, $Y = V$ in (24) we get

$$(Wb) = 0. \tag{28}$$

From the above we infer that b is also constant.
Putting $Y = U$ in (24) and using the fact that a, b are constants, we have

$$(\nabla_W B)(X) = 0. \tag{29}$$

This means that the vector field V corresponding to the 1-form B is parallel.
Again putting $Y = V$ in (24) and using the fact that a, b are constants, we obtain

$$(\nabla_W A)(X) = 0. \tag{30}$$

This means that the vector field U corresponding to the 1-form A is parallel.
Thus we are in a position to state the following:

Theorem 4.1. *In a Ricci symmetric $M(QE)_4$, the associated generators U and V are parallel.*

On the other hand, in view of (30), $(\nabla_W A)(Y) = g(\nabla_W U, Y) = 0$ for all Y, W. That is, $\nabla_W U = 0$ and hence $\nabla_U U = 0$.
Thus the integral curves of the vector field U are geodesics.
Similarly, in view of (29), $(\nabla_W B)(Y) = g(\nabla_W V, Y) = 0$ for all Y, W. That is, $\nabla_W V = 0$ and hence $\nabla_V V = 0$. Hence the integral curves of the vector field V are geodesics. Hence we have

Theorem 4.2. *In a Ricci symmetric $M(QE)_4$, the integral curves of the vector fields U and V are geodesics.*

Remark 4.3. *The vector field U is the velocity of a particle moving along the geodesic. If a particle moves in a force free field then the particle is always along the geodesic and the tangent vector can be identified as the velocity of the particle.*

By virtue of Theorem 4.1, the Riemannian curvature tensor satisfies

$$R(X, Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U = 0, \tag{31}$$

for all X, Y. Contracting (31), we have $S(Y, U) = 0$ for all Y.
Now we consider the matter distribution in perfect fluid whose velocity vector field is the vector field U corresponding to the 1-form A of the spacetime, that is, $g(U, U) = -1$. Therefore the energy momentum tensor T of type $(0, 2)$ is of the form ([18]):

$$T(X, Y) = pg(X, Y) + (\sigma + p)A(X)A(Y), \tag{32}$$

where σ and p are the energy density and the isotropic pressure respectively.
Hence from the Einstein's field equation we get

$$S(X, Y) - \frac{r}{2} g(X, Y) = \kappa[pg(X, Y) + (\sigma + p)A(X)A(Y)]. \tag{33}$$

Substituting $Y = U$ in (33) we have

$$-\frac{r}{2} A(X) = -\kappa\sigma A(X). \tag{34}$$

By hypothesis the spacetime is Ricci symmetric and hence r is constant, therefore σ is constant. Again taking a frame field and contracting X and Y in (33) we have $\sigma - 3p = -\frac{r}{k}$ = constant. Thus p is a constant. Thus in view of the above we can state the following:

Theorem 4.4. *In a Ricci symmetric perfect fluid $M(QE)_4$ spacetime obeying Einstein's field equation without cosmological constant the energy density and the isotropic pressure are constants.*

Further we have the energy and force equations for a perfect fluid [18], as follows:

$$U\sigma = g(grad\sigma, U) = -(\sigma + p)divU \tag{35}$$

and

$$\begin{aligned}(\sigma + p)(\nabla_U U) = -grad_{\perp}p &= -gradp - g(gradp, U)U \\ &= -gradp - (Up)U,\end{aligned} \tag{36}$$

where the spatial pressure gradient $grad_{\perp}p$ is the component of $gradp$ orthogonal to U. From the above theorem we get a Ricci symmetric perfect fluid $M(QE)_4$ spacetime satisfying Einstein's field equation without cosmological constant, both σ and p constants. Then from (35) and (36), we have

$$U\sigma = -(\sigma + p)divU = 0 \quad \text{and} \quad (\sigma + p)(\nabla_U U) = 0. \tag{37}$$

Since $\sigma + p = \frac{r}{k} =\neq 0$, we have

$$divU = 0 \quad \text{and} \quad \nabla_U U = 0. \tag{38}$$

Remark 4.5. *In cosmology we know that such a choice $\sigma + p = 0$ leads to rapid expansion of the spacetime which is now termed as inflation.*

It is known that $divU$ represents the expansion scalar and $\nabla_U U$ represents the acceleration vector. Thus in view of (38), both of them vanish. This leads to following result:

Theorem 4.6. *In a Ricci symmetric perfect fluid $M(QE)_4$ spacetime obeying Einstein's field equation without cosmological constant the expansion scalar and the acceleration vector vanish.*

