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Results on Initial Value Problems for Random Fuzzy Fractional Functional Differential Equations

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Abstract. In this paper random fuzzy fractional functional differential equations (RFFFDEs) with Caputo generalized Hukuhara differentiability are introduced. We present existence and uniqueness results for RFFFDEs using the idea of successive approximations. The behaviour of solutions when the data of the equation are subjected to errors is discussed. Furthermore, the solution to random fuzzy fractional functional initial value problem under Caputo-type fuzzy fractional derivatives by a modified fractional Euler method (MFEM) is presented. The results are illustrated with examples.

1. Introduction

Fractional differential equations are used in modeling many physical and chemical processes and in engineering; see the monographs of Podlubny [42] and Kilbas et al. [19] and the papers [12, 23] and the references therein. Agarwal et al. [1] proposed the concept of solutions for fractional differential equations with uncertainty and many other authors considered existence and uniqueness of solutions to fuzzy fractional equations; we refer the reader to [4–6, 10, 25, 46, 47].Various approaches and methods, based on Hukuhara differentiability or generalized Hukuhara differentiability were then considered to investigate interval or fuzzy fractional differential equations in a number of papers in literature (see for example [7]-[9], [14]-[17] and [25, 29, 30]).

Random fuzzy differential equations (RFDEs) consider the phenomena of randomness and also fuzziness. Puri and Ralescu introduced fuzzy set-valued random variables in [45], and gave the concept of differentiability by the Hukuhara difference in [43] (the concept of a fuzzy random variable was proposed by Kwakernaak [22] and used by Kruse and Meyer [21]). Two notions of measurability of fuzzy mappings

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appear in [20, 45] and the relations between different concepts of measurability for fuzzy random variables were discussed in the papers of Colubi et al. [13], Terán Agraz [3], Puri and Ralescu [43]. In this paper, we will use the definition of fuzzy random variables introduced by Puri and Ralescu [44]. In [31], the author considered random fuzzy differential equations with the fuzzy derivative in the sense of Puri and Ralescu [43] and Malinowski [32, 33] studied two kinds of solutions to RFDEs and under Lipschitz type conditions he presented local and global existence and uniqueness results for RFDEs using the method of successive approximations. In [35]-[37], RFDE was extended to stochastic fuzzy differential equations and for other results on random fuzzy fractional equations we refer the reader to [18, 28, 34].

In this paper we initiate a study on random fuzzy fractional functional differential equations where we use a concept of a Caputo-type fuzzy fractional derivative. They can be viewed as an extension of random fuzzy differential equations and deterministic fractional functional differential equations and we discuss the behaviour of solutions to RFFFDEs with generalized fuzzy Caputo derivatives. Our paper was motivated partly by results of Bede and Stefanini [11], Agarwal et al. [1, 2], Arshad et al. [10], Lupulescu [16, 17, 27], Allahviranloo et al. [5, 6, 46], Salahshour [47], Mazandarani et al. [29], Malinowski [33]. In this paper our aim is to

- show the equivalence of the random fuzzy fractional functional differential equation and the random fuzzy fractional functional integral equation under suitable conditions.
- prove the existence and uniqueness of solutions of fuzzy fractional integral and differential equations, and to present the solutions' behaviour which changes continuously with the initial conditions.
- discuss the modified fractional Euler method for solving random fuzzy functional differential equations of fractional order with a Caputo-type fuzzy fractional derivative.

In Section 2, we present basic notations of the Riemann-Liouville fractional integral and the Caputo fractional derivative for fuzzy functions. In Section 3, we study existence and uniqueness of solutions to random fuzzy fractional functional differential equations with Caputo generalized Hukuhara differentiability. The solution to random fuzzy fractional functional initial value with a Caputo-type fuzzy fractional derivatives by a modified fractional Euler method is also presented.

2. Preliminaries

In this section, we give some notations and properties related to the fuzzy set space, and summarize the major results for integration and differentiation of fuzzy-set-valued mappings. We recall also the notations of fuzzy random variables and fuzzy stochastic processes. Let $K_c(\mathbb{R}^d)$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $K_c(\mathbb{R}^d)$ are defined as usual, i.e., for $A, B \in K_c(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$,

$$A + B = \{a + b \,|\, a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \,|\, a \in A\}.$$

The Hausdorff distance or Pompeiu-Hausdorff distance d_H in $K_c(\mathbb{R}^d)$ is defined as follows:

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

where $A, B \in K_c(\mathbb{R}^d)$, and $\|.\|$ denotes usual Euclidean norm in $\mathbb{R}^d$. It is known that $K_c(\mathbb{R}^d)$ is a complete, separable and locally compact metric space with respect to d_H. The basic definition of fuzzy numbers is mentioned in [24]. Let E denote the set of fuzzy subsets of the real axis, if $u : \mathbb{R} \to [0, 1]$, satisfying the following properties:

(i) u is normal, that is, there exists $z_0 \in \mathbb{R}$ such that $u(z_0) = 1$;

(ii) u is fuzzy convex, that is, for $0 \le \lambda \le 1$

$$u(\lambda z_1 + (1 - \lambda)z_2) \ge \min\{u(z_1), u(z_2)\}, \text{ for any } z_1, z_2 \in \mathbb{R};$$

(iii) u is upper semicontinuous on $\mathbb{R}$;

(iv) $[u]^0 = cl\{z \in \mathbb{R} : u(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}, |\cdot|)$.

Then E is called the space of fuzzy numbers. For $r \in (0,1]$, denote $[u]^r = \{z \in \mathbb{R} \,|\, u(z) \geq r\} = [\underline{u}(r), \overline{u}(r)]$. Then from (i) to (iv), it follows that the $r-$level set $[u]^r$ is a closed interval for all $r \in [0,1]$. For addition and scalar multiplication in fuzzy set space E, we have $[u_1 + u_2]^r = [u_1]^r + [u_2]^r, [\lambda u_1]^r = \lambda[u_1]^r$. The notation $[u]^r = [\underline{u}(r), \overline{u}(r)]$. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively. For $u \in E$, we define the diameter of the $r-$level set of u as $diam[u]^r = \overline{u}(r) - \underline{u}(r)$. The Hausdorff distance between fuzzy numbers is given by

$$D_0[u_1, u_2] = \sup_{0 \leq r \leq 1} \{|\underline{u}_1(r) - \underline{u}_2(r)|, |\overline{u}_1(r) - \overline{u}_2(r)|\}.$$

The metric space (E, D_0) is complete. The following properties of the metric D_0 are valid (see [24]): $D_0[u_1 + u_3, u_2 + u_3] = D_0[u_1, u_2], D_0[\lambda u_1, \lambda u_2] = |\lambda| D_0[u_1, u_2], D_0[u_1, u_2] \leq D_0[u_1, u_3] + D_0[u_3, u_2]$, for all $u_1, u_2, u_3 \in E$ and $\lambda \in \mathbb{R}$. Let $u_1, u_2 \in E$. If there exists $u_3 \in E$ such that $u_1 = u_2 + u_3$ then u_3 is called the H-difference of u_1, u_2. We denote the u by $u_1 \ominus u_2$. Let us remark that $u_1 \ominus u_2 \neq u_1 + (-1)u_2$.

Definition 2.1. [11] The generalized Hukuhara difference of two fuzzy numbers $u_1, u_2 \in E$ (gH-difference for short) is defined as follows

$$u_1 \ominus_{gH} u_2 = u_3 \Leftrightarrow \begin{cases} \quad (i)\ u_1 = u_2 + u_3, \\ \text{or } (ii)\ u_2 = u_1 + (-1)u_3. \end{cases}$$

A fuzzy function $x : [a,b] \to E$ is called d-increasing (d-decreasing) on $[a,b]$ if for every $r \in [0,1]$ the real function $t \mapsto diam[x(t)]^r$ is nondecreasing (nonincreasing) on $[a,b]$. If x is d-increasing or d-decreasing on $[a,b]$, then we say that d is *d-monotone* on $[a,b]$.

Lemma 2.2. Let $x : [a,b] \to E$ *be a* d -monotone fuzzy function and let $\omega \in E$ be given. Also, let $y : [a,b] \to E$ be the fuzzy function defined by $y(t) = \omega \ominus_{gH} x(t)$, $t \in [a,b]$.

(a) If $diam[x(t)]^r \leq diam[\omega]^r$ for every $r \in [0,1]$ and for all $t \in [a,b]$, then y and x are differently d-monotonic on $[a,b]$.

(b) If $diam[x(t)]^r \geq diam[\omega]^r$ for every $r \in [0,1]$ and for all $t \in [a,b]$, then y and x are equally d-monotonic on $[a,b]$.

Remark 2.3. Let $x : [a,b] \to E$ be a d-monotone fuzzy function and let $y : [a,b] \to E$ be the fuzzy function defined by $y(t) = x(t) \ominus_{gH} x(a)$, $t \in [a,b]$. If x is d-monotone in $[a,b]$, then y is $d-$increasing.

The generalized Hukuhara differentiability was introduced in [11].

Definition 2.4. Let $t \in (a,b)$ and h be such that $t + h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy-valued function $x : (a,b) \to E$ at t is defined as

$$D_{gH}x(t) = \lim_{h \to 0} \frac{x(t+h) \ominus_{gH} x(t)}{h}. \tag{1}$$

The fuzzy gH-fractional Caputo differentiability of fuzzy-valued functions was introduced in [5, 29]. For a detailed discussion on fractional derivatives and fuzzy fractional derivatives, we refer the reader to [1, 5, 6, 42]. Let $C([a,b], E)$ denote the set of continuous fuzzy-valued functions from $[a,b]$ into E. Then $C([a,b], E)$ is a complete metric space with respect to the metric D_0^*, where $D_0^*[x_1, x_2] = \sup_{a \leq t \leq b} D_0[x_1(t), x_2(t)]$. For $\gamma \in [0,1)$, we introduce the space $C_\gamma([a,b], E)$ of interval-valued functions $x : (a,b] \to E$ such that the function $(\cdot - a)^\gamma x(\cdot) \in C([a,b], E)$. The space $C_\gamma([a,b], E)$ is a complete metric space with respect to the metric D_0^γ, where

$$D_0^\gamma[x_1, x_2] = \sup_{a \leq t \leq b} t^\gamma D_0[x_1(t), x_2(t)].$$

In this paper, the space $C_\gamma([a,b],E)$ is called the space of $\gamma-$continuous fuzzy functions. The space of all Lebesgue integrable fuzzy valued functions on the bounded interval $[a,b]$ is denoted by $L([a,b],E)$. A fuzzy function $x:[a,b]\to E$ is said to be absolutely continuous if, for each $\varepsilon>0$, there exists $\delta>0$ such that, for each family $\{(s_k,t_k);k=1,2,...,n\}$ of disjoint open intervals in $[a,b]$ with $\sum\limits_{k=1}^{n}(t_k-s_k)<\delta$, we have $\sum\limits_{k=1}^{n}D_0[x(t_k),x(s_k)]<\varepsilon$ (here D_0 is the Hausdorff–Pompeiu distance between fuzzy numbers). Let $AC([a,b],E)$ denote the set of all absolutely continuous fuzzy functions from $[a,b]$ into E. A fuzzy function $x:[a,b]\to E$ is absolutely continuous if and only if $\underline{x}(r,t)$ and $\overline{x}(r,t)$ are both absolutely continuous for every $r\in[0,1]$ (see Proposition 4 in Lupulescu [27]).

