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Oscillatory Behaviour of Higher-Order Nonlinear Neutral Delay Dynamic Equations on Time Scales

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Abstract. In this paper, new sufficient conditions are established for the oscillation of solutions of the higher order dynamic equations

$$\left[r(t)(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta} + q(t)f(x(\delta(t))) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

where $z(t) := x(t) + p(t)x(\tau(t))$, $n \geq 2$ is an even integer and $\alpha \geq 1$ is a quotient of odd positive integers. Under less restrictive assumptions for the neutral coefficient, we employ new comparison theorems and Generalized Riccati technique.

1. Introduction

In this paper, we introduce new sufficient conditions for the oscillation of solutions to the nonlinear neutral delay dynamic equation

$$\left[r(t)(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta} + q(t)f(x(\delta(t))) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1}$$

where $z(t) := x(t) + p(t)x(\tau(t))$ and $\alpha \geq 1$ is a quotient of odd positive integers. We assume that the following conditions hold.

(H$_1$) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t) > 0$, $r^{\Delta}(t) > 0$;

(H$_2$) $\tau, \delta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau^{\Delta}(t) \geq \tau_0 > 0$, $\delta^{\sigma}(t) \leq t\ \delta^{\Delta}(t) > 0$, $\tau \circ \delta = \delta \circ \tau$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \delta(t) = \infty$;

(H$_3$) $p, q \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $0 \leq p(t) \leq p_0 < \infty$, and $q(t) > 0$, where $p_0 > 0$ is a constant;

(H$_4$) $f \in C(\mathbb{T}, \mathbb{T})$, $xf(x) > 0$ for all $x \neq 0$, and there exists a positive constant k such that $\frac{f(x)}{x} \geq k$ for all $x \neq 0$.

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Throughout this paper, we will consider the following two cases:

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s = \infty, \tag{2}$$

and

$$\zeta(t) = \int_{t}^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s < \infty. \tag{3}$$

The theory of time scales was introduced by Hilger [11] in 1988 to unify continuous and discrete analysis. A *time scale*, which inherits standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, the *forward jump operator* is defined as $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf(t, \infty)_{\mathbb{T}}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined as $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise, it is called *right-scattered*. Similarly *left-dense* and *left-scattered points* are defined with respect to the backward jump operator.

The set of all such *rd*-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is an *rd*-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.The Delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^{\Delta}(t) = \begin{cases} \frac{f^{\sigma}(t)-f(t)}{\mu(t)}, & \mu(t) > 0 \\ \lim_{s\to t} \frac{f(t)-f(s)}{t-s}, & \mu(t) = 0 \end{cases}$$

The derivative of the product of two differentiable functions f and g is defined by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

and the derivative of the quotient of two differentiable functions f and $g \neq 0$: is given by

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{g(t)f^{\Delta}(t) - f(t)g^{\Delta}(t)}{g(\sigma(t))g(t)}.$$

F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^{\Delta} = f$ holds on $\mathbb{T}^k$. In this case integration of f is defined by

$$\int_{s}^{t} f(\tau)\Delta\tau = F(t) - F(s), \quad where \;\; s, t \in \mathbb{T}.$$

An antiderivative of 0 is 1 and the antiderivative of 1 is t; however it is not possible to find a polynomial that is an antiderivative of t .The role of t^2 is therefore played in the time scales calculus by

$$\int_{0}^{t} \sigma(\tau)\Delta\tau \quad and \quad \int_{0}^{t} \tau\Delta\tau.$$

In general, the functions

$$g_0(t,s) \equiv 1, \;\; and \;\; g_{k+1}(t,s) = \int_{s}^{t} g_k(\sigma(\tau), s)\Delta\tau, \quad k \geq 0,$$

and

$$h_0(t,s) \equiv 1, \text{ and } h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \quad k \geq 0,$$

may be considered as the polynomials on $\mathbb{T}$. The relationship between g_k and h_k is

$$g_k(t,s) = (-1)^k h_k(t,s) \quad for\ all\ k \in \mathbb{N}.$$

The following is the dynamic generalization of the well-known Taylor's formula.

Lemma 1.1. *Let $n \in \mathbb{N}$, $s \in \mathbb{T}$, and $f \in C^n_{rd}(\mathbb{T},\mathbb{R})$. Then,*

$$f(t) = \sum_{k=0}^{n-1} h_k(t,s) f^{\Delta k}(s) + \int_s^t h_{n-1}(t,\sigma(\eta)) f^{\Delta n}(\eta)\Delta\eta \ \ for \ \ t \in \mathbb{T}.$$

By a solution of (1), we mean a nontrivial function $x \in C_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, where $T_x \in [t_0,\infty)_{\mathbb{T}}$, which has the property that $\left[r(t)(z^{\Delta n-1}(t))^\alpha\right] \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies (1) identically on $[T_x,\infty)_{\mathbb{T}}$. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is called oscillatory if all its solutions oscillate.

In recent years considerable researchs has been completed on oscillatory theory, see [1, 4, 9, 12, 16–18, 21, 22, 24].

For instance, in 2015 Karpuz [13] studied the qualitative behavior of solutions to the higher-order delay dynamic equations of the form

$$\left[x(t) + A(t)x(\alpha(t))\right]^{\Delta n} + B(t)x(\beta(t)) = 0, \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}},$$

where $n \in \mathbb{N}$, $A \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $\alpha(t), \beta(t) \leq t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Chen [8] established sufficient conditions for the oscillation and asymptotic behavior of solutions of the nth-order nonlinear neutral delay dynamic equations

$$\{a(t)\psi(x(t))[|(x(t)+p(t)x(\tau(t)))^{\Delta n-1}|^{\alpha-1}|(x(t)+p(t)x(\tau(t)))^{\Delta n-1}|]^\gamma\}^\Delta + \lambda f(t,x(\delta(t))) = 0,$$

where $\alpha > 0$ is a constant, $\gamma > 0$ is a quotient of odd positive integers, $\lambda = \pm 1$; $p(t) \in C_r d(\mathbb{T},\mathbb{R})$ and $0 \leq p(t) \leq 1$.

In the last two decades, several special cases of (1) have been discussed by numerous authors in the literature, we mention for instance Li et al. [15] established a new oscillation criteria for the neutral delay differential equations

$$\left[x(t) + p(t)x(\tau(t))\right]^{(n)} + q(t)f(x(\sigma(t))) = 0, \ \ t \geq t_0,$$

where $0 \leq p(t) \leq p_0 < \infty$.