By the hypothesis the spacetime under consideration is Ricci symmetric, that is, $\nabla S = 0$. Hence the scalar curvature r is constant, that is, $dr(X) = 0$, for all X. It is known from [9] that

$$\begin{aligned}(divC)(X, Y)W &= \frac{1}{2}[\{(\nabla_X S)(Y, W) - (\nabla_Y S)(X, W)\} \\ &- \frac{1}{6}\{dr(X)g(Y, W) - dr(Y)g(X, W)\}],\end{aligned} \tag{39}$$

where C is the Weyl conformal curvature tensor. Hence from $\nabla S = 0$ and $dr(X) = 0$ we get $divC = 0$. The conditions $divC = 0$ and $dr(X) = 0$ are equivalent to have a "Yang Pure Space" (see ref. [8], Eq. 2). In [8], Theorem 4.1 the authors proved that a 4-dimensional perfect fluid spacetime with $\sigma + p \neq 0$ is a Yang Pure space if and only if it is a Robertson-Walker spacetime. Thus we can state the following:

Theorem 4.7. *A Ricci symmetric $M(QE)_4$ spacetime is a Robertson-Walker spacetime.*

It is known from [18] that the curl of a vector field U is given by

$$(curlU)(X, Y) = g(\nabla_X U, Y) - g(\nabla_Y U, X). \tag{40}$$

If we denote the projection tensor h, then $h(X) = X + A(X)U$. Since the vorticity tensor ω is the projection of curl of U, from (40) we get

$$\omega(X, Y) = g(\nabla_{hX} U, hY) - g(\nabla_{hY} U, hX) = 0, \tag{41}$$

by Theorem 4.6. Thus the space-time under consideration is vorticity-free.
Again it is known from [6] that the shear tensor σ is given by

$$\begin{aligned}\sigma &= \frac{1}{2}[g(\nabla_{hX} U, hY) - g(\nabla_{hY} U, hX)] - \frac{1}{3}(divU)g(hX, Y) \\ &= 0,\end{aligned} \tag{42}$$

by Theorem 4.6. Thus the spacetime under consideration is also shear-free.
Also such a spacetime the four-velocity vector U is constant over the spacelike hypersurface orthogonal to U.
But, as described and classified by Barnes [1], perfect fluid spacetimes that are vorticity free and shear-free are of type I, D or O (conformally flat). Thus we can conclude the following theorem:

Theorem 4.8. *The local cosmological structure of a Ricci symmetric $M(QE)_4$ perfect fluid spacetime can be identified as Petrov type I, D or O.*

5. The generators U and V as concurrent vector fields

In a recent paper Mallick and De [10] proved that if the associated vector fields of a $M(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a quasi-Einstein manifold. Hence the mixed quasi-Einstein spacetime under the above conditions reduces to a quasi-Einstein spacetime. For the completeness we give the proof here.

A vector field ξ is said to be concurrent if [20]

$$\nabla_X \xi = \rho X, \tag{43}$$

where ρ is a non-zero constant. If $\rho = 0$, the vector field reduces to a parallel vector field.

Example 5.1. *[3] Let M be a real vector space of dimension n and choose a basis $E_1, E_2, ..., E_n$ for M. A vector $v \in M$ can be expressed uniquely as*

$$v = \sum_{i=1}^{n} x^i(v) E_i,$$

and the standard chart $(x^1, x^2, ..., x^n)$ defines a manifold structure on M which is independent of the particular basis chosen. The vector field $\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$ is independent of the chosen basis and we call it the radial vector field on M. The conditions

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0, \quad i, j = 1, 2, ..., n.$$

determine a complete linear connection on M which we call the standard connection on M. The radial vector field is concurrent with respect to the standard connection.

In this section we consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X, Y) \tag{44}$$

and

$$(\nabla_X B)(Y) = \beta g(X, Y), \tag{45}$$

where α and β are non-zero constants and assume that $\alpha \neq \beta$.
Now using (43) and (44) in (22) we get

$$\begin{aligned}(\nabla_W S)(X, Y) = b[&\alpha g(X, W)B(Y) + \beta g(Y, W)A(X) \\ &+\beta g(X, W)A(Y) + \alpha g(Y, W)B(X)].\end{aligned} \tag{46}$$