Fuzzy-valued Riemann-Liouville fractional integral. Recall that if a real function $\varphi\in L[a,b]$, then the Riemann-Liouville fractional integral $I^\alpha_{a^+}\varphi$ of order $\alpha\in(0,1]$ is defined by (see [19])

$$\left(I^\alpha_{a^+}\varphi\right)(t)=\frac{1}{\Gamma(\alpha)}\int\limits_a^t(t-s)^{\alpha-1}\varphi(s)ds,\ \text{for } t\geq a.$$

The Riemann-Liouville derivative of order $\alpha\in(0,1]$ for a real function $\varphi\in AC[a,b]$ is defined by (see [19])

$$\left(D^\alpha_{a^+}\varphi\right)(t)=\frac{d}{dt}I^{1-\alpha}_{a^+}\varphi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t(t-s)^{-\alpha}\varphi(s)ds,\quad t\geq a.$$

The Caputo derivative of order $\alpha\in(0,1]$ for a real function φ is defined by (see [19])

$$\left({}^CD^\alpha_{a^+}\varphi\right)(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-s)^{-\alpha}\varphi'(s)ds,\quad t\geq a.$$

Let $x\in L([a,b],E)$. The Riemann-Liouville fractional integral of order α of the fuzzy-valued function x is defined as follows:

$$(\mathfrak{I}^\alpha_{a^+}x)(t)=\frac{1}{\Gamma(\alpha)}\int\limits_a^t(t-s)^{\alpha-1}x(s)ds,\quad t\geq a,$$

where $\Gamma(\alpha)$ is the well-known Gamma function. Since $[x(t)]^r=[x^-_r(t),x^+_r(t)]$ and $\alpha\in(0,1]$, then we can denote the fuzzy-valued Riemann-Liouville integral of the fuzzy-valued function x, $(\mathfrak{I}^\alpha_{a^+}x)(t)$, based on lower and upper functions, that is,

$$\left[(\mathfrak{I}^\alpha_{a^+}x)(t)\right]^r=\left[\frac{1}{\Gamma(\alpha)}\int\limits_a^t(t-s)^{\alpha-1}x^-_r(s)ds,\frac{1}{\Gamma(\alpha)}\int\limits_a^t(t-s)^{\alpha-1}x^+_r(s)ds\right],\quad t\geq a.$$

Fuzzy Riemann-Liouville fractional derivative. For a given fuzzy function $x\in L([a,b],E)$ and $\alpha\in(0,1]$, we define the fuzzy function $x_{1-\alpha}:[a,b]\to E$ by

$$x_{1-\alpha}(t)=\left(\mathfrak{I}^{1-\alpha}_{a^+}x\right)(t):=\int\limits_a^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}x(s)ds,\ \text{ for } t\in[a,b].$$

If the gH-derivative $D_{gH}x_{1-\alpha}(t)$ exists for $t\in[a,b]$, then $D_{gH}x_{1-\alpha}(t)$ is called the *fuzzy Riemann-Liouville fractional derivative (or Riemann-Liouville gH-fractional derivative) of order* $\alpha\in(0,1]$. The Riemann-Liouville gH-fractional derivative of x will be denoted by ${}^{RL}\mathcal{D}^\alpha_{a^+}x$. Therefore,

$$\left({}^{RL}\mathcal{D}^\alpha_{a^+}x\right)(t):=D_{gH}\left(\mathfrak{I}^{1-\alpha}_{a^+}x\right)(t),\ t\in[a,b].$$

Lemma 2.5. [16, 17] Let $x \in L([a,b],E)$ be a d-monotone fuzzy function such that $({}^{RL}\mathcal{D}^{\alpha}_{a^+}x)(t)$ exists for $t \in [a,b]$ and $y(t) := x(t) \ominus_{gH} x(a)$, $t \in [a,b]$. Then, $\mathfrak{I}^{1-\alpha}_{a^+}y \in AC([a,b],E)$ and $\frac{d}{dt}diam[\left(\mathfrak{I}^{1-\alpha}_{a^+}y\right)(t)]^r \geq 0$ for $t \in [a,b]$.

Proposition 2.6. *[16] If $x \in L([a,b],E)$, then*

$$\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\mathfrak{I}^{\alpha}_{a^+}x\right)(t) = x(t),\ t \in [a,b]. \tag{2}$$

Proposition 2.7. [16] Let $x \in L([a,b],E)$ be such that $x_{1-\alpha} \in AC([a,b],E)$. If either $\frac{d}{dt}diam[x_{1-\alpha}(t)]^r \geq 0$ for $t \in [a,b]$ or $\frac{d}{dt}diam[x_{1-\alpha}(t)]^r \leq 0$ for $t \in [a,b]$, then the gH-difference $x(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}x_{1-\alpha}(a)$ exists for $t \in [a,b]$, and

$$\left(\mathfrak{I}^{\alpha}_{a^+}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+}x\right)(t) = x(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}x_{1-\alpha}(a)\ \ for\ \ t \in [a,b]. \tag{3}$$

Remark 2.8. [16] For $0 \leq \gamma < 1$, if $x \in C_{\gamma}([a,b],E)$, then (2) holds for any $t \in (a,b]$. In particular, if $x \in C([a,b],E)$, then (2) holds for any $t \in [a,b]$.

Fuzzy-valued Caputo fractional derivative. Let $x \in L([a,b],E)$ be a fuzzy function such that the Riemann-Liouville gH-fractional derivative ${}^{RL}\mathcal{D}^{\alpha}_{a^+}x$, $\alpha \in (0,1]$, exists on $[a,b]$. In this case we can define the *fuzzy Caputo fractional derivative of order* $\alpha \in (0,1]$ of x by

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}x\right)(t) := \left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\left[x(\cdot) \ominus_{gH} x(a)\right]\right)(t),\ t \in [a,b].$$

We also notice that if $x \in AC([a,b],E)$ is a d-monotone fuzzy function and $\alpha \in (0,1]$, then

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}x\right)(t) = \left(\mathfrak{I}^{1-\alpha}_{a^+}D_{gH}x\right)(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}D_{gH}x(s)ds,\ t \in [a,b]. \tag{4}$$

Furthermore, if $t \mapsto diam[x(t)]^r$ is increasing on $[a,b]$ or decreasing on $[a,b]$ for every $r \in [0,1]$, then

$$\left(\mathfrak{I}^{\alpha}_{a^+}\,{}^{C}\mathcal{D}^{\alpha}_{a^+}x\right)(t) = x(t) \ominus_{gH} x(a), \quad t \in [a,b].$$

Remark 2.9. Let $x : [a,b] \to E$ be a fuzzy-valued function on $[a,b]$.

(i) If x is d-increasing for all $t \in [a,b]$ then $[({}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}x)(t)]^r = \left[{}^{C}D^{\alpha}_{a^+}\underline{x}(t,r), {}^{C}D^{\alpha}_{a^+}\overline{x}(t,r)\right]$.

(ii) If x is d-decreasing for all $t \in [a,b]$ then $[({}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}x)(t)]^r = \left[{}^{C}D^{\alpha}_{a^+}\overline{x}(t,r), {}^{C}D^{\alpha}_{a^+}\underline{x}(t,r)\right]$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. A function $x : \Omega \to E$ is called a fuzzy random variable, if the set-valued mapping $[x(\cdot)]^{\alpha} : \Omega \to K_c(\mathbb{R})$ is a measurable multiplication for all $\alpha \in [0,1]$, i.e. $\{\omega \in \Omega |\, [x(\omega)]^{\alpha} \cap B \neq \emptyset\} \in \mathcal{F}$ for every closed set $B \subset \mathbb{R}$. A mapping $x : [a,b] \times \Omega \to E$ is said to be a fuzzy stochastic process if $x(\cdot,\omega)$ is a fuzzy set-valued function with any fixed $\omega \in \Omega$, and $x(t,\cdot)$ is a fuzzy random variable for any fixed $t \in [a,b]$. In [31], the $x(\cdot,\omega)$ function is called a trajectory. A fuzzy stochastic process $x(t,\omega) \in E$ is called continuous if for almost all $\omega \in \Omega$ the trajectory $x(\cdot,\omega)$ is a continuous function on $[a,b]$ with respect to the metric D_0.

For convenience, from now on, the fact that there that exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ we have $x(\omega) = y(\omega)$, where x, y are random elements, will be written as $x(\omega) \overset{\mathbb{P}.1}{=} y(\omega)$. Similarly, for inequalities. Also if there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and for every fixed $\omega \in \Omega_0$ we have $x(t,\omega) = y(t,\omega)$ for every $t \in [a,b]$, where x, y are stochastic processes, then we will write $x(t,\omega) \overset{[a-\sigma,a+p],\,\mathbb{P}.1}{=} y(t,\omega)$ in short, or $x(t,\omega) = y(t,\omega)$ for every $t \in [a,b]$ with $\mathbb{P}.1$. Similarly, for inequality.

3. Main results

In this section, we give existence and uniqueness theorems for a solution of a fuzzy fractional functional integral equation and these results are used to investigate existence and uniqueness results for solutions of fuzzy fractional functional differential equations.

For $\sigma > 0$, we denote by C_σ the space $C([-\sigma, 0], E)$ equipped with the metric defined by

$$D_\sigma[x, y] = \sup_{t\in[-\sigma,0]} D_0[x(t), y(t)]$$

Define $I = [a, b]$, $J = [a-\sigma, a] \cup I = [a-\sigma, b]$. Then, for each $t \in I$ we denote by x_t the element of C_σ defined by $x_t(s) = x(t+s)$ for $s \in [-\sigma, 0]$.

We assume that $f : \Omega \times I \times C_\sigma \to E$ satisfies the following hypotheses:

(A1) $f_\cdot(t, \varphi) : \Omega \to E$ is a fuzzy random variable for $(t, \varphi) \in I \times C_\sigma$ and $f_\omega(\cdot, \varphi) : I \to E$ is measurable for any $\varphi \in C_\sigma$,

(A2) with $\mathbb{P}.1$ the mapping $f_\omega(\cdot,\cdot) : (a, b] \times C_\sigma \to E$ is a $\gamma-$continuous fuzzy mapping at every point $(t_0, \varphi_0) \in I \times C_\sigma$, i.e., there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega$ the following is true: for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $t \in I$ and $\varphi \in C_\sigma$ we have

$$\max\{|t-t_0|, D_\sigma[\varphi, \varphi_0]\} < \delta \Rightarrow D_0^\gamma[f_\omega(t, \varphi), f_\omega(t_0, \varphi_0)] < \epsilon.$$

Fuzzy fractional functional integral equation: Consider the following random fuzzy fractional functional integral equation

$$\begin{cases} x(t,\omega) \overset{\mathbb{P}.1}{=} \varphi(t-a,\omega) \text{ for } t \in [a-\sigma, a] \\ x(t,\omega) \ominus_{gH} \varphi(0,\omega) \overset{\mathbb{P}.1}{=} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f_\omega(s, x_s) ds, \quad t \in I. \end{cases} \tag{1}$$

If x is a continuous fuzzy stochastic process such that $diam[x(t,\omega)]^r \geq diam[\varphi(0,\omega)]^r$ for all $t \in [a, b]$, for $\mathbb{P}-$a.a. ω and for every $r \in [0, 1]$, then (1) can be written as

$$\begin{cases} x(t,\omega) \overset{\mathbb{P}.1}{=} \varphi(t-a,\omega) \text{ for } \ t \in [a-\sigma, a] \\ x(t,\omega) \ominus \varphi(0,\omega) \overset{\mathbb{P}.1}{=} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f_\omega(s, x_s) ds, \ t \in [a, b]. \end{cases} \tag{2}$$