More recently , Baculíková et al. [3] combined new generalization of the classical Philos and Staikos lemma (see[19, 20]) together with a suitable comparison technique to introduce new oscillation criteria for the *nth*-order differential equation

$$\left[r(t)(z'(t))^\gamma\right]^{(n-1)} + q(t)x^\gamma(\sigma(t)) = 0,$$

where γ is the ratio of two positive odd integers, $n \geq 3$, $p(t) \geq 0$ and $q(t) > 0$.

In 2016, Karpuz and Öcalan [14] presented new sufficient conditions for the oscillation of first-order delay dynamic equation on time scales

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{4}$$

provided that $p(t) > 0$ and $\tau(\sigma(t)) \leq t$.

For completeness, we outline some known results, which will be useful for proving our main results.

Theorem 1.2. [6] *Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $y : \tilde{\mathbb{T}} \to \mathbb{R}$. If $y^{\tilde{\Delta}}[v(t)]$ and $v^{\Delta}(t)$ exist for $t \in \mathbb{T}_k$, then*

$$(y[v(t)])^{\Delta} = y^{\tilde{\Delta}}[v(t)]v^{\Delta}(t).$$

Lemma 1.3. [6] *Let $n \in N$, $f \in C^n_{rd}(\mathbb{T}, \mathbb{R})$ and $\sup \mathbb{T} = \infty$. Suppose that f is either positive or negative, $f^{\Delta n}$is not identically zero and is either nonnegative or nonpositive on $[t_0, \infty)_{\mathbb{T}}$for some $t_0 \in \mathbb{T}$. Then, there exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and $m \in [0, n)_{\mathbb{Z}}$ such that $(-1)^{n-m} f(t) f^{\Delta n}(t) \geq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ with*

- *$f(t)f^{\Delta j}(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and all $j \in [0, m)_{\mathbb{Z}}$;*
- *$(-1)^{m+j} f(t) f^{\Delta j}(t) \geq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and all $j \in [m, n)_{\mathbb{Z}}$.*

Lemma 1.4. [13] *Let $\sup \mathbb{T} = \infty$, $n \in \mathbb{N}$ and $f \in C^n_{rd}([t_0, \infty), \mathbb{R}^+_0)$ with $f^{\Delta n} \leq 0$ on $[t_0, \infty)_{\mathbb{T}}$. Let Lemma 1.3 hold with $m \in [0, n)_{\mathbb{Z}}$ and $s \in [t_0, \infty)_{\mathbb{T}}$. Then*

$$f(t) \geq h_m(t, s) f^{\Delta m}(t) \quad for\ all\ t \in [s, \infty)_{\mathbb{T}} \tag{5}$$

Lemma 1.5. [10] *Let $\sup \mathbb{T} = \infty$ and $f \in C^n_{rd}(\mathbb{T}, \mathbb{R}^+)$ as well as $(n \geq 2)$. Suppose that Kneser's theorem holds with $m \in [1, n)_{\mathbb{N}}$ and $f^{\Delta n}(t) \leq 0$on $\mathbb{T}$. Then there exists a sufficiently large $t_1 \in \mathbb{T}$ such that*

$$f^{\Delta}(t) \geq h_{m-1}(t, t_1) f^{\Delta m}(t), \quad for \quad all \ \ t \in [t_1, \infty)_{\mathbb{T}}.$$

In this article, we introduce new comparison theorems in which we compare the higher-order dynamic equation (1) with first order dynamic equations of the form (4). The obtained results supplement and improve those reported in the literature.

2. Main results

We begin with the following lemma.

Lemma 2.1. *Let the conditions* (H_1)-(H_3) *be satisfied . If $x(t)$ is an eventually positive solution of* (1), *then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$z^{\Delta n-1}(t) > 0, \ \ z^{\Delta n}(t) \leq 0, \ \ z^{\Delta}(t) > 0, \quad t > t_1. \tag{6}$$

Proof. Since $x(t)$ is an eventually positive solution of (1), then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \ \ x(\delta(t)) > 0 \ \ and \ \ x(\tau(t)) > 0, \qquad for \quad t \geq t_1.$$

Now from (1) and the assumptions (H_3) and (H_4), we have $z(t) \geq x(t) > 0$. Then (1) implies that

$$\left[r(t)(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta} \leq -kq(t)x^{\alpha}(\delta(t)) < 0, \quad t \geq t_1. \tag{7}$$

Therefore, $r(t)(z^{\Delta n-1}(t))^{\alpha}$ is decreasing and either $z^{\Delta n-1}(t) > 0$ or $z^{\Delta n-1}(t) < 0$ eventually for $t \geq t_1$. If $z^{\Delta n-1}(t) < 0$, then there exists a constant c such that

$$z^{\Delta n-1}(t) \leq -c\frac{1}{r^{\frac{-1}{\alpha}}(t)}.$$

Integrating from t_1 to t, we obtain

$$z^{\Delta n-2}(t) \leq -c\int_{t_1}^{t}\frac{1}{r^{\frac{-1}{\alpha}}(s)}\Delta s.$$

Letting $t \to \infty$, it follows from (2), that $\lim_{t\to\infty} z^{\Delta n-2}(t) = -\infty$. Therefore, $\lim_{t\to\infty} z(t) = -\infty$ which is a contradiction. Consequently, $z^{\Delta n-1}(t) > 0$ for $t \geq t_1$.
Now, we prove that $z^{\Delta n}(t) \leq 0$. Since

$$\left[r(t)(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta} = r^{\Delta}(t)(z^{\Delta n-1}(t))^{\alpha} + r^{\sigma}(t)\left[(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta} \leq 0. \tag{8}$$

Using the Pötzche chain rule [6] with fact that $\alpha \geq 1$, we obtain

$$\begin{aligned}[(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} &= \left\{\alpha\int_0^1\left[z^{\Delta n-1}(t) + \mu h z^{\Delta n}(t)\right]^{\alpha-1}dh\right\}z^{\Delta n} \\ &\geq \alpha(z^{\Delta n-1}(t))^{\alpha-1}z^{\Delta n}(t).\end{aligned}$$

This with (8), leads to

$$r^{\Delta}(t)(z^{\Delta n-1}(t))^{\alpha} + \alpha r^{\sigma}(t)(z^{\Delta n-1}(t))^{\alpha-1}z^{\Delta n}(t) \leq 0,$$

since $z^{\Delta n-1}(t) > 0$, $r^{\Delta}(t)$ and $r(t) > 0$, we then obtain

$$z^{\Delta n}(t) \leq 0.$$

Applying Lemma1.3 and Lemma1.5, we obtain

$$z^{\Delta n-1}(t) > 0,\ \ z^{\Delta n}(t) \leq 0,\ \ z^{\Delta}(t) > 0,\quad t > t_1.$$