Taking a frame field and after contraction over X, Y in (46), we obtain

$$dr(W) = 2b[\alpha B(W) + \beta A(W)], \tag{47}$$

where r is the scalar curvature of the spacetime.
Now taking a frame field and after contraction over X, Y in (4) over X and Y we obtain that $r = an$. Since $a \in \mathbb{R}$, we obtain that $dr(X) = 0$, for all X.
Thus, equation (47) yields

$$\alpha B(W) + \beta A(W) = 0. \tag{48}$$

Since α and β are non-zero constants, using (48) in (4), we finally obtain

$$S(X,Y) = ag(X,Y) - \frac{2b\beta}{\alpha}A(X)A(Y). \tag{49}$$

Thus the spacetime under consideration reduces to a quasi-Einstein spacetime.
Applying Ricci identity to (44) we get

$$R(X,Y)U = 0, \tag{50}$$

which implies

$$S(X,U) = 0, \tag{51}$$

for all vector field X. Similarly we can obtain $S(X,V) = 0$, for all vector field X. This implies

$$0 = S(X,U) = (a + \frac{2b\beta}{\alpha})A(X). \tag{52}$$

Since $A \neq 0$, it follows that

$$(a + \frac{2b\beta}{\alpha}) = 0. \tag{53}$$

Now we consider perfect fluid $(QE)_4$ spacetime. Therefore we have

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa[pg(X,Y) + (\sigma + p)A(X)A(Y)]. \tag{54}$$

This implies

$$S(X,Y) = (\kappa p + \frac{r}{2})g(X,Y) + \kappa(\sigma + p)A(X)A(Y). \tag{55}$$

Thus

$$a = \kappa p + \frac{r}{2}, \qquad b = \kappa(\sigma + p). \tag{56}$$

Taking a frame field and after contraction over X, Y in (55) we have

$$r = k(\sigma - 3p). \tag{57}$$

Using (56), (57) in (53) we get

$$p = \frac{\alpha + 2\beta}{2(\alpha - \beta)}\sigma. \tag{58}$$

In view of the above discussions we can state the following:

Theorem 5.2. *If the associated vector fields of a $M(QE)_4$ spacetime are concurrent vector fields and the associated scalars are constants, then the $M(QE)_4$ spacetime reduces to a quasi-Einstein spacetime and a perfect fluid $M(QE)_4$ spacetime obeying Einstein's field equation without cosmological constant is isentropic.*

Also from the Einstein's field Equation (7) and (49) we have

$$T(X,Y) = \frac{1}{k}(a - \frac{r}{2})g(X,Y) - \frac{2b\beta}{k\alpha}A(X)A(Y). \tag{59}$$

Thus, this spacetime can be considered as a model of perfect fluid spacetime, in general relativity.
Moreover, taking a frame field and after contraction over X, Y in (49), we have

$$r = 4a + \frac{2b\beta}{\alpha} \tag{60}$$

Again putting $Y = U$ in (54) we have

$$S(X, U) - \frac{r}{2}A(X) = -k\sigma A(X). \tag{61}$$

Also from (49) we have

$$S(X, U) = (a + \frac{2b\beta}{\alpha})A(X). \tag{62}$$

Therefore from (61) and (62) we have

$$a + \frac{2b\beta}{\alpha} - \frac{r}{2} = -k\sigma, \tag{63}$$

from which it follows that

$$\sigma = \frac{1}{k}(-\frac{r}{2} + 3a). \tag{64}$$

Also (60) and (62) yields

$$S(X, U) = (r - 3a)A(X). \tag{65}$$

From (65) it follows that $(r - 3a)$ is an eigen value of the Ricci tensor and U is an eigen vector corresponding to this eigenvalue.
Let V be another eigenvector of S different from U. Then V must be orthogonal to U.
Putting $Y = V$ in (54) we have

$$S(X, V) - \frac{r}{2}g(X, V) = p\kappa g(X, V). \tag{66}$$

It follows that from (66)

$$S(X, V) = (\frac{r}{2} + p\kappa)g(X, V). \tag{67}$$

Therefore from (56) and (67) we have

$$S(X, V) = ag(X, V). \tag{68}$$

From (68) it follows that a is another eigenvalue of S and V is an eigenvector corresponding to this eigenvalue. Since for a given eigenvector there is only one eigenvalue and $r - 3a$ and a are different, it follows that the Ricci tensor has only two district eigenvalues, namely, $r - 3a$ and a.
Let the multiplicity of $r - 3a$ be m. Then the multiplicity of a must be $4 - m$, because the dimension of the spacetime is 4.
Hence

$$m(r - 3a) + (4 - m)a = r. \tag{69}$$

From this we get $m = 1$, because $r - 4a = \frac{2b\beta}{\alpha} \neq 0$. Therefore the multiplicity of $r - 3a$ be 1 and the multiplicity of a must be 3. Hence the Segré characteristic of S is $[(1, 1, 1), 1]$ as given in [19]. Therefore under the conditions stated in the Theorem 5.1 we conclude the following:

Theorem 5.3. *A perfect fluid $M(QE)_4$ spacetime of non zero scalar curvature with the basic vector field of quasi-Einstein spacetime as velocity vector field of the fluid is of Segré characteristic* $[(1, 1, 1), 1]$.