If x is a continuous fuzzy stochastic process such that $diam[x(t,\omega)]^r \leq diam[\varphi(0,\omega)]^r$ for all $t \in [a, b]$, for $\mathbb{P}-$a.a. ω and for every $r \in [0, 1]$, then (1) can be written as

$$\begin{cases} x(t,\omega) \overset{\mathbb{P}.1}{=} \varphi(t-a,\omega) \text{ for } \ t \in [a-\sigma, a] \\ \varphi(0,\omega) \ominus x(t,\omega) \overset{\mathbb{P}.1}{=} \dfrac{(-1)}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f_\omega(s, x_s) ds, \ t \in [a, b]. \end{cases} \tag{3}$$

Theorem 3.1. Let $f : \Omega \times [a, b] \times C_\sigma \to E$ satisfies the conditions (A1)-(A2) and assume that there exist positive constants L, M such that for every $\psi, \xi \in C_\sigma$

$$D_0[f_\omega(t,\xi), f_\omega(t,\psi)] \overset{[a,b],\mathbb{P}.1}{\leq} L D_\sigma[\xi, \psi], \quad D_0\left[f_\omega(t,\xi), \hat{\mathbf{0}}\right] \overset{[a,b],\mathbb{P}.1}{\leq} M.$$

Then, the following successive approximations given by

$$\begin{cases} x^0(t,\omega) \overset{[a-\sigma,a],\mathbb{P}.1}{=} \varphi(t-a,\omega), \\ x^0(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \varphi(0,\omega) \end{cases}$$

and for $n = 1, 2, ...$

$$\begin{cases} x^n(t,\omega) \overset{[a-\sigma,a],\mathbb{P}.1}{=} \varphi(t-a,\omega), \\ x^n(t,\omega) \ominus_{gH} \varphi(0,\omega) \overset{[a,b],\mathbb{P}.1}{=} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f_\omega(s, x_s^{n-1}) ds \end{cases} \tag{4}$$

converge uniformly to a unique solution of the random fractional functional integral equation (1) on some intervals $[a, \mathbb{T}]$ for some $\mathbb{T} \in (a, b]$.

Proof. Let $\rho > 0$ be a given real number, and let $\mathbb{B}_\rho(x_0) := \{x \in E \,|\, D_0[x, x_0] \le \rho\}$. Choose $t^* > a$ such that $t^* - a \le \Big(\dfrac{\rho\Gamma(1+\alpha)}{M}\Big)^{1/\alpha}$ and put $\mathbb{T} := \min\{t^*, b\}$. Let $\mathbb{B}$ be a set of continuous fuzzy stochastic processes x such that $x(t,\omega) = \varphi(t-a,\omega)$ for $(t,\omega) \in [a-\sigma, a] \times \Omega$ and $x(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\in} \mathbb{B}_\rho(x_0)$. Next, we consider the sequence of continuous fuzzy stochastic processes $\{x^n\}_{n=0}^\infty$ given by: $x^0(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{:=} x_0$, where

$$\begin{cases} x^0(t,\omega) \overset{[a-\sigma,a],\mathbb{P}.1}{=} \varphi(t-a,\omega), \\ x^0(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{=} \varphi(0,\omega) \end{cases}$$

and for $n = 1, 2, ...$

$$\begin{cases} x^n(t,\omega) \overset{[a-\sigma,a],\mathbb{P}.1}{=} \varphi(t-a,\omega), \\ x^n(t,\omega) \ominus_{gH} \varphi(0,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{=} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f_\omega(s, x_s^{n-1}) ds. \end{cases} \tag{5}$$

For all $n \ge 0$, it follows that $x^n(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\in} \mathbb{B}_\rho(x_0)$ if and only if $x^n(t,\omega) \ominus_{gH} \varphi(0,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\in} B_\rho(\hat{\mathbf{0}})$. If we suppose that $x^{n-1}(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\in} \mathbb{B}$ for a given $n \ge 2$, then from

$$\begin{aligned} D_0[x^n(t,\omega) \ominus_{gH} \varphi(0,\omega), \hat{\mathbf{0}}] &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\le} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0[f_\omega(s, x_s^{n-1}), \hat{\mathbf{0}}] ds \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\le} \frac{M(t-a)^\alpha}{\Gamma(1+\alpha)} \le \rho \end{aligned}$$

it follows that $x^n(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\in} \mathbb{B}$. Hence, by mathematical induction, we have that $x^n(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\in} \mathbb{B}$ for all $n \ge 1$. For $(t,\omega) \in [a,\mathbb{T}] \times \Omega$ and $n \ge 1$, let us denote $f_\omega(t, x^n(t,\omega))$ by $\mathbb{F}_\omega^n(t)$. Next, for any $t_1, t_2 \in [a, \mathbb{T}]$ with $t_1 < t_2$, we have

$$\begin{aligned} &\Gamma(\alpha) D_0[x^n(t_1,\omega) \ominus_{gH} \varphi(0,\omega), x^n(t_2,\omega) \ominus_{gH} \varphi(0,\omega)] \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{=} D_0\Big[\int_a^{t_1} (t_1-s)^{\alpha-1} \mathbb{F}_\omega^{n-1}(s) ds, \int_a^{t_2} (t_2-s)^{\alpha-1} \mathbb{F}_\omega^{n-1}(s) ds\Big] \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{=} D_0\Big[\int_a^{t_1} (t_1-s)^{\alpha-1} \mathbb{F}_\omega^{n-1}(s) ds, \int_a^{t_1} (t_2-s)^{\alpha-1} \mathbb{F}_\omega^{n-1}(s) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \mathbb{F}_\omega^{n-1}(s) ds\Big]. \end{aligned}$$

Since

$$diam\Big[(t_1-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)\Big]^r - diam\Big[(t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)\Big]^r$$
$$= ((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})diam[\mathbb{F}_\omega^{n-1}(s)]^r \overset{\mathbb{P}.1}{\geq} 0$$

for $a \leq s \leq t_1 \leq t_2$, that is, the difference $diam\Big[(t_1-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)\Big]^r - diam\Big[(t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)\Big]^r$ has a constant sign on $[a,\mathbb{T}]$ with $\mathbb{P}.1$, then we have

$$\int_a^{t_1}(t_1-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)ds \ominus_{gH} \int_a^{t_1}(t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)ds$$
$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{=} \int_a^{t_1}\Big[(t_1-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s) \ominus_{gH} (t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)\Big]ds$$
$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{=} \int_a^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]\mathbb{F}_\omega^{n-1}(s)ds.$$

Next, we obtain that

$$\Gamma(\alpha)D_0[x^n(t_1,\omega)\ominus_{gH}\varphi(0,\omega), x^n(t_2,\omega)\ominus_{gH}\varphi(0,\omega)]$$
$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{=} D_0\Big[\int_a^{t_1}(t_1-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)ds \ominus_{gH} \int_a^{t_1}(t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)ds, \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)ds\Big]$$
$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{=} D_0\Big[\int_a^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]\mathbb{F}_\omega^{n-1}(s)ds, \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{F}_\omega^{n-1}(s)ds\Big]$$
$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \int_a^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]D_0[\mathbb{F}_\omega^{n-1}(s),\hat{\mathbf{0}}]ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}D_0[\mathbb{F}_\omega^{n-1}(s),\hat{\mathbf{0}}]ds$$
$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{M}{\alpha}[2(t_2-t_1)^\alpha+(t_1-a)^\alpha-(t_2-a)^\alpha] \leq \frac{2M}{\alpha}(t_2-t_1)^\alpha.$$

Therefore, for any $\varepsilon > 0$ and any $n \geq 1$, we have that

$$D_0[x^n(t_1,\omega)\ominus_{gH}\varphi(0,\omega), x^n(t_2,\omega)\ominus_{gH}\varphi(0,\omega)] < \varepsilon,$$

provided that $|t_2-t_1| < \delta_0 := \Big(\frac{\varepsilon\Gamma(1+\alpha)}{2M}\Big)^{1/\alpha}$. It then follows that the functions $x^n(\cdot,\omega)\ominus_{gH}\varphi(0,\omega)$ are continuous with $\mathbb{P}.1$. Now, for $n \geq 0$ and $t \in [a-\sigma,\mathbb{T}]$ the functions $x^n(t,\cdot) : \Omega \to E$ are fuzzy random variables. Indeed, since $\varphi \in C_\sigma$ is a fuzzy random variable, for every $t \in [a-\sigma,a]$, $[\varphi(t,\omega)]^r$ is a measurable multifunction for every $r \in [0,1]$. Thus $x^n(t,\cdot)$ is random variable for any $t \in [a-\sigma,a]$. Next, for every $t \in [a,\mathbb{T}]$ and for $n \geq 0, r \in [0,1]$ the mappings $\omega \mapsto \Big[\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\mathbb{F}_\omega^n(s)ds\Big]^r$ is a measurable multifunction. Let $r \in [0,1]$ be fixed. By virtue of the definition of fuzzy integral and a theorem of Nguyen [38], for every $t \in [a,\mathbb{T}]$ we have

$$\Big[\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\mathbb{F}_\omega^n(s)ds\Big]^r = \frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\Big[\mathbb{F}_\omega^n(s)\Big]^r ds.$$

As the integrand is a multifunction which is continuous in s and measurable in ω, the mapping $\omega \mapsto \dfrac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big[\mathbb{F}^n_\omega(s)\Big]^r ds$ is a measurable multifunction for each $r \in [0,1]$. Therefore, for every $t \in [a,\mathbb{T}]$, the sequence $\{x^n(t,\cdot)\}_{n=0}^{\infty}$ is a sequence of fuzzy random variable. Consequently, $\{x^n\}_{n=0}^{\infty}$ is a sequence of fuzzy stochastic process.

Note that for $(t,\omega) \in [a,\mathbb{T}] \times \Omega$

$$\begin{aligned} D_0[x^1(t,\omega) \ominus_{gH} \varphi(0,\omega), x^0(t,\omega) \ominus_{gH} \varphi(0,\omega)] &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^t \frac{D_0[\mathbb{F}^0_\omega(s), \hat{0}]}{(t-s)^{1-\alpha}} ds \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{M(\mathbb{T}-a)^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

and

$$\begin{aligned} \sup_{\varsigma\in[-\sigma,0]} D_0[x^1(t+\varsigma,\omega), x^0(t+\varsigma,\omega)] &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{1}{\Gamma(\alpha)} \sup_{\varsigma\in[-\sigma,0]} \int_a^{t+\varsigma} (t+\varsigma-s)^{\alpha-1} D_0[f_\omega(s,x^0_s),\hat{0}] ds \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{1}{\Gamma(\alpha)} \sup_{\theta\in[t-\sigma,t]} \int_a^{\theta} (\theta-s)^{\alpha-1} D_0[f_\omega(s,x^0_s),\hat{0}] ds \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{M}{\Gamma(\alpha+1)} \sup_{\theta\in[t-\sigma,t]} (\theta-a)^\alpha \leq \frac{M(\mathbb{T}-a)^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Moreover, for $n \in \{2,3,4,...\}$ and $(t,\omega) \in [a,\mathbb{T}] \times \Omega$ we obtain that

$$\begin{aligned} D_0[x^2(t,\omega) \ominus_{gH} \varphi(0,\omega), x^1(t,\omega) \ominus_{gH} \varphi(0,\omega)] &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^t \frac{D_0[\mathbb{F}^1_\omega(s), \mathbb{F}^0_\omega(s)]}{(t-s)^{1-\alpha}} ds \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{L}{\Gamma(\alpha)} \int_a^t \frac{D_0[x^1_s, x^0_s]}{(t-s)^{1-\alpha}} ds \\ &\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} ML\Big[\frac{(\mathbb{T}-a)^\alpha}{\Gamma(\alpha+1)}\Big]^2. \end{aligned}$$