□

Theorem 2.2. *Suppose that* (2) *and* (H_1)-(H_3) . *If*

$$\int_{t_0}^{\infty} Q(s)\Delta s = \infty, \tag{9}$$

where $Q(t) = \min\{kq(t), kq(\tau(t))\}$, *then every solution of* (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)$, such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ on $[T_0, \infty)$. Then by Lemma 2.1, we have $z(t) > 0$, $z^{\Delta}(t) > 0$, $z^{\Delta n-1}(t) > 0$, and $z^{\Delta n}(t) \leq 0$. Then we obtain

$$\left[r(t)(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta} \leq -kq(t)x^{\alpha}(\delta(t)) < 0,\quad t \geq t_1. \tag{10}$$

It follows from Theorem 1.2 and $\left[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\right]^{\Delta} = \left[r(t)(z^{\Delta n-1}(t))^{\alpha}\right]^{\Delta}\tau^{\Delta}(t)$, that there exists a $t_2 \geq T$ such that

$$p_0^{\alpha}\frac{\left[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\right]^{\Delta}}{\tau^{\Delta}(t)} \leq -kp_0^{\alpha}q(\tau(t))x^{\alpha}(\delta(\tau(t))).$$

But since $\tau^{\Delta}(t) \geq \tau_0 > 0$, for $t \geq t_2$, we have

$$\frac{p_0^{\alpha}}{\tau_0}\Big[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\Big]^{\Delta} \leq -kp_0^{\alpha}q(\tau(t))x^{\alpha}(\delta(\tau(t))). \tag{11}$$

Combining (10) and (11) and using the assumption that $\delta \circ \tau = \tau \circ \delta$, we obtain

$$\begin{aligned}
\Big(r(t)(z^{\Delta n-1}(t))^{\alpha}\Big)^{\Delta} &+ \frac{p_0^{\alpha}}{\tau_0}\Big(r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\Big)^{\Delta} \\
&\leq -\Big(kq(t)x^{\alpha}(\delta(t)) + p_0^{\alpha}kq(\tau(t))x^{\alpha}(\delta(\tau(t)))\Big) \\
&\leq -\min\{kq(t), kq(\tau(t))\}\Big(x^{\alpha}(\delta(t)) + p_0^{\alpha}x^{\alpha}(\tau(\delta(t)))\Big) \\
&= -Q(t)\Big(x^{\alpha}(\delta(t)) + p_0^{\alpha}x^{\alpha}(\tau(\delta(t)))\Big).
\end{aligned} \tag{12}$$

Since $0 \leq p(t) < p_0 < \infty$, then by the following inequality (see[5],Lemma1)

$$x_1^{\alpha} + x_2^{\alpha} \geq \frac{1}{2^{\alpha-1}}(x_1 + x_2)^{\alpha}, \tag{13}$$

where $\alpha \geq 1$, $x_1 \geq 0$ and $x_2 \geq 0$, we have

$$x^{\alpha}(\delta(t)) + p_0^{\alpha}x^{\alpha}(\tau(\delta(t))) \geq \frac{1}{2^{\alpha-1}}\Big(x(\delta(t)) + p_0x(\tau(\delta(t)))\Big)^{\alpha} \geq \frac{z^{\alpha}(\delta(t))}{2^{\alpha-1}}. \tag{14}$$

Substituting (14) into (12), for $t \geq t_2$, we obtain

$$\Big(r(t)(z^{\Delta n-1}(t))^{\alpha}\Big)^{\Delta} + \frac{p_0^{\alpha}}{\tau_0}\Big(r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\Big)^{\Delta} + Q(t)\frac{z^{\alpha}(\delta(t))}{2^{\alpha-1}} \leq 0. \tag{15}$$

Integrating from t_1 to t, we have

$$\int_{t_1}^{t}\Big(r(s)(z^{\Delta n-1}(s))^{\alpha}\Big)^{\Delta}\Delta s + \frac{p_0^{\alpha}}{\tau_0}\int_{t_1}^{t}\Big(r(\tau(s))(z^{\Delta n-1}(\tau(s)))^{\alpha}\Big)^{\Delta}\Delta s + \frac{1}{2^{\alpha-1}}\int_{t_1}^{t} Q(s)z^{\alpha}(\delta(s))\Delta s \leq 0, \tag{16}$$

i.e.,

$$\begin{aligned}
\frac{1}{2^{\alpha-1}}\int_{t_1}^{t} Q(s)z^{\alpha}(\delta(s)) &\leq -\int_{t_1}^{t}\Big(r(s)(z^{\Delta n-1}(s))^{\alpha}\Big)^{\Delta}\Delta s - \frac{p_0^{\alpha}}{\tau_0^2}\int_{t_1}^{t}\Big(r(\tau(s))(z^{\Delta n-1}(\tau(s)))^{\alpha}\Big)^{\Delta}\Delta(\tau(s)) \\
&\leq r(t_1)z^{\Delta n-1}(t_1) - r(t)z^{\Delta n-1}(t) \\
&\quad + \frac{p_0^{\alpha}}{\tau_0^2}\Big(r(\tau(t_1))(z^{\Delta n-1}(\tau(t_1)))^{\alpha} - r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\Big).
\end{aligned} \tag{17}$$

Since $z^{\Delta}(t) > 0$ for $t \geq t_1$, then there exists a constant $c > 0$ such that $z(\delta(t)) \geq c$, $t \geq t_1$. Using the fact that $r(t)z^{\Delta n-1}(t)$ is decreasing, we obtain from (17)

$$\int_{t_1}^{\infty} Q(s)\Delta s < \infty.$$

This contradicts (9), and completes the proof. □

Theorem 2.3. *Assume that for all sufficiently large $s \in [t_0, \infty)_{\mathbb{T}}$, (2) holds and $\tau(t) \geq t$. If the first-order dynamic equation*

$$u^{\Delta}(t) + Q(t, s)u(\delta(t)) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{18}$$

where $Q(t, s) = Q(t)\frac{h_{n-1}^{\alpha}(\delta(t),s)}{2^{\alpha-1}r(\delta(t))}$, is oscillatory, then (1) is also oscillatory.