On the other hand putting $Y = V$ in (55) and making use of $S(X, V) = 0$ for all X, we get

$$r = -2\kappa p. \tag{70}$$

Since $\kappa \neq 0$. In view of (57) and (70) we have

$$p = \sigma. \tag{71}$$

Thus under the conditions stated in the Theorem 5.1 we are in a position to state the following:

Theorem 5.4. *A perfect fluid $M(QE)_4$ spacetime obeying Einstein's field equation without cosmological constant represents stiff matter.*

6. Heat Flux in a $M(QE)_4$ spacetime

If in a $M(QE)_4$ spacetime the matter distribution is a fluid with the basic vector field as the velocity vector field, can this distribution be described by the following form of the energy-momentum tensor?

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y) + A(X)B(Y) + A(Y)B(X), \tag{72}$$

where $g(X,V) = B(X)$ for all X and V is the heat flux vector field.
Therefore from the Einstein's field equation we have

$$S(X,Y) - \frac{r}{2}g(X,Y) = k[pg(X,Y) + (\sigma + p)A(X)A(Y) \\ +A(X)B(Y) + A(Y)B(X)], \tag{73}$$

where $B(X) = g(X,V)$, $A(X) = g(X,U)$ and V is the heat flux vector. Therefore from (72) we have

$$S(X,Y) = (\frac{r}{2} + kp)g(X,Y) + k(\sigma + p)A(X)A(Y) \\ +kA(X)B(Y) + kA(Y)B(X). \tag{74}$$

This implies

$$ag(X,Y) + b[A(X)B(Y) + A(Y)B(X)] = (\frac{r}{2} + kp)g(X,Y) \\ +k(\sigma + p)A(X)A(Y) + kA(X)B(Y) + kA(Y)B(X). \tag{75}$$

Putting $Y = U$ in (75) we have

$$aA(X) - bB(X) = (\frac{r}{2} + kp)A(X) - k(\sigma + p)A(X) - kB(X). \tag{76}$$

It follows that

$$\{a + k\sigma - \frac{r}{2}\}A(X) = (b - k)B(X), \tag{77}$$

for all X. Remove X from the above equation we have

$$\{a + k\sigma - \frac{r}{2}\}U = (b - k)V, \tag{78}$$

Taking inner product in (78) by U yields

$$a + k\sigma - \frac{r}{2} = 0. \tag{79}$$

Using (79) in (77) we get $B = 0$ provided $b \neq k$.
In view of the above we can state the following:

Theorem 6.1. *A $M(QE)_4$ spacetime can not admit heat flux provided the smooth function b is not equal to the cosmological constant k.*

Remark 6.2. *This means that such a spacetime describe a Universe which has already attained thermal equilibrium.*

7. Example of a $M(QE)_4$ spacetime

In this section we prove the existence of a $M(QE)_4$ spacetime by constructing a non-trivial concrete example.

We consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{80}$$

where $i,\ j = 1,2,3,4$ and x^1, x^2 are non zero.
The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1,\ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2},\ \Gamma^2_{12} = \frac{1}{x^1},\ \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1},\ S_{12} = -\frac{1}{x^1 x^2}.$$

We shall now show that this M^4 is a $M(QE)_4$ spacetime i.e., it satisfies the defining relation (4).
We take the associated scalars as follows:

$$a = \frac{1}{x^1(x^2)^2},\ b = -\frac{2}{(x^1)^2 x^2}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} x^1, & \text{for i=2} \\ 0, & \text{for i=1,3,4} \end{cases}$$

and

$$B_i(x) = \begin{cases} \frac{1}{2}, & \text{for i=1} \\ \frac{3^{1/2}x^2}{2}, & \text{for i=3} \\ 0, & \text{for i=2,4} \end{cases}$$

at any point $x \in M$. In our (M^4, g), (4) reduces with these associated scalars and 1-forms to the following equation:

$$S_{12} = ag_{12} + b[A_1 B_2 + A_2 B_1] \tag{81}$$

It can be easily proved that the equation (81) is true.
We shall now show that the 1-forms are unit and orthogonal.
Here,

$$g^{ij}A_i A_j = 1,\ g^{ij}B_i B_j = 1,\ g^{ij}A_i B_j = 0.$$

So, the manifold under consideration is a $M(QE)_4$.
Thus we can state the following:

Theorem 7.1. *Let* $(\mathbb{R}^4, g)$ *be a 4-dimensional Lorentzian manifold with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2,$$

where $i,\ j = 1,2,3,4$ *and* x^1, x^2 *are non zero. Then* $(\mathbb{R}^4, g)$ *is a* $M(QE)_4$ *spacetime.*

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