Further, if we assume that

$$\max\left\{ D_0[x^n(t,\omega), x^{n-1}(t,\omega)], \sup_{\varsigma\in[-\sigma,0]} D_0[x^n(t+\varsigma,\omega), x^{n-1}(t+\varsigma,\omega)] \right\} \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{M}{L}\Big[\frac{(\mathbb{T}-a)^\alpha L}{\Gamma(\alpha+1)}\Big]^n,$$

then we have

$$\begin{aligned} &D_0[x^{n+1}(t,\omega) \ominus_{gH} \varphi(0,\omega), x^n(t,\omega) \ominus_{gH} \varphi(0,\omega)] \\ &\qquad \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^t L(t-s)^{\alpha-1} \frac{M}{L}\Big[\frac{(\mathbb{T}-a)^\alpha L}{\Gamma(\alpha+1)}\Big]^n ds \\ &\qquad \leq \frac{M}{L}\Big[\frac{(\mathbb{T}-a)^\alpha L}{\Gamma(\alpha+1)}\Big]^{n+1}. \end{aligned} \tag{6}$$

Hence, for $n > m > 0$ we have

$$\sup_{t\in[a,\mathbb{T}]} D_0[x^n(t,\omega) \ominus_{gH} \varphi(0,\omega), x^m(t,\omega) \ominus_{gH} \varphi(0,\omega)] \overset{\mathbb{P}.1}{\leq} \frac{M}{L} \sum_{i=m+1}^{n} \Big[\frac{(\mathbb{T}-a)^\alpha L}{\Gamma(\alpha+1)}\Big]^i.$$

The convergence of the series $\sum_{i=1}^{\infty}\Big[\frac{(\mathbb{T}-a)^{\alpha}L}{\Gamma(\alpha+1)}\Big]^{i}$ implies that for any $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ large enough such that for $n, m > n_0$

$$\sup_{t\in[a,\mathbb{T}]} D_0[x^n(t,\omega)\ominus_{gH}\varphi(0,\omega), x^m(t,\omega)\ominus_{gH}\varphi(0,\omega)] \overset{\mathbb{P}.1}{\leq} \varepsilon. \tag{7}$$

As (E, D_0) is a complete metric space and (7) holds, there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ the sequence $\{x^n(\cdot,\omega)\}$ is uniformly convergent with $\mathbb{P}.1$. For $\omega \in \Omega_0$ let $\hat{x}(\cdot,\omega)$ denote its limit. Let us define a mapping $x : [a-\sigma, \mathbb{T}] \times \Omega \to E$ as

$$x(t,\omega) = \begin{cases} \varphi(t-a,\omega) & \text{for} \quad (t,\omega) \in [a-\sigma, a]\times\Omega_0 \\ \hat{x}(t,\omega) & \text{for} \quad (t,\omega)\in[a,\mathbb{T}]\times\Omega_0 \\ \hat{\mathbf{0}} & \text{for} \quad (t,\omega)\in[a-\sigma,\mathbb{T}]\times(\Omega\backslash\Omega_0). \end{cases}$$

Since $\varphi \in C_\sigma$ is a fuzzy random variable and $\sup_{t\in[a,\mathbb{T}]} D_0[x^n(t,\omega)\ominus_{gH}\varphi(0,\omega), x(t,\omega)\ominus_{gH}\varphi(0,\omega)] \overset{\mathbb{P}.1}{\to} 0$ as $n \to \infty$, we see that $[x(t,\cdot)]^r$ with $r \in [0,1]$ and $t \in [a-\sigma, \mathbb{T}]$ is a measurable multifunction. Therefore $x : [a-\sigma,\mathbb{T}]\times\Omega \to E$ is a continuous fuzzy stochastic process. In the sequel we show that x is a solution of the random fuzzy fractional functional integral equation (1). For $\varepsilon > 0$, there is a n_0 large enough such that for every $n \geq n_0$ we obtain

$$\begin{aligned}
&\sup_{t\in[a,\mathbb{T}]} D_0\Big[x(t,\omega)\ominus_{gH}\varphi(0,\omega), \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f_\omega(s,x_s)ds\Big] \\
&\overset{\mathbb{P}.1}{\leq} \sup_{t\in[a,\mathbb{T}]} D_0[x^n(t,\omega)\ominus_{gH}\varphi(0,\omega), x(t,\omega)\ominus_{gH}\varphi(0,\omega)] \\
&+ \sup_{t\in[a,\mathbb{T}]} \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D_0[f_\omega(s,x_s^{n-1}), f_\omega(s,x_s)]ds \\
&\overset{\mathbb{P}.1}{\leq} \sup_{t\in[a,\mathbb{T}]} D_0[x^{n-1}(t,\omega), x(t,\omega)] + \sup_{t\in[a,\mathbb{T}]} \frac{L}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D_\sigma[x_s(\cdot,\omega), x_s^{n-1}(\cdot,\omega)]ds \\
&\overset{\mathbb{P}.1}{=} \sup_{t\in[a,\mathbb{T}]} D_0[x^{n-1}(t,\omega), x(t,\omega)] + \sup_{t\in[a,\mathbb{T}]} \frac{L}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} \sup_{\varsigma\in[-\sigma,0]} D_0[x_s(\varsigma,\omega), x_s^{n-1}(\varsigma,\omega)]ds \\
&\overset{\mathbb{P}.1}{=} \sup_{t\in[a,\mathbb{T}]} D_0[x^{n-1}(t,\omega), x(t,\omega)] + \sup_{t\in[a,\mathbb{T}]} \frac{L}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} \sup_{\theta\in[s-\sigma,s]} D_0[x(\theta,\omega), x^{n-1}(\theta,\omega)]ds \\
&\overset{\mathbb{P}.1}{\to} 0 \quad \text{as } n\to\infty \quad \text{for any } t\in[a,\mathbb{T}],
\end{aligned}$$

because the sequence $\{x^n(\cdot,\omega)\}$ is uniformly convergent to $x(\cdot,\omega)$. Therefore we get

$$D_0\Big[x(t,\omega)\ominus_{gH}\varphi(0,\omega), \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f_\omega(s,x_s)ds\Big] \overset{[a,\mathbb{T}],\mathbb{P}.1}{=} 0.$$

Hence (1) is satisfied. To prove the uniqueness, let $z : [a-\sigma,\mathbb{T}]\times\Omega \to E$ be a second solution for (1) on

$[a-\sigma, \mathbb{T}]$. Then for every $t \in [a, \mathbb{T}]$ we have

$$D_0[x(t,\omega) \ominus_{gH} \varphi(0,\omega), z(t,\omega) \ominus_{gH} \varphi(0,\omega)] \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_\sigma[x_s(\cdot,\omega), z_s(\cdot,\omega)]ds$$

$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{\theta \in [s-\sigma,s]} D_0[x(\theta,\omega), z(\theta,\omega)]ds$$

$$= \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{\theta \in [s-\sigma,s]} D_0[x(\theta,\omega) \ominus_{gH} \varphi(0,\omega), z(\theta,\omega) \ominus_{gH} \varphi(0,\omega)]ds.$$

If we let $\psi(s,\omega) = \sup_{\theta \in [s-\sigma,s]} D_0[x(\theta,\omega) \ominus_{gH} \varphi(0,\omega), z(\theta,\omega) \ominus_{gH} \varphi(0,\omega)]$ for any $s \in [a, \mathbb{T}]$, then we have

$$\psi(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \psi(s,\omega)ds.$$

Applying Gronwall's inequality we can infer that $\sup_{t \in [a,\mathbb{T}]} D_0[x(t,\omega) \ominus_{gH} \varphi(0,\omega), z(t,\omega) \ominus_{gH} \varphi(0,\omega)] \overset{\mathbb{P}.1}{=} 0$ which leads us to the conclusion $x(t,\omega) \overset{[a,\mathbb{T}],\mathbb{P}.1}{=} z(t,\omega)$. This proves the uniqueness of the solution for (1) on the interval $[a, \mathbb{T}]$. □

Theorem 3.2. Let $f : \Omega \times [a, \mathbb{T}] \times C_\sigma \to E$ satisfy the assumptions of Theorem 3.1. If $x(t,\omega)$ and $z(t,\omega)$ are solutions of (1) with $x(t,\omega) = \varphi(t-a,\omega)$ and $z(t,\omega) = \xi(t-a,\omega)$ for $t \in [a-\sigma, a]$, then we have

$$D_0[x(t,\omega), z(t,\omega)] \overset{[a-\sigma,\mathbb{T}],\mathbb{P}.1}{\leq} D_\sigma[\varphi(t-a,\omega), \xi(t-a,\omega)] \exp\left\{\frac{L(\mathbb{T}-a)^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Proof. Observe that for $(t,\omega) \in [a, \mathbb{T}] \times \Omega$,

$$D_0[x(t,\omega) \ominus_{gH} \varphi(0,\omega), z(t,\omega) \ominus_{gH} \xi(0,\omega)] \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq}$$

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0\Big[f_\omega(s, x_s), f_\omega(s, z_s)\Big]ds$$

$$\overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{\theta \in [s-\sigma,s]} D_0[x(\theta,\omega), z(\theta,\omega)]ds$$

or

$$D_0[x(t,\omega), z(t,\omega)] \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} D_0[\varphi(0,\omega), \xi(0,\omega)] + \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{\theta \in [s-\sigma,s]} D_0[x(\theta,\omega), z(\theta,\omega)]ds.$$

Putting $\psi(s,\omega) = \sup_{\theta \in [s-\sigma,s]} D_0[x(\theta,\omega), z(\theta,\omega)]$ for any $s \in [a, \mathbb{T}]$ and due to Gronwall's inequality we obtain

$$D_0[x(t,\omega), z(t,\omega)] \overset{[a,\mathbb{T}],\mathbb{P}.1}{\leq} D_0[\varphi(0,\omega), \xi(0,\omega)] \exp\left\{\frac{L(\mathbb{T}-a)^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Therefore we get the inequality

$$D_0[x(t,\omega), z(t,\omega)] \overset{[a-\sigma,\mathbb{T}],\mathbb{P}.1}{\leq} D_\sigma[\varphi(t-a,\omega), \xi(t-a,\omega)] \exp\left\{\frac{L(\mathbb{T}-a)^\alpha}{\Gamma(\alpha+1)}\right\}.$$

The proof is complete. □

Remark 3.3. With the assumptions of Theorem 3.2, the comparison between any two solutions of (1) can be obtained by using the generalized Gronwall inequality which can be used in a fractional differential equation. Indeed, if $x(t,\omega)$ and $z(t,\omega)$ are solutions of (11) with $x(t,\omega) = \varphi(t,\omega)$ and $z(t,\omega) = \xi(t,\omega)$ for $t \in [a-\sigma, a]$, then we have

$$D_0[x(t,\omega), z(t,\omega)] \overset{[a-\sigma,\mathbb{T}],\mathbb{P}.1}{\leq} D_\sigma[\varphi(t,\omega), \xi(t,\omega)] + \int_a^t \Big(\sum_{k=1}^{\infty} \frac{L^k}{\Gamma(k\alpha)}(t-s)^{k\alpha-1} D_\sigma[\varphi,\xi]\Big) ds.$$

In the sequel, we want to show that the solutions of (1) depend continuous on the initial condition and the right-hand side of equation. Let us again consider the equation (1) of the form

$$\begin{cases} x(t,\omega) \overset{\mathbb{P}.1}{=} \varphi(t-a,\omega) \text{ for } t \in [a-\sigma, a] \\ x(t,\omega) \ominus_{gH} \varphi(0,\omega) \overset{\mathbb{P}.1}{=} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f_\omega(s, x_s) ds, \quad \text{for } t \in [a,b] \end{cases} \tag{8}$$

and a problem with another initial value and another right-hand side, i.e.,

$$\begin{cases} x^\varepsilon(t,\omega) \overset{\mathbb{P}.1}{=} \varphi^\varepsilon(t-a,\omega) \text{ for } t \in [a-\sigma, a] \\ x^\varepsilon(t,\omega) \ominus_{gH} \varphi^\varepsilon(0,\omega) \overset{\mathbb{P}.1}{=} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f^\varepsilon_\omega(s, x^\varepsilon_s) ds, \quad \text{for } t \in [a,b], \end{cases} \tag{9}$$

where $f, f^\varepsilon : \Omega \times [a,b] \times C_\sigma \to E$ satisfy, such as in Theorem 3.1, conditions (A1)-(A2) and a Lipschitz condition with positive constants L and L^ε, respectively, and also a bounded condition holds with positive constants M and M^ε, respectively.