Proof. Assume that (1) is nonoscillatory. Without loss of generality there is a solution x of (1) and $t_1 \in [t_0, \infty)_{\mathbb{T}}$ with $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.2, we arrive (15). By Lemma 1.4 for all $t \in [s, \infty)_{\mathbb{T}}$, we obtain

$$\left(r(t)(z^{\Delta n-1}(t))^{\alpha}\right)^{\Delta} + \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\right)^{\Delta} + Q(t)\frac{h_{n-1}^{\alpha}(\delta(t), s)}{2^{\alpha-1}}(z^{\Delta n-1}(\delta(t)))^{\alpha} \leq 0. \tag{19}$$

Let $y(t) = r(t)(z^{\Delta n-1}(t))^{\alpha} > 0$. Then

$$\left(y(t) + \frac{p_0^{\alpha}}{\tau_0}y(\tau(t))\right)^{\Delta} + Q(t)\frac{h_{n-1}^{\alpha}(\delta(t), s)}{2^{\alpha-1}r(\delta(t))}y(\delta(t) \leq 0. \tag{20}$$

Now, define

$$u(t) := y(t) + \frac{p_0^{\alpha}}{\tau_0}y(\tau(t)), \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{21}$$

since $y(t)$ is decreasing and $\tau(t) \geq t$. Then

$$u(t) \leq \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right)y(t), \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{22}$$

Using (20), we obtain

$$u^{\Delta}(t) + Q(t)\frac{h_{n-1}^{\alpha}(\delta(t), s)}{2^{\alpha-1}r(\delta(t))}u(\delta(t) \leq 0. \tag{23}$$

Therefore,

$$u^{\Delta}(t) + Q(t, t_1)u(\delta(t)) \leq 0 \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{24}$$

By [7, Theorem 3.1], Eq.(18) also presents a nonoscillatory solution. This contradiction proves that (1) is oscillatory. □

In view of Theorem 1 and Theorem 2 in [14] as well as Theorem 2.2, we obtain the following oscillation criteria for (1).

Corollary 2.4. *If*

$$\limsup_{t\to\infty} \frac{\int_{\delta(t)}^{\sigma(t)} Q(s, t_1)\Delta s}{1 - [1 - \mu(\delta(t))Q(\delta(t), t_1)]\mu^{\sigma}(t)Q^{\sigma}(t, t_1)} > 1, \tag{25}$$

then every solution of (1) *is oscillatory.*

Corollary 2.5. *If there exists* $\gamma \in [0, 1]_{\mathbb{R}}$ *such that*

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} Q(s, t_1)\Delta s > \gamma \quad \textit{and} \quad \limsup_{t\to\infty} \int_{\delta(t)}^{\sigma(t)} Q(s, t_1)\Delta s > 1 - \left(1 - \sqrt{1-\gamma}\right)^2, \tag{26}$$

then every solution of (1) *is oscillatory.*

The following theorem introduces a new oscillation criterion when $\delta(t) \leq \tau(t) \leq t$.

Theorem 2.6. *Assume that* (2) *holds. If there exists a real-valued function* $\rho \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\Big(\frac{1}{2^{\alpha-1}}\rho(s)Q(s)-\frac{(1+\frac{p_0^\alpha}{\tau_0})}{(\alpha+1)^{\alpha+1}}\frac{(\rho^\Delta(s))^{\alpha+1}r(\delta(s))}{(\delta^\Delta(s))^\alpha h_{n-2}^\alpha(\delta(s),t_1)\rho^\alpha(s)}\Big)\Delta s=\infty, \tag{27}$$

Then (1)*is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\delta(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Define a Riccati substitution as

$$\omega(t)=\rho(t)\frac{r(t)(z^{\Delta n-1}(t))^\alpha}{z^\alpha(\delta(t))},\quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{28}$$

Clearly $\omega(t)>0$, and

$$\begin{aligned}\omega^\Delta(t) &= \frac{\rho(t)}{z^\alpha(\delta(t))}[r(t)(z^{\Delta n-1}(t))^\alpha]^\Delta+[r(t)(z^{\Delta n-1}(t))^\alpha]^\sigma\Big[\frac{\rho(t)}{z^\alpha(\delta(t))}\Big]^\Delta\\ &= \rho(t)\frac{[r(t)(z^{\Delta n-1}(t))^\alpha]^\Delta}{z^\alpha(\delta(t))}+\frac{\rho^\Delta(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\alpha\delta^\Delta(t)\frac{\rho(t)}{\rho(\sigma(t))}\frac{z^\Delta(\delta(t))}{z(\delta(t))}\omega(\sigma(t)),\end{aligned} \tag{29}$$

since by Lemma 1.5 , we have

$$z^\Delta(\delta(t))\ge h_{n-2}(\delta(t),t_1)z^{\Delta n-1}(\delta(t)). \tag{30}$$

Substituting (30) into (29), we get

$$\omega^\Delta(t)\le\rho(t)\frac{[r(t)(z^{\Delta n-1}(t))^\alpha]^\Delta}{z^\alpha(\delta(t))}+\frac{\rho^\Delta(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\alpha\delta^\Delta(t)\frac{\rho(t)}{\rho(\sigma(t))}\frac{h_{n-2}(\delta(t),t_1)z^{\Delta n-1}(\delta(t))}{z(\delta(t))}\omega(\sigma(t)). \tag{31}$$

Since

$$\omega^{\frac{1}{\alpha}}(\sigma(t))=\rho^{\frac{1}{\alpha}}(\sigma(t))\frac{r^{\frac{1}{\alpha}}(\sigma(t))(z^{\Delta n-1}(\sigma(t)))}{z(\delta(\sigma(t)))}. \tag{32}$$

Since $\delta^\Delta>0$ and $\delta(t)\le t\le\sigma(t)$. In view of the fact $r(t)(z^\Delta(t))^\alpha$ is decreasing and $z^\Delta>0$, then we obtain

$$\frac{\omega^{\frac{1}{\alpha}}(\sigma(t))}{\rho^{\frac{1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\le\frac{z^{\Delta n-1}(\delta(t))}{z(\delta(t))}. \tag{33}$$

Substituting (33) into (30) , we obtain

$$\omega^\Delta(t)\le\rho(t)\frac{[r(t)(z^{\Delta n-1}(t))^\alpha]^\Delta}{z^\alpha(\delta(t))}+\frac{\rho^\Delta(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\alpha\delta^\Delta(t)h_{n-2}(\delta(t),t_1)\frac{\rho(t)}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{34}$$

Define another function $\nu(t)$ by

$$\nu(t):=\rho(t)\frac{r(\tau(t))(z^{\Delta n-1}(\tau(t)))^\alpha}{z^\alpha(\delta(t))},\quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{35}$$