Theorem 3.4. *Let $f, f^\varepsilon : \Omega \times [a,b] \times C_\sigma \to E$ satisfy the assumptions of Theorem 3.1. Assume also that there exists constant ε^* such that $D_0[f_\omega(t,\xi_t), f^\varepsilon_\omega(t,\xi_t)] \leq \varepsilon^*$ with $\mathbb{P}.1$ for every $t \in [a,b]$.*
Suppose that the solutions $x, x_\varepsilon : [a-\sigma, b] \times \Omega \to E$ to problems (8) and (9), respectively, do exist. Then

$$D_0[x(t,\omega), x^\varepsilon(t,\omega)] \overset{[a-\sigma,b],\,\mathbb{P}.1}{\leq} \Big[D_\sigma[\varphi,\varphi^\varepsilon] + \frac{\varepsilon^*(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] \exp\left\{\frac{L^\varepsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\right\}. \tag{10}$$

Proof. Let $x(t,\omega), x^\varepsilon(t,\omega)$ denote the solutions to problems (8) and (9), respectively. For $t \in [a,b]$ we get $\omega \in \Omega$

$$\begin{aligned} D_0[f_\omega(t,x_t), f^\varepsilon_\omega(t,x^\varepsilon_t)] &\overset{[a,b],\mathbb{P}.1}{\leq} D_0[f_\omega(t,x_t), f^\varepsilon_\omega(t,x_t)] + D_0[f^\varepsilon_\omega(t,x_t), f^\varepsilon_\omega(t,x^\varepsilon_t)] \\ &\leq \varepsilon^* + L^\varepsilon D_\sigma[x_t, x^\varepsilon_t]. \end{aligned}$$

Therefore, we obtain

$$D_0[x(t,\omega)\ominus_{gH}\varphi(0,\omega), x^\varepsilon(t,\omega)\ominus_{gH}\varphi^\varepsilon(0,\omega)]$$
$$\overset{[a,a+p],\,\mathbb{P}.1}{\leq} \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\left(\varepsilon^* + L^\varepsilon D_\sigma[x_s(\cdot,\omega), x_s^\varepsilon(\cdot,\omega)]\right)}{(t-s)^{1-\alpha}}ds$$
$$\overset{[a,a+p],\,\mathbb{P}.1}{\leq} \frac{\varepsilon^*(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{L^\varepsilon}{\Gamma(\alpha)}\int_a^t \frac{\sup_{\theta\in[s-\sigma,s]} D_0[x(\theta,\omega), x^\varepsilon(\theta,\omega)]}{(t-s)^{1-\alpha}}ds$$

or

$$D_0[x(t,\omega), x^\varepsilon(t,\omega)] \overset{[a,a+p],\,\mathbb{P}.1}{\leq} D_0[\varphi(0,\omega), \varphi^\varepsilon(0,\omega)] + \frac{\varepsilon^*(b-a)^\alpha}{\Gamma(\alpha+1)}$$
$$+ \frac{L^\varepsilon}{\Gamma(\alpha)}\int_a^t \frac{\sup_{\theta\in[s-\sigma,s]} D_0[x(\theta,\omega), x^\varepsilon(\theta,\omega)]}{(t-s)^{1-\alpha}}ds$$

Thus from Gronwall's lemma we get

$$D_0[x(t,\omega), x^\varepsilon(t,\omega)] \overset{[a,b],\,\mathbb{P}.1}{\leq} \left[D_0[\varphi(0,\omega), \varphi^\varepsilon(0,\omega)] + \frac{\varepsilon^*(b-a)^\alpha}{\Gamma(\alpha+1)}\right]\exp\left\{\frac{L^\varepsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\right\}.$$

Consequently, we have

$$D_0[x(t,\omega), x^\varepsilon(t,\omega)] \overset{[a-\sigma,b],\,\mathbb{P}.1}{\leq} \left[D_\sigma[\varphi, \varphi^\varepsilon] + \frac{\varepsilon^*(b-a)^\alpha}{\Gamma(\alpha+1)}\right]\exp\left\{\frac{L^\varepsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\right\}.$$

The proof is complete. □

Remark 3.5. From the estimate (10), it is easy to see that if $D_\sigma[\varphi,\varphi^\varepsilon] \overset{[a-\sigma,b],\mathbb{P}.1}{\rightarrow} 0$ and $D_0[f_\omega(t,x_t), f_\omega^\varepsilon(t,x_t)] \to 0$ for all $t\in[a,b]$ with $\mathbb{P}.1$, then $\sup_{t\in[a-\sigma,b]} D_0[x(t,\omega), x^\varepsilon(t,\omega)] \overset{\mathbb{P}.1}{\rightarrow} 0$.

Fuzzy fractional functional differential equation: Consider the following random fuzzy fractional functional differential equation (RFFFDE) of order $\alpha\in(0,1)$ with the initial condition:

$$\begin{cases} {}^C\mathcal{D}^\alpha_{a^+}x(t,\omega) \overset{[a,b],\,\mathbb{P}.1}{=} f_\omega(t,x_t), \\ x(t,\omega) \overset{[a-\sigma,a],\,\mathbb{P}.1}{=} \varphi(t-a,\omega), \end{cases} \tag{11}$$

where ${}^C\mathcal{D}^\alpha_{a^+}$ is the Caputo's generalized Hukuhara derivative, $f:\Omega\times[a,b]\times C_\sigma\to E$ is a fuzzy stochastic function. A function $x:[a-\sigma,b]\times\Omega\to E$ is said to be a solution of (11) if x is a continuous fuzzy stochastic process, $x(t,\omega) \overset{[a-\sigma,a],\,\mathbb{P}.1}{=} \varphi(t-a,\omega)$ and ${}^C\mathcal{D}^\alpha_{a^+}x(t,\omega) \overset{[a,b],\,\mathbb{P}.1}{=} f_\omega(t,x_t)$, $t\in[a,b]$. A solution x of (11) is said to be $d-$monotone on $[a,b]$ if it is $d-$increasing or $d-$decreasing on $[a,b]$ for $\mathbb{P}-$a.a. ω.

Remark 3.6. Let us suppose that x is a $d-$monotone continuous fuzzy stochastic process on $[a,b]$ and satisfies (1). Since x is $d-$ monotone on $[a,b]$, then by Remark 2.3 it follows that $x(t,\omega)\ominus_{gH}\varphi(0,\omega)$ is $d-$increasing on $[a,b]$ with $P-$a.a. ω. Hence, from (1) it follows that the fuzzy functions $t\to\Im^\alpha_{a^+}f_\omega(t,x_t)$ must be $d-$increasing on $[a,b]$ with $P-$a.a. ω. Therefore, the d-monotone fuzzy stochastic processes satisfying (1) must be sought in the set of all d-monotone continuous fuzzy stochastic processes x for which the function $t\mapsto\Im^\alpha_{a^+}f(t,x_t)$ is d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω .

The following lemma shows the equivalence between a fractional fuzzy differential equation and an fractional fuzzy integral equation.

Lemma 3.7. Let the function f satisfy (A1) and (A2) . Then a $d-$monotone fuzzy stochastic process x is a solution of initial value problem (11), if and only if x satisfies (1) and the function $t \mapsto \mathfrak{I}^{\alpha}_{a^+} f_\omega(t, x_t)$ is d-increasing on $[a, b]$ for $\mathbb{P}-$a.a. ω.

Proof. Let a d-monotone continuous fuzzy stochastic process x be a solution of the initial value problem (11) and let $y(t,\omega) := x(t,\omega) \ominus_{gH} x(a,\omega)$, $(t,\omega) \in [a,b]\times\Omega$, $\mathbb{F}_\omega(t) := f_\omega(t, x_t)$, $t \in [a,b]$. As x is d-monotone on $[a,b]$, then by Remark 2.3 it follows that $t \mapsto y(t,\omega)$ is d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω. From (11) and $\left({}^{C}\mathcal{D}^{\alpha}_{a^+} x\right)(t,\omega) := \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} y\right)(t,\omega) = D_{gH}\left(\mathfrak{I}^{1-\alpha}_{a^+} y\right)(t,\omega)$, $y_{1-\alpha}(a,\omega) := \left(\mathfrak{I}^{1-\alpha}_{a^+} y\right)(a,\omega) = \hat{\mathbf{0}}$, $t \in [a,b]$ for $\mathbb{P}-$a.a. ω, by Lemma 2.5, we get $\frac{d}{dt} diam[\mathfrak{I}^{1-\alpha}_{a^+} y(t,\omega)]^r \geq 0$ for all $t \in [a,b]$ and for $\mathbb{P}-$a.a. ω. Hence, by Proposition 2.7, we have that

$$\left(\mathfrak{I}^{\alpha}_{a^+}\, {}^{C}\mathcal{D}^{\alpha}_{a^+} x\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \left(\mathfrak{I}^{\alpha}_{a^+}\, {}^{RL}\mathcal{D}^{\alpha}_{a^+}[x(\cdot,\omega)\ominus_{gH} x(a,\omega)]\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \left(\mathfrak{I}^{\alpha}_{a^+}\, {}^{RL}\mathcal{D}^{\alpha}_{a^+} y\right)(t,\omega)$$
$$\overset{[a,b],\mathbb{P}.1}{=} y(t,\omega) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} y_{1-\alpha}(a,\omega) \overset{[a,b],\mathbb{P}.1}{=} y(t,\omega).$$

Since $\mathbb{F}_\omega$ satisfies (A2) and from $\left({}^{C}\mathcal{D}^{\alpha}_{a^+} x\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \mathbb{F}_\omega(t)$, it follows that $\left(\mathfrak{I}^{\alpha}_{a^+}\, {}^{C}\mathcal{D}^{\alpha}_{a^+} x\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \left(\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega\right)(t)$, and thus $y(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \left(\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega\right)(t)$. Therefore, we obtain that

$$y(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathbb{F}_\omega(s)ds,$$

that is, x satisfies (1). Since $t \mapsto y(t,\omega) := x(t,\omega) \ominus_{gH} \varphi(0,\omega)$ is d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω, it follows that $t \mapsto (\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega)(t)$ is also d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω. Conversely, suppose that x is a d-monotone continuous fuzzy stochastic process satisfying (1) and such that $t \mapsto (\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega)(t)$ is d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω. Since $\mathbb{F}_\omega$ satisfies (A2), the function $t \mapsto (\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega)(t)$ is continuous on $[a,b]$ with $\mathbb{P}.1$ and $(\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega)(a) \overset{\mathbb{P}.1}{=} \hat{\mathbf{0}}$. Then $y(a,\omega) \overset{\mathbb{P}.1}{=} \hat{\mathbf{0}}$, and thus $x(a,\omega) \overset{\mathbb{P}.1}{=} \varphi(0,\omega)$. Further, since $t \mapsto (\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega)(t)$ is d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω, then applying ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ in (1) we have that $\left({}^{RL}\mathcal{D}^{\alpha}_{a^+} y\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} ({}^{RL}\mathcal{D}^{\alpha}_{a^+}\mathfrak{I}^{\alpha}_{a^+}\mathbb{F}_\omega)(t)$. From Proposition 2.6 it follows that $\left({}^{RL}\mathcal{D}^{\alpha}_{a^+} y\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} \mathbb{F}_\omega(t)$, and thus $\left({}^{C}\mathcal{D}^{\alpha}_{a^+} x\right)(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} f_\omega(t, x_t)$, that is, (11) holds. □

Corollary 3.8. If a $d-$monotone fuzzy stochastic process x is a solution of (1) such that the function $t \mapsto \mathfrak{I}^{\alpha}_{a^+} f_\omega(t, x_t)$ is d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω, then x is a $d-$monotone solution of (11).

Remark 3.9. Observe that:

(i) if a continuous fuzzy stochastic process x is a unique $d-$monotone solution of (1) on $[a,b]$, then the function $y(t,\omega) := x(t,\omega) \ominus_{gH} \varphi(0,\omega)$ is $d-$increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω. Further, the function y may create two solutions of (1): a unique $d-$increasing solution of (1) and a unique $d-$decreasing solution of (1) on $[a,b]$ for $\mathbb{P}-$a.a. ω.

(ii) if a continuous fuzzy stochastic process x is a $d-$monotone solution of (11) on $[a,b]$, then x is a solution of (1) on $[a,b]$, but the converse is not true if the function $t \mapsto \mathfrak{I}^{\alpha}_{a^+} f_\omega(t, x_t)$ is not d-increasing on $[a,b]$ for $\mathbb{P}-$a.a. ω.

4. Numerical illustration

In the sequel, we consider a population of species. Recall the framework of the population growth model in the classical situation where every quantity is precisely described. Under simplified conditions such as a constant environment (and with no migration), it can be shown that the change in population size z through time t (the time horizon is from zero to $b > 0$) will depend on the difference between individual birth rate r and death rate m, and given by:

$$z'(t) = (r - m)z(t), \quad z(0) = z_0, \tag{12}$$

where r, m are the constants which describe an instantaneous birth rate (births per individual per time period t) and an instantaneous death rate (deaths per individual per time period t), respectively. The symbol z_0 denotes the initial number of individuals and z denotes the current population size. For convenience, let us denote $\lambda = r - m$. It is reasonable to take $\lambda < 0$ in the case of a population when decreasing, and $a > 0$ if the number of individuals of the population is increasing. By solving the differential equation (12), we get a formula to estimate a population size at any time $z(t) = z_0 e^{\lambda t}$. The above considerations correspond to the perfect knowledge of the parameters of the system and the precise description of the state of the system at each instant t.

Remark 4.1. In the classical population models (12), it is considered that the birth rate changes immediately as soon as a change in the number of individuals is produced. However, the members of the population must reach a certain degree of development to give birth to new individuals and this suggests an introduction of a delay term into the system.

Remark 4.2. In modelling of real-world problems (for example in the classical population models (12)) only partial information of the system may be known or there may be uncertainty in the parameters used in the model or some measurements may be imprecise. In such situation, random fuzzy differential equation (RFDE), random fuzzy functional differential equation (RFFDE) and stochastic fuzzy differential equations (SFDE) are natural ways to model dynamic systems subject to uncertainties. Several authors have discussed the theory of RFDE, RFFDE and SFDE (see [31]-[33], [41], [35]-[37]).

In the sequel, we present some examples being are simple illustrations of the theory of RFFFDE and we solve them using two types of $d-$monotone. We consider again the following RFFFDE described by

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0^+} x(t,\omega) \overset{[0,b],\mathbb{P}.1}{=} f_\omega(t, x(t-\sigma,\omega)), \\ x(t,\omega) \overset{[-\sigma,0],\mathbb{P}.1}{=} \varphi(t,\omega), \end{cases} \tag{13}$$

where $\sigma > 0, b > 0$ are such that $b = l.\sigma$ for given $l \in \mathbb{N}^*$, $f : \Omega \times [0,b] \times C_\sigma \to E$, $\alpha \in (0,1)$ is the order of the differential equation, $\varphi(t,\omega)$ is the initial value. We observe that Theorem 2.9 gives us a useful procedure to solve the RFFFDE (13). Let us denote the $r-$levels ($r \in [0,1]$) of x and φ as

$$[x(t,\omega)]^r = [\underline{x}(t,\omega,r), \overline{x}(t,\omega,r)], t \in [0,b], \quad [\varphi(t,\omega)]^r = [\underline{\varphi}(t,\omega,r), \overline{\varphi}(t,\omega,r)], t \in [-\sigma,0],$$

respectively. Obviously, $\underline{x}(\cdot,\cdot,r), \overline{x}(\cdot,\cdot,r) : [0,b] \times \Omega \to \mathbb{R}$. By using Zadeh's extension principle, we obtain $[f_\omega(t, x(t-\sigma,\omega))]^r = [\underline{f}_\omega(t,u_1,u_2,r), \overline{f}_\omega(t,u_1,u_1,r)]$, where

$$\underline{f}_\omega(t,u_1,u_2,r) = \underline{f}_\omega(t, \underline{x}(t-\sigma,\omega,r), \overline{x}(t-\sigma,\omega,r), r),$$
$$\overline{f}_\omega(t,u_1,u_2,r) = \overline{f}_\omega(t, \underline{x}(t-\sigma,\omega,r), \overline{x}(t-\sigma,\omega,r), r),$$

for $r \in [0,1]$. In this equation (13) we shall solve it by two types of $d-$monotone. Consequently, based on the types of monotone, we have the following two cases.
Case 1. If x is $d-$increasing for $\mathbb{P}-$a.a. ω, then

$\left[({}^{C}\mathcal{D}^{\alpha}_{0^+}x)(t,\omega)\right]^r = \left[{}^{C}D^{\alpha}_{0^+}\underline{x}(t,\omega,r), {}^{C}D^{\alpha}_{0^+}\overline{x}(t,\omega,r)\right]$ and (13) is translated into the following random fractional functional differential system:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \underline{f}_{\omega}(t,\underline{x}(t-\sigma,\omega,r),\overline{x}(t-\sigma,\omega,r),r), \ t \in [0,b] \\ {}^{C}D^{\alpha}_{0^+}\overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \overline{f}_{\omega}(t,\underline{x}(t-\sigma,\omega,r),\overline{x}(t-\sigma,\omega,r),r), \ t \in [0,b] \\ \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \underline{\varphi}(t,\omega,r), \ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \overline{\varphi}(t,\omega,r), \ t \in [-\sigma,0]. \end{cases} \tag{14}$$

Case 2. If x is $d-$decreasing for $\mathbb{P}-$a.a. ω, then
$\left[({}^{C}\mathcal{D}^{\alpha}_{0^+}x)(t,\omega)\right]^r = \left[{}^{C}D^{\alpha}_{0^+}\overline{x}(t,\omega,r), {}^{C}D^{\alpha}_{0^+}\underline{x}(t,\omega,r)\right]$ and (13) is translated into the following fractional functional differential system:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \underline{f}_{\omega}(t,\underline{x}(t-\sigma,\omega,r),\overline{x}(t-\sigma,\omega,r),r), \ t \in [0,b] \\ {}^{C}D^{\alpha}_{0^+}\underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \overline{f}_{\omega}(t,\underline{x}(t-\sigma,\omega,r),\overline{x}(t-\sigma,\omega,r),r), \ t \in [0,b] \\ \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \underline{\varphi}(t,\omega,r), \ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \overline{\varphi}(t,\omega,r), \ t \in [-\sigma,0]. \end{cases} \tag{15}$$

Now, we consider an example for the population model with two kinds of uncertainties (i.e., fuzziness and randomness simultaneously) in the following form of random fuzzy fractional functional differential equations.

Example 4.3. Let $\Omega = (0,1), \mathcal{F}-$Borel $\delta-$field of subsets of $\Omega, \mathbb{P}-$Lebesgue measure on $(\Omega,\mathcal{F})$. Let us consider the random fuzzy fractional functional differential equation as follows:

$$\begin{cases} ({}^{C}\mathcal{D}^{\alpha}_{0^+}x)(t,\omega) \overset{\mathbb{P}.1}{=} \lambda x(t-1,\omega), \quad 1 \geq t \geqslant 0 \\ x(t,\omega) \overset{\mathbb{P}.1}{=} \omega(1-t, 2-t, 3-t), \quad t \in [-1,0] \end{cases} \tag{16}$$

and its associated integral equation

$$\begin{cases} x(t,\omega) \overset{\mathbb{P}.1}{=} \omega(1-t, 2-t, 3-t), \quad t \in [-1,0] \\ x(t,\omega) \ominus_{gH} x(0,\omega) \overset{\mathbb{P}.1}{=} \dfrac{\lambda}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} x(s-1,\omega)ds, \quad \text{for } t \in [0,1], \end{cases} \tag{17}$$

where $x \in B_3(x_0) = \{x \in C_\sigma : D_\sigma[x,x_0] \leq 3\}$, $\lambda \in [-1,1]\backslash\{0\}$.

For problem (17), we check the validity of the hypotheses in Theorem 3.1. Indeed, it can be checked that $f : \Omega \times [0,1] \times B_3(x_0) \to E$ in problem (17) is a continuous mapping and satisfies the conditions of Theorem 3.1. In particular

- for $(t,\xi) \in [0,1] \times B_3(x_0)$

$$D_0[f_\omega(t,\xi),\hat{\mathbf{0}}] = |\lambda| D_\sigma[\xi,\hat{\mathbf{0}}] \overset{[0,1],\mathbb{P}.1}{\leq} D_\sigma[\xi,\hat{\mathbf{0}}] \overset{[0,1],\mathbb{P}.1}{\leq} 6,$$

- for $(t,\omega) \in [0,1] \times \Omega, \xi, \psi \in B_3(x_0)$

$$D_0[f_\omega(t,\xi), f_\omega(t,\psi)] \overset{[0,1],\mathbb{P}.1}{\leq} |\lambda| D_\sigma[\xi,\psi].$$

Hence we can easilyy show that f satisfies the assumptions of Theorem 3.1. Based on the types of monotone, we solve problem (17) in two cases.
Case 1. Consider $\lambda \in (0,1]$ and x is $d-$increasing for $\mathbb{P}-$a.a. ω. For $t \in [0,1]$ the sequence of successive approximations

$$x^0(t,\omega) \overset{[0,1],\mathbb{P}.1}{:=} (\omega, 2\omega, 3\omega) \text{ and } x^n(t,\omega) \overset{[0,1],\mathbb{P}.1}{:=} x^0(t,\omega) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^{n-1}(s-1,\omega) ds$$

is well defined. Also, by recursion we obtain that $x^n(t,\omega) \in B_3(x_0)$ for $t \in [0,1]$ with $\mathbb{P}.1$. Therefore, by Theorem 3.1, there exists a unique $d-$increasing solution to the problem (17). Using (14) and after some manipulations, we obtain systems of random fractional functional integral equations

$$\begin{cases} \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \lambda I^\alpha_{0^+} \underline{x}(t-1,\omega,r) + \underline{\varphi}(0,\omega,r), \quad 1 \geq t \geqslant 0, \\ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \lambda I^\alpha_{0^+} \overline{x}(t-1,r) + \overline{\varphi}(0,\omega,r), \quad 1 \geq t \geqslant 0, \\ \underline{x}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \underline{\varphi}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \omega(1+r-t), \\ \overline{x}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \overline{\varphi}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \omega(3-r-t). \end{cases} \tag{18}$$