Then $\nu(t)>0$, and

$$\begin{aligned}\nu^\Delta(t) &= \frac{\rho(t)}{z^\alpha(\delta(t))}[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^\alpha]^\Delta+[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^\alpha]^\sigma\Big[\frac{\rho(t)}{z^\alpha(\delta(t))}\Big]^\Delta\\ &= \rho(t)\frac{[r(t)(z^{\Delta n-1}(t))^\alpha]^\Delta}{z^\alpha(\delta(t))}+\frac{\rho^\Delta(t)}{\rho(\sigma(t))}\nu(\sigma(t))-\alpha\delta^\Delta(t)\frac{\rho(t)}{\rho(\sigma(t))}\frac{z^\Delta(\delta(t))}{z(\delta(t))}\nu(\sigma(t)).\end{aligned} \tag{36}$$

This with (30), leads to

$$\nu^{\Delta}(t) \leq \rho(t)\frac{[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}]^{\Delta}}{z^{\alpha}(\delta(t))} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\nu(\sigma(t)) - \alpha\delta^{\Delta}(t)\frac{\rho(t)}{\rho(\sigma(t))}\frac{h_{n-2}(\delta(t),t_1)z^{\Delta n-1}(\delta(t))}{z(\delta(t))}\nu(\sigma(t)). \tag{37}$$

From the definition of $\nu(t)$, with the facts that $\tau^{\Delta} > 0$, $\delta^{\Delta} > 0$, $z^{\Delta} > 0$ and $r(t)(z^{\Delta}(t))^{\alpha}$ is decreasing, we get

$$\nu^{\frac{1}{\alpha}}(\sigma(t)) = \rho^{\frac{1}{\alpha}}(\sigma(t))\frac{[r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t))]^{\sigma}}{z(\delta(\sigma(t)))} \leq \rho^{\frac{1}{\alpha}}(\sigma(t))\frac{r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t))}{z(\delta(t))}, \tag{38}$$

But since $\delta(t) \leq \tau(t)$ and $r(t)(z^{\Delta}(t))^{\alpha}$ is decreasing, (38) takes the form

$$\nu^{\frac{1}{\alpha}}(\sigma(t)) \leq \rho^{\frac{1}{\alpha}}(\sigma(t))\frac{r^{\frac{1}{\alpha}}(\delta(t))z^{\Delta n-1}(\delta(t))}{z(\delta(t))} \tag{39}$$

i.e.,

$$\frac{\nu^{\frac{1}{\alpha}}(\sigma(t))}{\rho^{\frac{1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))} \leq \frac{z^{\Delta n-1}(\delta(t))}{z(\delta(t))}. \tag{40}$$

Substituting (40) into (37), we obtain

$$\nu^{\Delta}(t) \leq \rho(t)\frac{[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}]^{\Delta}}{z^{\alpha}(\delta(t))} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\nu(\sigma(t)) - \alpha\delta^{\Delta}(t)h_{n-2}(\delta(t),t_1)\frac{\rho(t)}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{41}$$

Combining (41) and (34), we obtain

$$\begin{aligned}\omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}\nu^{\Delta}(t) \leq{}& \rho(t)\frac{[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} + \frac{p_0^{\alpha}}{\tau_0}[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}]^{\Delta}}{z^{\alpha}(\delta(t))}\\ &+ \left(\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \frac{\alpha\delta^{\Delta}(t)h_{n-2}(\delta(t),t_1)\rho(t)}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t))\right)\\ &+ \frac{p_0^{\alpha}}{\tau_0}\left(\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\nu(\sigma(t)) - \frac{\alpha\delta^{\Delta}(t)h_{n-2}(\delta(t),t_1)\rho(t)}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t))\right).\end{aligned} \tag{42}$$

Applying the following inequality

$$Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}. \tag{43}$$

on (42), we obtain

$$\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \frac{\alpha\delta^{\Delta}(t)h_{n-2}(\delta(t),t_1)\rho(t)}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\delta(t))}{(\delta^{\Delta}(t))^{\alpha}h_{n-2}^{\alpha}(\delta(t),t_1)\rho^{\alpha}(t)}, \tag{44}$$

and

$$\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\nu(\sigma(t)) - \frac{\alpha\delta^{\Delta}(t)h_{n-2}(\delta(t),t_1)\rho(t)}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\delta(t))}{(\delta^{\Delta}(t))^{\alpha}h_{n-2}^{\alpha}(\delta(t),t_1)\rho^{\alpha}(t)}. \tag{45}$$

This with (15), (44) and (42) leads to

$$\omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}\nu^{\Delta}(t) \leq \frac{-1}{2^{\alpha-1}}\rho(t)Q(t) + \frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\delta(t))}{(\delta^{\Delta}(t))^{\alpha}h_{n-2}^{\alpha}(\delta(t),t_1)\rho^{\alpha}(t)}. \tag{46}$$

Integrating from $t_2 > t_1$ to t, we obtain

$$\int_{t_2}^{t}\Big(\frac{1}{2^{\alpha-1}}\rho(s)Q(s) - \frac{(1+\frac{p_0^\alpha}{\tau_0})}{(\alpha+1)^{\alpha+1}}\frac{(\rho^\Delta(s))^{\alpha+1}r(\delta(s))}{(\delta^\Delta(s))^\alpha h_{n-2}^\alpha(\delta(s),t_1)\rho^\alpha(s)}\Big)\Delta s \le \omega(t_2) + \frac{p_0^\alpha}{\tau_0}\nu(t_2). \tag{47}$$

Taking $\limsup_{t\to\infty}$, we get a contradiction with (27). This completes the proof □

Now, we present new oscillation criteria for (1) under the case (3).

Theorem 2.7. *Assume that* (3) *holds and* $\tau(t) \ge t$ *.If*

$$\int_{t_1}^{\infty}\Big[\frac{h_{n-2}(\delta(s),t_1)\zeta^\alpha(s)Q(s)}{2^{\alpha-1}} - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^\alpha}{\tau_0})(\tau^\Delta(s))^{\alpha+1}}{r^{\frac{1}{\alpha}}(s)\zeta(\tau(s))}\Big]\Delta s = \infty, \tag{48}$$

then every solution of (1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.2, it is clear that $[r(t)(z^{\Delta n-1}(t))^\alpha]$ is a decreasing function. Thus $z^{\Delta n-1}(t)$ is either eventually positive or eventually negative for $t \ge t_2 \ge t_1$.