By solving (18), we obtain the exact solution as follows:

$$[x(t,\omega)]^r = \left[\omega(1+r) + \frac{\lambda\omega(2+r)t^\alpha}{\Gamma(\alpha+1)} - \frac{\lambda t^{\alpha+1}}{\Gamma(\alpha+2)}, \omega(3-r) + \frac{\lambda\omega(4-r)t^\alpha}{\Gamma(\alpha+1)} - \frac{\lambda t^{\alpha+1}}{\Gamma(\alpha+2)}\right],$$

where $t \in [0,1]$. This solution is shown in Figure 1.
Case 2. Consider $\lambda \in [-1,0)$ and x is $d-$decreasing for $\mathbb{P}-$a.a. ω. For $t \in [0,\mathbb{T}]$ the the sequence of successive approximations

$$x^0(t,\omega) \overset{[0,1],\mathbb{P}.1}{:=} (\omega, 2\omega, 3\omega) \text{ and } x^n(t,\omega) \overset{[0,1],\mathbb{P}.1}{:=} x^0(t,\omega) \ominus \frac{(-\lambda)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^{n-1}(s-1,\omega) ds$$

is well defined. Indeed, firstly observe that for $(t,\xi) \in [0,1] \times S(x^0,3)$ we have

$$diam[f_\omega(t,\xi)]^r \overset{[0,1],\mathbb{P}.1}{=} |\lambda| diam[\xi]^r \overset{[0,1],\mathbb{P}.1}{=} 8|\lambda|.$$

Note that

$$diam[\varphi(0)]^r \overset{[0,1],\mathbb{P}.1}{=} 2\omega(1-r) \geq \frac{1}{\Gamma(\alpha)} \int_0^t |\lambda|(t-s)^{\alpha-1} diam[\xi_s]^r ds = \frac{|\lambda|}{\Gamma(\alpha+1)} 8t^\alpha,$$

which implies that the Hukuhara differences

$$x^0(t,\omega) \ominus \frac{(-\lambda)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x^{n-1}(s-1,\omega) ds$$

exist in the case $t \in [0,\mathbb{T}]$ for $\mathbb{P}-$a.a. ω, where $\mathbb{T} \overset{\mathbb{P}.1}{=} \min\left\{1, \left[\frac{2\omega(1-r)\Gamma(\alpha+1)}{8|\lambda|}\right]^{\frac{1}{\alpha}}\right\}$. Also, by recursion we obtain that $x^n(t,\omega) \in B_3(x_0)$ for $t \in [0,\mathbb{T}]$ with $\mathbb{P}.1$. Thus, by Theorem 3.1, there exists a unique $d-$decreasing solution (defined on interval $[0, \mathbb{T}]$) to the problem (17). Using (15) and after some manipulations, we obtain systems of random fractional functional integral equations

$$\begin{cases} \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \lambda I_{0^+}^{\alpha} \underline{x}(t-1,\omega,r) + \underline{\varphi}(0,\omega), & \mathbb{T} \geq t \geqslant 0, \\ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \lambda I_{0^+}^{\alpha} \overline{x}(t-1,\omega,r) + \overline{\varphi}(0,\omega), & \mathbb{T} \geq t \geqslant 0, \\ \underline{x}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \underline{\varphi}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \omega(1+r-t), \\ \overline{x}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \overline{\varphi}(t,\omega,r) \overset{[-1,0],\mathbb{P}.1}{=} \omega(3-r-t). \end{cases} \tag{19}$$

By solving (19), we obtain the exact solution as follows:

$$[x(t,\omega)]^r = \left[\omega(1+r) + \frac{\lambda\omega(2+r)t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\lambda t^{\alpha+1}}{\Gamma(\alpha+2)}, \omega(3-r) + \frac{\lambda\omega(4-r)t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\lambda t^{\alpha+1}}{\Gamma(\alpha+2)}\right],$$

where $t \in [0, \mathbb{T}]$, $\lambda \in [-1, 0)$. The graph of the solution is drawn in Figure 2.

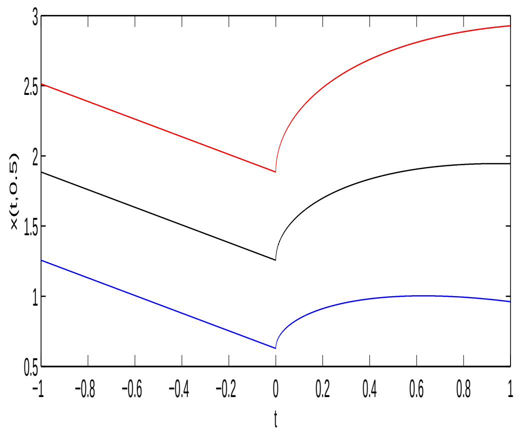

Figure 1: Solution of Example 4.3, Case 1. ($\lambda = 0.5, \alpha = 0.5, \omega = 0.5$)

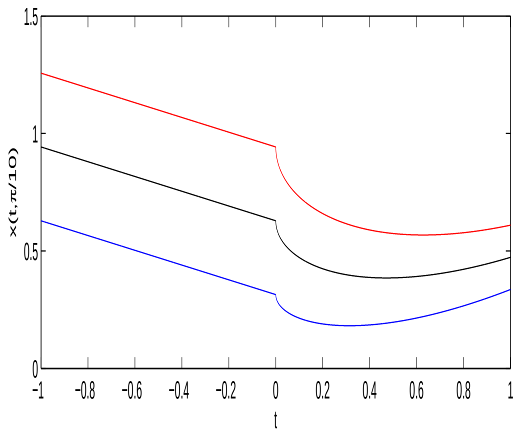

Figure 2: Solution of Example 4.3, Case 2. ($\lambda = -0.5, \alpha = 0.5, \omega = \pi/10$)

Remark 4.4. In Example 4.3, we solved explicitly and obtained exact solutions of the random fuzzy fractional functional differential equation. However, in general it is not easy to derive the analytical solutions to most RFFFDEs. Therefore, it is vital to develop some reliable and efficient techniques to solve random fuzzy functional fractional differential equations. Then it will be possible to simulate trajectory of solutions. Mazandarani and Kamyad [29] proposed the modified fractional Euler method (MFEM) for solving fuzzy fractional initial value problem under Caputo type fuzzy fractional derivatives of order $\alpha \in (0,1)$. In [14], the author proposed the modified Adams-Bashforth-Moulton method (MABMM) for solving fuzzy delay differential equations of fractional order with Caputo-type fuzzy fractional derivative. We notice that with the help of MFEM and MABMM one can approximate the solutions of RFFFDEs.

In the sequel the modified fractional Euler method for solving random fuzzy functional differential equation of fractional order under the Caputo-type fuzzy fractional derivative will be investigated. The MFEM based on a generalized Taylor's formula [39] and a modified trapezoidal rule [40] is used for solving random fuzzy functional fractional differential equation of order $\alpha \in (0,1)$. We now give a generalization of the trapezoidal rule to approximation the fractional integral $I^{\alpha}_{0^+}g(t)$ of order $\alpha > 0$.

Theorem 4.5. [40] Suppose that the interval $[0,b]$ is subdivided into N subintervals $[t_j, t_{j+1}]$ of equal width $h = \dfrac{b}{N}$ by using the nodes $t_j = jh$, for $j = 0, 1, ..., N$. The modified trapezoidal rule

$$\begin{aligned} T(g,h,\alpha) &= \Big((N-1)^{\alpha+1} - (N-\alpha-1)N^{\alpha}\Big)\frac{h^{\alpha}g(0)}{\Gamma(\alpha+2)} + \frac{h^{\alpha}g(b)}{\Gamma(\alpha+2)} \\ &\quad + \sum_{j=1}^{N-1}\Big((N-j+1)^{\alpha+1} - 2(N-j)^{\alpha+1} + (N-j-1)^{\alpha+1}\Big)\frac{h^{\alpha}g\left(t_j\right)}{\Gamma\left(\alpha+2\right)} \end{aligned}$$

is an approximation to the fractional integral

$$I^{\alpha}_{0^+}g(b) = T(g,h,\alpha) - O(h^2).$$

$t\backslash r$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
0.1	1.166	1.280	1.394	1.508	1.621	1.735	1.849	1.963	2.077	2.190	2.304
0.2	1.261	1.384	1.508	1.631	1.754	1.877	2.000	2.123	2.247	2.370	2.493
0.3	1.346	1.478	1.609	1.741	1.872	2.004	2.136	2.267	2.399	2.530	2.662
0.4	1.424	1.563	1.702	1.841	1.981	2.120	2.259	2.398	2.538	2.677	2.816
0.5	1.495	1.641	1.788	1.934	2.080	2.226	2.373	2.519	2.665	2.811	2.958
0.6	1.561	1.714	1.867	2.019	2.172	2.325	2.478	2.630	2.783	2.936	3.089
0.7	1.623	1.781	1.940	2.099	2.258	2.416	2.575	2.734	2.893	3.051	3.210
0.8	1.680	1.844	2.008	2.172	2.337	2.501	2.665	2.830	2.994	3.158	3.322
0.9	1.733	1.902	2.071	2.241	2.410	2.580	2.749	2.918	3.088	3.257	3.426
1	1.782	1.956	2.130	2.304	2.479	2.653	2.827	3.001	3.175	3.349	3.523

Table 1: The approximation solution to (25) in Case 1 ($\omega = 0.5, \alpha = 0.75$) - $\underline{x}(t,\omega,r)$

$t\backslash r$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	3.000	2.900	2.800	2.700	2.600	2.500	2.400	2.300	2.200	2.100	2.000
0.1	3.499	3.385	3.271	3.157	3.043	2.929	2.816	2.702	2.588	2.474	2.304
0.2	3.783	3.660	3.537	3.413	3.290	3.167	3.044	2.921	2.797	2.674	2.493
0.3	4.038	3.906	3.775	3.643	3.511	3.380	3.248	3.117	2.985	2.853	2.662
0.4	4.271	4.131	3.992	3.852	3.713	3.574	3.435	3.295	3.156	3.017	2.816
0.5	4.485	4.338	4.192	4.046	3.899	3.753	3.607	3.461	3.314	3.168	2.958
0.6	4.683	4.530	4.378	4.225	4.072	3.919	3.766	3.614	3.461	3.308	3.089
0.7	4.868	4.709	4.550	4.391	4.232	4.074	3.915	3.756	3.597	3.439	3.210
0.8	5.038	4.874	4.710	4.546	4.381	4.217	4.053	3.888	3.724	3.560	3.322
0.9	5.198	5.028	4.859	4.689	4.520	4.351	4.181	4.012	3.843	3.673	3.426
1	5.346	5.172	4.998	4.824	4.650	4.476	4.302	4.128	3.954	3.780	3.523

Table 2: The approximation solution to (25) in Case 1 ($\omega = 0.5, \alpha = 0.75$) - $\overline{x}(t, \omega, r)$