Case(I): $z^{\Delta n-1}(t) > 0$, $t \ge t_2$. The proof of this case is similar to that of Theorem 2.6;

Case(II):$z^{\Delta n-1}(t) < 0$, $t \ge t_2$. Applying Lemma 1.3, we obtain $z^{\Delta n-2}(t) > 0$ and $z^\Delta(t) > 0$. Then $\lim_{t\to\infty} z(t) \ne 0$. Define the function

$$\nu(t) := \frac{r(\tau(t))(z^{\Delta n-1}(\tau(t)))^\alpha}{(z^{\Delta n-2}(t))^\alpha}, \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{49}$$

Since $[r(t)(z^{\Delta n-1}(t))^\alpha]$ is decreasing and $\tau^\Delta > 0$, we have

$$r^{\frac{1}{\alpha}}(\tau(s))z^{\Delta n-1}(\tau(s)) \le r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t)), \quad t \ge s > t_2.$$

i.e.,

$$z^{\Delta n-1}(\tau(s)) \le r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t))\frac{1}{r^{\frac{1}{\alpha}}(\tau(s))}.$$

Integrating from t to l, we obtain

$$z^{\Delta n-2}(\tau(l)) \le z^{\Delta n-2}(\tau(t)) + r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t))\int_{\tau(t)}^{\tau(l)}\frac{1}{r^{\frac{1}{\alpha}}(s)}\Delta s. \tag{50}$$

Letting $l \to \infty$, we get

$$0 \le z^{\Delta n-2}(\tau(t)) + r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t))\zeta(\tau(t)). \tag{51}$$

Using the facts that $z^{\Delta n-1} < 0$ and $\tau(t) \ge t$, we have

$$z^{\Delta n-2}(\tau(t)) \le z^{\Delta n-2}(t), \quad t \ge t_2.$$

Hence,

$$-1 \le \frac{r^{\frac{1}{\alpha}}(\tau(t))z^{\Delta n-1}(\tau(t))}{z^{\Delta n-2}(t)}\zeta(\tau(t)),$$

i.e.,

$$-1 \leq \nu(t)\zeta^{\alpha}(\tau(t)) \leq 0, \quad t \geq t_2. \tag{52}$$

Next, define the function

$$\omega(t) := \frac{r(t)(z^{\Delta n-1}(t))^{\alpha}}{(z^{\Delta n-2}(t))^{\alpha}}, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{53}$$

Thus clearly $\omega < 0$ and by the facts $[r(t)(z^{\Delta n-1}(t))^{\alpha}]$ is decreasing and $\tau^{\Delta} > 0$, we get

$$\omega(t) \geq \nu(t).$$

i.e.,

$$-1 \leq \omega(t)\zeta^{\alpha}(\tau(t)) \leq 0, \quad t \geq t_2, \tag{54}$$

From (49), we have

$$\nu^{\Delta}(t) = \frac{[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}]^{\Delta}}{(z^{\Delta n-2}(t))^{\alpha}} - \frac{\alpha}{r^{\frac{1}{\alpha}}(\tau(t))}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{55}$$

Similarly we can obtain the following from (53)

$$\omega^{\Delta}(t) = \frac{[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta}}{(z^{\Delta n-2}(t))^{\alpha}} - \frac{\alpha}{r^{\frac{1}{\alpha}}(t)}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{56}$$

Combining (54) and (56), we get

$$\omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}\nu^{\Delta}(t) \leq -Q(t)\frac{z^{\alpha}(\delta(t))}{2^{\alpha-1}(z^{\Delta n-2}(t))^{\alpha}} - \frac{\alpha}{r^{\frac{1}{\alpha}}(t)}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)) - \frac{p_0^{\alpha}}{\tau_0}\frac{\alpha}{r^{\frac{1}{\alpha}}(\tau(t))}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{57}$$

Using Lemma 1.4, for $m = n - 2$, we have

$$z(t) \geq h_{n-2}(t, t_1)z^{\Delta n-2}(t). \tag{58}$$

Since $z^{\Delta n-1}(t) < 0$ and $\delta(t) \leq t$, then $z^{\Delta n-2}(t) \leq z^{\Delta n-2}(\delta(t))$, consequently by (58). the inequality (57) takes the form

$$\omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}\nu^{\Delta}(t) + \frac{h_{n-2}(\delta(t), t_1)}{2^{\alpha-1}}Q(t) + \alpha r^{\frac{-1}{\alpha}}(t)\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)) + \frac{\alpha p_0^{\alpha}}{\tau_0}r^{\frac{-1}{\alpha}}(\tau(t))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)) \leq 0. \tag{59}$$

Multiplying the above inequality by $\zeta^{\alpha}(\tau(t))$ and integrating it from t_1 to t, we obtain

$$\begin{aligned}
&\omega(t)\zeta^{\alpha}(\tau(t)) - \omega(t_1)\zeta^{\alpha}(\tau(t_1)) + \frac{p_0^{\alpha}}{\tau_0}\nu(t)\zeta^{\alpha}(\tau(t)) - \frac{p_0^{\alpha}}{\tau_0}\nu(t_1)\zeta^{\alpha}(\tau(t_1)) \\
&+ \alpha\int_{t_1}^{t}\Big[r^{\frac{-1}{\alpha}}(s)\zeta^{\alpha-1}(\tau(s))\tau^{\Delta}(s)\omega(\sigma(s)) + r^{\frac{-1}{\alpha}}(s)\zeta^{\alpha}(\tau(s))\omega^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Big]\Delta s \\
&+ \frac{\alpha p_0^{\alpha}}{\tau_0}\int_{t_1}^{t}\Big[r^{\frac{-1}{\alpha}}(s)\zeta^{\alpha-1}(\tau(s))\tau^{\Delta}(s)\nu(\sigma(s)) + r^{\frac{-1}{\alpha}}(\tau(s))\zeta^{\alpha}(\tau(s))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Big]\Delta s \\
&+ \frac{1}{2^{\alpha-1}}\int_{t_1}^{t}h_{n-2}(\delta(s), t_1)\zeta^{\alpha}(\tau(s))Q(s)\Delta s + \leq 0.
\end{aligned} \tag{60}$$

Applying the inequality (43), we get

$$\begin{aligned}&\omega(t)\zeta^{\alpha}(\tau(t))-\omega(t_1)\zeta^{\alpha}(\tau(t_1))+\frac{p_0^{\alpha}}{\tau_0}\nu(t)\zeta^{\alpha}(\tau(t))-\frac{p_0^{\alpha}}{\tau_0}\nu(t_1)\zeta^{\alpha}(\tau(t_1))+\\&\int_{t_1}^{t}\Big[\frac{h_{n-2}(\delta(s),t_1)\zeta^{\alpha}(\tau(s))Q(s)}{2^{\alpha-1}}-\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^{\alpha}}{\tau_0})(\tau^{\Delta}(s))^{\alpha+1}}{r^{\frac{1}{\alpha}}(s)\zeta(\tau(s))}\Big]\Delta s\le 0.\end{aligned}\tag{61}$$

Therefore from (52) and (53), we have

$$\begin{aligned}&\int_{t_1}^{t}\Big[\frac{h_{n-2}(\delta(s),t_1)\zeta^{\alpha}(s)Q(s)}{2^{\alpha-1}}-\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^{\alpha}}{\tau_0})(\tau^{\Delta}(s))^{\alpha+1}}{r^{\frac{1}{\alpha}}(s)\zeta(\tau(s))}\Big]\Delta s\\&\quad\le\omega(t_1)\zeta^{\alpha}(t_1)+\frac{p_0^{\alpha}}{\tau_0}\nu(t_1)\zeta^{\alpha}(t_1)+1+\frac{p_0^{\alpha}}{\tau_0}.\end{aligned}\tag{62}$$

Which contradicts (48). This completes the proof. □

Theorem 2.8. *Assume that* (3) *holds and* $\delta(t)\le\tau(t)\le t$. *If*

$$\int_{t_0}^{\infty}\Big[\frac{h_{n-2}(\delta(s),t_1)\zeta^{\alpha}(s)Q(s)}{2^{\alpha-1}}-\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{r^{\frac{1}{\alpha}}(s)\zeta(s)}\Big]\Delta s=\infty,\tag{63}$$

then every solution of (1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\delta(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.2, $[r(t)(z^{\Delta n-1}(t))^{\alpha}]$ is clearly a decreasing function. This means $z^{\Delta n-1}(t)$ is either eventually positive or eventually negative for $t\ge t_2\ge t_1$.

Case(I): $z^{\Delta n-1}(t)>0,\ t\ge t_2$. The proof of this case is similar to that of Theorem 2.6;

Case(II):$z^{\Delta n-1}(t)<0,\ t\ge t_2$. Applying Lemma 1.3, we get $z^{\Delta n-2}(t)>0$ and $z^{\Delta}(t)>0$. Then $\lim_{t\to\infty}z(t)\ne 0$. Define the function $\omega(t)$ as defined in (53). Since $[r(t)(z^{\Delta n-1}(t))^{\alpha}]$ is a decreasing function, then we have for $s\ge t\ge t_1$

$$r(s)(z^{\Delta n-1}(s))^{\alpha}\le r(t)(z^{\Delta n-1}(t))^{\alpha}.\tag{64}$$

Dividing (64) by $r(s)$ and integrating from t to l, we have

$$z^{\Delta n-2}(l)\le z^{\Delta n-2}(t)+r^{\frac{1}{\alpha}}(t)z^{\Delta n-1}(t)\int_t^l r^{-\frac{1}{\alpha}}(s)\Delta s.\tag{65}$$

Letting $l\to\infty$, we get

$$0\le z^{\Delta n-2}(t)+r^{\frac{1}{\alpha}}(t)z^{\Delta n-1}(t)\int_t^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s.\tag{66}$$

That is,

$$-1\le\frac{r^{\frac{1}{\alpha}}(t)z^{\Delta n-1}(t)}{z^{\Delta n-2}(t)}\zeta(t).$$

Therefore,

$$-1\le\omega(t)\zeta^{\alpha}(t)\le 0,\quad t\ge t_1.\tag{67}$$

Define another Riccati transformation as defined in (49). Clearly, $\nu(t) < 0$, $[r(t)(z^{\Delta n-1}(t))^{\alpha}]$ is decreasing and $\tau(t) \leq t$, we have

$$r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha} \geq r(t)(z^{\Delta n-1}(t))^{\alpha},$$

then

$$\nu(t) \geq \omega(t)$$

i.e.,

$$-1 \leq \nu(t)\zeta^{\alpha}(t) \leq 0, \quad t \geq t_1. \tag{68}$$

Proceeding as in the proof of Theorem 2.7 we arrive to (59). Multiplying (59) by $\zeta(t)$ and integrating from t_1 to t, we get

$$\begin{aligned}
&\omega(t)\zeta^{\alpha}(t) - \omega(t_1)\zeta^{\alpha}(t_1) + \frac{p_0^{\alpha}}{\tau_0}\nu(t)\zeta^{\alpha}(t) - \frac{p_0^{\alpha}}{\tau_0}\nu(t_1)\zeta^{\alpha}(t_1) \\
&+ \alpha\int_{t_1}^{t}\Big[r^{\frac{-1}{\alpha}}(s)\zeta^{\alpha-1}(s)\omega(\sigma(s)) + r^{\frac{-1}{\alpha}}(s)\zeta^{\alpha}(s)\omega^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Big]\Delta s \\
&+ \frac{\alpha p_0^{\alpha}}{\tau_0}\int_{t_1}^{t}\Big[r^{\frac{-1}{\alpha}}(s)\zeta^{\alpha-1}(s)\nu(\sigma(s)) + r^{\frac{-1}{\alpha}}(\tau(s))\zeta^{\alpha}(s)\nu^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Big]\Delta s \\
&+ \frac{1}{2^{\alpha-1}}\int_{t_1}^{t} h_{n-2}(\delta(s), t_1)\zeta^{\alpha}(s)Q(s)\Delta s + \leq 0.
\end{aligned} \tag{69}$$

Applying the inequality (43), we get

$$\begin{aligned}
&\omega(t)\zeta^{\alpha}(t) - \omega(t_1)\zeta^{\alpha}(t_1) + \frac{p_0^{\alpha}}{\tau_0}\nu(t)\zeta^{\alpha}(t) - \frac{p_0^{\alpha}}{\tau_0}\nu(t_1)\zeta^{\alpha}(t_1) + \\
&\int_{t_1}^{t}\Big[\frac{h_{n-2}(\delta(s), t_1)\zeta^{\alpha}(s)Q(s)}{2^{\alpha-1}} - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{r^{\frac{1}{\alpha}}(s)\zeta(s)}\Big]\Delta s \leq 0.
\end{aligned} \tag{70}$$