After some manipulations, the initial value problems (14) and (15) can be equivalent to the following random fractional integral equations

$$\begin{cases} \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} I^{\alpha}_{0^+} \underline{f}_{\omega}(t, \underline{x}(t-\sigma,\omega,r), \overline{x}(t-\sigma,\omega,r), r) + \underline{\varphi}(0,\omega,r), \;\; t \in [0,b] \\ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} I^{\alpha}_{0^+} \overline{f}_{\omega}(t, \underline{x}(t-\sigma,\omega,r), \overline{x}(t-\sigma,\omega,r), r) + \overline{\varphi}(0,\omega,r), \;\; t \in [0,b] \\ \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \underline{\varphi}(t,\omega,r), \;\; \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \overline{\varphi}(t,\omega,r), \;\; t \in [-\sigma, 0]. \end{cases} \tag{20}$$

for Case 1, and

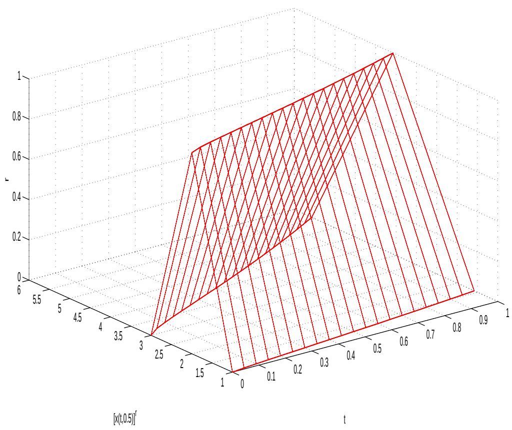

Figure 3: The approximation (i)-solution of Example 4.6 ($\alpha = 0.75, \omega = 0.5$)

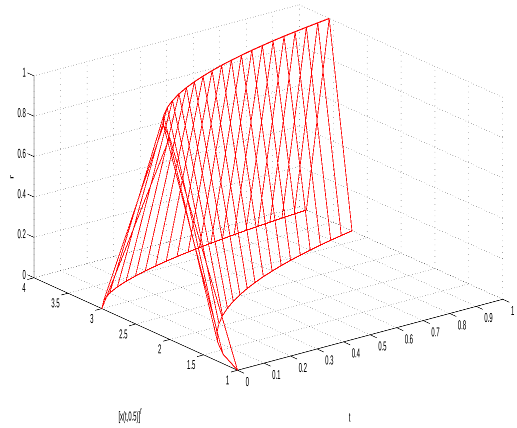

Figure 4: The approximation (ii)-solution of Example 4.6 ($\alpha = 0.75, \omega = 0.5$)

$$\begin{cases} \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} I^{\alpha}_{0^+}\overline{f}_{\omega}(t,\underline{x}(t-\sigma,\omega,r),\overline{x}(t-\sigma,\omega,r),r)+\underline{\varphi}(0,\omega,r), \ t\in[0,b] \\ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} I^{\alpha}_{0^+}\underline{f}_{\omega}(t,\underline{x}(t-\sigma,\omega,r),\overline{x}(t-\sigma,\omega,r),r)+\overline{\varphi}(0,\omega,r), \ t\in[0,b] \\ \underline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \underline{\varphi}(t,\omega,r), \ \overline{x}(t,\omega,r) \overset{\mathbb{P}.1}{=} \overline{\varphi}(t,\omega,r), \ t\in[-\sigma,0]. \end{cases} \tag{21}$$

for Case 2.

Consider a uniform grid $\{t_n = nh : n = -k, -k+1, ..., -1, 0, 1, ..., N\}$ where k and N are integers such that $h = T/N$ and $h = \sigma/k$. Now the key problem is to establish an approximation to the delayed terms $\underline{x}(t-\sigma,\omega,r)$ and $\overline{x}(t-\sigma,\omega,r)$ with any fixed $\omega \in \Omega$, respectively. Since the establishment is similar for the two terms, we only establish it for the case of the delayed term $\underline{x}(t-\sigma,\omega,r)$. Let $\omega\in\Omega$ be fixed. Let $\underline{x}(t_j,\omega,r) = \varphi(t_j,\omega,r)$, $j = -k, -k+1, ..., -1, 0$. Therefore, $\underline{x}(t-\sigma,\omega,r)$ can be approximated by

$$\underline{x}(t_j-\sigma,\omega,r) \approx \begin{cases} \underline{x}_{j-k}(\omega,r), & \text{if} \quad j = 0,1,...,N, \\ \underline{\varphi}_j(\omega,r) & \text{if} \quad j = -k,-k+1,...,-1,0. \end{cases} \tag{22}$$

Thus, we have the following relations:

$$\begin{aligned} &\text{for} \quad j = 1,....,N \quad \underline{x}(t_j-\sigma,\omega,r) \to \underline{x}_{j-k}(\omega,r) \\ &\text{for} \quad j = -k,-k+1,...,0 \quad \underline{x}(t_j,\omega,r) \to \underline{\varphi}_j(\omega,r). \end{aligned}$$

$t\backslash r$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
0.1	1.499	1.585	1.671	1.758	1.844	1.930	2.016	2.102	2.189	2.275	2.361
0.2	1.783	1.860	1.937	2.014	2.091	2.167	2.244	2.321	2.398	2.475	2.551
0.3	2.038	2.107	2.175	2.243	2.312	2.380	2.449	2.517	2.585	2.654	2.722
0.4	2.271	2.331	2.392	2.453	2.514	2.574	2.635	2.696	2.757	2.817	2.878
0.5	2.485	2.539	2.592	2.646	2.700	2.754	2.807	2.861	2.915	2.969	3.022
0.6	2.684	2.731	2.778	2.825	2.872	2.920	2.967	3.014	3.061	3.109	3.156
0.7	2.868	2.909	2.950	2.991	3.033	3.074	3.115	3.156	3.198	3.239	3.280
0.8	3.039	3.075	3.110	3.146	3.182	3.217	3.253	3.289	3.325	3.360	3.396
0.9	3.198	3.229	3.259	3.290	3.321	3.351	3.382	3.412	3.443	3.474	3.504
1	3.347	3.373	3.398	3.424	3.450	3.476	3.502	3.528	3.554	3.508	3.606

Table 3: The approximation solution to (25) in Case 2 ($\omega = 0.5, \alpha = 0.75$) - $\underline{x}(t, \omega, r)$

$t\backslash r$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	3.000	2.900	2.800	2.700	2.600	2.500	2.400	2.300	2.200	2.100	2.000
0.1	3.166	3.080	2.993	2.907	2.821	2.735	2.649	2.562	2.476	2.390	2.361
0.2	3.261	3.184	3.107	3.030	2.953	2.877	2.800	2.723	2.646	2.569	2.551
0.3	3.346	3.277	3.209	3.140	3.072	3.004	2.935	2.867	2.798	2.730	2.722
0.4	3.423	3.362	3.302	3.241	3.180	3.119	3.059	2.998	2.937	2.876	2.878
0.5	3.495	3.441	3.387	3.333	3.280	3.226	3.172	3.118	3.065	3.011	3.022
0.6	3.561	3.513	3.466	3.419	3.372	3.325	3.277	3.230	3.183	3.136	3.156
0.7	3.622	3.581	3.540	3.498	3.457	3.416	3.375	3.333	3.292	3.251	3.280
0.8	3.679	3.643	3.608	3.572	3.536	3.501	3.465	3.429	3.393	3.358	3.396
0.9	3.732	3.702	3.671	3.640	3.610	3.579	3.548	3.518	3.487	3.457	3.504
1.0	3.782	3.756	3.730	3.704	3.678	3.652	3.626	3.600	3.574	3.548	3.606

Table 4: The approximation solution to (25) in Case 2 ($\omega = 0.5, \alpha = 0.75$) - $\overline{x}(t, \omega, r)$

Using the modified trapezoidal rule in Theorem 4.5, the numerical scheme for (20), (21) can be depicted as:

$$\begin{cases} \underline{x}_n(\omega, r) \overset{\mathbb{P}.1}{=} \underline{\varphi}(0, \omega, r) + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \underline{f}_\omega(t_n, r, \underline{x}_{n-k}(\omega, r), \overline{x}_{n-k}(\omega, r)) \\ \qquad\qquad + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \displaystyle\sum_{j=0}^{n-1} a_{j,n} \underline{f}_\omega(t_j, r, \underline{x}_{j-k}(\omega, r), \overline{x}_{j-k}(\omega, r)) \\ \overline{x}_n(\omega, r) \overset{\mathbb{P}.1}{=} \overline{\varphi}(0, \omega, r) + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \overline{f}_\omega(t_n, r, \underline{x}_{n-k}(\omega, r), \overline{x}_{n-k}(\omega, r)) \\ \qquad\qquad + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \displaystyle\sum_{j=0}^{n-1} a_{j,n} \overline{f}_\omega(t_j, r, \underline{x}_{j-k}(\omega, r), \overline{x}_{j-k}(\omega, r)) \\ a_{j,n} = \begin{cases} (n-1)^{\alpha+1} - (n-\alpha-1)n^\alpha, & j = 0 \\ (n-j+1)^{\alpha+1} + (n-j-1)^{\alpha+1} - 2(n-j)^{\alpha+1}, & j \in [1, n-1] \end{cases} \end{cases} \tag{23}$$

for Case 1, and

$$\begin{cases} \underline{x}_n(\omega, r) \overset{\mathbb{P}.1}{=} \underline{\varphi}(0, \omega, r) + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \overline{f}_\omega(t_n, r, \underline{x}_{n-k}(\omega, r), \overline{x}_{n-k}(\omega, r)) \\ \qquad + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n-1} a_{j,n} \overline{f}_\omega(t_j, r, \underline{x}_{j-k}(\omega, r), \overline{x}_{j-k}(\omega, r)) \\ \overline{x}_n(\omega, r) \overset{\mathbb{P}.1}{=} \overline{\varphi}(0, \omega, r) + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \underline{f}_\omega(t_n, r, \underline{x}_{n-k}(\omega, r), \overline{x}_{n-k}(\omega, r)) \\ \qquad + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n-1} a_{j,n} \underline{f}_\omega(t_j, r, \underline{x}_{j-k}(\omega, r), \overline{x}_{j-k}(\omega, r)) \\ a_{j,n} = \begin{cases} (n-1)^{\alpha+1} - (n-\alpha-1)n^\alpha, & j = 0 \\ (n-j+1)^{\alpha+1} + (n-j-1)^{\alpha+1} - 2(n-j)^{\alpha+1}, & j \in [1, n-1] \end{cases} \end{cases} \tag{24}$$

for Case 2.

Example 4.6. Let $\Omega = (0,1), \mathcal{F}-$Borel $\delta-$field of subsets of Ω, $\mathbb{P}-$Lebesgue measure on $(\Omega, \mathcal{F})$. Consider the following random fuzzy fractional functional initial value problem

$$\begin{cases} ({}^{C}_{gH}\mathcal{D}^\alpha_{0^+} x)(t, \omega) \overset{\mathbb{P}.1}{=} \dfrac{\omega(4-t)}{4} x(t-1, \omega) + \omega\Big(\cos(t), 2\cos(t), 3\cos(t)\Big), \quad t \in [0,1], \\ x(t, \omega) = \varphi(t, \omega) \overset{\mathbb{P}.1}{=} (\dfrac{2\omega}{1-t}, \dfrac{4\omega}{1-t}, \dfrac{6\omega}{1-t}), \quad t \in [-1, 0]. \end{cases} \tag{25}$$

Since the exact solution cannot be found analytically, we use the numerical method proposed in this study. **Case 1.** Consider x is $d-$increasing for $\mathbb{P}-$a.a. ω. Using the modified fractional Euler method (23), the approximation solution $[x(t, \omega)]^r = [\underline{x}(t, \omega, r), \overline{x}(t, \omega, r)]$, $t \in [0,1]$ to (25) is shown in Figure 3 and Tables 1, 2. **Case 2.** Consider x is $d-$decreasing for $\mathbb{P}-$a.a. ω. Using the modified fractional Euler method (24), the approximation solution $[x(t, \omega)]^r = [\underline{x}(t, \omega, r), \overline{x}(t, \omega, r)]$, $t \in [0,1]$ to (25) is shown in Figure 4 and Tables 3,4.

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