Therefore from (67) and (68), we have

$$\begin{aligned}
&\int_{t_1}^{t}\Big[\frac{h_{n-2}(\delta(s), t_1)\zeta^{\alpha}(s)Q(s)}{2^{\alpha-1}} - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{r^{\frac{1}{\alpha}}(s)\zeta(s)}\Big]\Delta s \\
&\quad \leq \omega(t_1)\zeta^{\alpha}(t_1) + \frac{p_0^{\alpha}}{\tau_0}\nu(t_1)\zeta^{\alpha}(t_1) + 1 + \frac{p_0^{\alpha}}{\tau_0}.
\end{aligned} \tag{71}$$

This contradicts (63) and completes the proof. □

Example 2.9. *Consider for $\mathbb{T} = \mathbb{R}$, the fourth-order differential equation*

$$\Big(t^5 x'''(t)\Big)' + \beta t x(t) = 0, \quad t \geq 1, \tag{72}$$

where $\beta > 0$ is constant. Here $\alpha = 1$, $r(t) = t^5$, $q(t) = \beta t$, $\tau(t) = \delta(t) = t$.
Since $\mathbb{T} = \mathbb{R}$, then $h_k(t,s) = \frac{(t-s)^k}{k!}$. Clearly

$$\zeta(t) = \int_t^{\infty} r^{-1/\alpha}(s)ds = \int_t^{\infty} s^{-5}ds = \frac{t^4}{4} < \infty.$$

Now, since

$$\int_{t_0}^{\infty}\Big[\frac{h_{n-2}(\delta(s),t_1)\zeta^{\alpha}(s)Q(s)}{2^{\alpha-1}}-\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{r^{\frac{1}{\alpha}}(s)\zeta(s)}\Big]\Delta s=\infty$$

Hence, by Theorem 2.8, every solution of (72) *is oscillatory. One can observe that by Theorem 2.1 [25], every solution of Eq.* (72) *is oscillatory for* $\beta > 12$ *if* $\beta > 4$. *Therefore, our criteria are more general than the equation in [25].*

Example 2.10. *Consider the second order neutral delay differential equation*

$$[t^{1/2}[x(t)+p_0x(\lambda t)]']'+\frac{a}{t^{3/2}}x(\beta t)=0, \tag{73}$$

where $0 < p_0 < \infty$, $0 < \lambda < \infty$, $0 < \beta < 1$, *and* $a > 0$.
It is clear that $\alpha = 1$,$n = 2$, $r(t) = t^{1/2}$, $p(t) = p_0 < \infty$, $\tau(t) = \lambda t$, $\delta(t) = \beta t$, $q(t) = \frac{a}{t^{3/2}}$, *and* $r(t) = t^{\frac{1}{2}}$, $\int_{t_0}^{\infty} r^{\frac{-1}{\alpha}}(s)ds = \infty$
If $\lambda \geq 1$, $0 < \beta < 1$, *then* $\tau(t) \geq \delta(t)$, *and* $Q(t) = \frac{a}{(\lambda t)^{3/2}}$. *From corollary 2.4, and* $\mathbb{T} = \mathbb{R}$, (25) *takes the form*

$$\begin{aligned}\limsup_{t\to\infty}\int_{\delta(t)}^{t}Q(s,t_1)\Delta s&=\limsup_{t\to\infty}\int_{\delta(t)}^{t}Q(s)\frac{h_{n-1}^{\alpha}(\delta(s),t_1)}{2^{\alpha-1}r(\delta(s))}\\&=\limsup_{t\to\infty}\int_{\beta t}^{t}\frac{a}{(\lambda s)^{3/2}}\frac{1}{\sqrt{\beta s}}ds=\frac{a\beta^{1/2}}{2\lambda^{3/2}}\ln\Big(\frac{1}{\beta}\Big).\end{aligned} \tag{74}$$

Using corollary 2.4, then Eq.(73) *is oscillatory if* $\frac{a\beta^{1/2}}{2\lambda^{3/2}}\ln\Big(\frac{1}{\beta}\Big) > 1$ *for any* $\lambda \geq 1$ *and* $0 < \beta < 1$.
If $0 < \beta < \lambda \leq 1$, *then* $\tau(t) \leq \delta(t)$, *and* $Q(t) = \frac{a}{t^{3/2}}$. *From Theorem 2.6 and* $\mathbb{T} = \mathbb{R}$, *we have*

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\Big(\frac{1}{2^{\alpha-1}}\rho(s)Q(s)-\frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{(\alpha+1)^{\alpha+1}}\frac{(n-2)!(\rho'(s))^{\alpha+1}r(\delta(s))}{(\delta'(s))^{\alpha}(\delta(s)-t_1)^{\alpha(n-2)}\rho^{\alpha}(s)}\Big)ds\\&\quad=\limsup_{t\to\infty}\int_{t_0}^{t}\Big(s^2\frac{a}{s^{3/2}}-\frac{1+\frac{p_0}{\lambda}}{4}\frac{(2s)^2\sqrt{\beta s}}{\beta s^2}\Big)ds=\infty.\end{aligned} \tag{75}$$

Provided that $a > \frac{\lambda+p_0^2}{\lambda\beta^{1/2}}$, (73) *is oscillatory for* $0 < \beta < \lambda \leq 1$. *Note that this example has been studied in [2] and to the best of our knowledge, we improve the oscillation criteria that mentioned in [2].*

Example 2.11. *Consider the second-order differential equation*

$$\Big(x(t)+\frac{9}{10}x\Big(\frac{t}{4}\Big)\Big)''+\frac{\lambda}{t^2}x\Big(\frac{t}{5}\Big)=0,\quad t\geq 1 \tag{76}$$

where $\lambda > 0$. *Here* $r(t) = 1$, $\alpha = 1$, $n = 2$, $p(t) = 0.9$, $\tau(t) = t/4$, $q(t) = \frac{\lambda}{t^2}$, *and* $\delta(t) = t/5$. *Then* $\tau_0 = 1/4$, $Q(t) = \frac{\lambda}{t^2}$, *and*

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\Big(\frac{1}{2^{\alpha-1}}\rho(s)Q(s)-\frac{(1+\frac{p_0^{\alpha}}{\tau_0})}{(\alpha+1)^{\alpha+1}}\frac{(n-2)!(\rho'(s))^{\alpha+1}r(\delta(s))}{(\delta'(s))^{\alpha}(\delta(s)-t_1)^{\alpha(n-2)}\rho^{\alpha}(s)}\Big)ds\\&\quad=\Big(\lambda-\frac{24}{3}\Big)\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{s}ds=\infty.\end{aligned} \tag{77}$$

Provided that $\lambda > \frac{24}{3}$. *This is consistent with the results of [23].*

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