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Different Behaviors of Rough Weighted Statistical Limit Set Under Unbounded Moduli

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Abstract. In this paper we introduce f-rough weighted statistical limit set and f-weighted statistical cluster points set which are natural generalizations of rough statistical limit set and f-statistical cluster points set of sequence respectively. Some new examples are constructed to ensure the deviation of basic results. So both the sets don't follow the nature of usual extension properties which will be discussed here.

1. Introduction

The concept of statistical convergence was introduced by Fast [7] and Steinhaus [25]. Later on reintroduced by Schonenberg [24] independently as follows: let K be a subset of the set of all natural numbers $\mathbb{N}$ and let us denote the set $\{k \leq n : k \in K\}$ by $K(n)$. Then the asymptotic density of K is given by $d(K) = \lim\limits_{n\to\infty} \frac{|K(n)|}{n}$, where $|K(n)|$ denotes the cardinality of $K(n)$. Clearly we know that $d(\mathbb{N} \setminus K) = 1 - d(K)$. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number c if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - c| \geq \varepsilon\}| = 0.$$

Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [8] and Šalát [23].

In the year 2014, the concept of f-statistical convergence was introduced by Aizpuru et al. [1, Definition 2] just by replacing $|\{k \leq n : |x_k - c| \geq \varepsilon\}|$ and $\frac{1}{n}$ by $f(|\{k \leq n : |x_k - c| \geq \varepsilon\}|)$ and $\frac{1}{f(n)}$ respectively, where f is an unbounded modulus function. The notion of a modulus function was introduced by Nakano [17]. We recall that a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ iff $x = 0$, (ii) $f(x + y) \leq f(x) + f(y)\ \forall\ x, y \geq 0$, (iii) f is increasing and f is continuous from the right at 0. The f-density of $K \subseteq \mathbb{N}$ is denoted and defined by $d_f(K) = \lim\limits_{n\to\infty} \frac{f(|K(n)|)}{f(n)}$. A sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is

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said to be f-statistically bounded [15, Definition 3.5] if there exists a positive real number M such that $d_f(\{k \in \mathbb{N} : |x_k| \geq M\}) = 0$.

In case of f-density, the relation $d_f(\mathbb{N} \setminus K) = 1 - d_f(K)$, exist only $d_f(K) = 0$, where f is any unbounded modulus function. In other all cases the relation can't be hold.

Example 1.1 [1, Example 2.1]. If $f(x) = \log(1 + x)$ and $K = \{2n : n \in \mathbb{N}\}$ then $d_f(K) = d_f(\mathbb{N} \setminus K) = 1$. This shows that asymptotic density and f-density are totally different.

The concept of weighted statistical convergence was initially introduced by Karakaya and Chishti [13] in the year 2009 and gradually developed by Mursaleen et al. [16], Ghosal [10] and Das et al. [6] by taking the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers such that $\liminf\limits_{n\to\infty} t_n > 0$ (this was also done to some extent in [9, 11]). However an important question remains unanswered: If the weighted sequence is properly divergent to $+\infty$ then $\liminf\limits_{n\to\infty} t_n$ does not exist. The definition of weighted statistical convergence can't consider this case. Also the definition of weighted statistical convergence was independently introduce by Braha [3], is not well in the sense that each bounded real sequence is weighted statistically convergent to any real number. So some problems are still there, therefore it will be modified in this paper and the question of uniqueness of limit value is proved. We primarily show that under some general assumptions of weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ we can generalize the definition of weighted statistical convergence by using the definition of f-density.

On the other hand, idea of rough convergence is a generalization of the ordinary convergence, which was first defined by Lim [14] in the year 1974 and most recently in 2001 this subject developed by Phu [20]. In Phu's papers [21] and [22] related to the subject, he defined the rough continuity of linear operators and rough convergence in infinite dimensional spaces respectively. In particular an interesting generalization of rough convergence was introduced by Aytar [2] as follows: a sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be rough statistically convergent to a real number x_* w.r.t the roughness of degree r (where $r \geq 0$), denoted by $x_n \xrightarrow[r]{st} x_*$, provided for any $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - x_*| \geq r + \varepsilon\}$ has asymptotic density zero. Here r is called the roughness degree. The set $st - LIM^r x = \{x_* \in \mathbb{R} : x_n \xrightarrow[r]{st} x_*\}$ is called the r-statistical limit set of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be r-statistically convergent if $st - LIM^r x \neq \emptyset$.

Theorem 1.2 [2, Theorem 2.2]. For a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ we have $\operatorname{diam}(st - LIM^r x) \leq 2r$. In general, $\operatorname{diam}(st - LIM^r x)$ has no smaller bound.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be statistically bounded [26, Definition 1] if there exists a positive real number G such that the set $\{n \in \mathbb{N} : |x_n| \geq G\}$ has the natural density zero.

Theorem 1.3 [2, Theorem 2.4]. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is statistically bounded if there exists a non-negative real number r such that $st - LIM^r x \neq \emptyset$.

A real number c is said to be a statistical cluster point [26, Definition 2] of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ provided that the asymptotic density of the set $\{k \in \mathbb{N} : |x_k - c| < \varepsilon\}$ is different from zero for every $\varepsilon > 0$. The set of all statistical cluster points of the sequence x is denoted by Γ_x. More result on this convergence can be found in [4, 12, 19].

Theorem 1.4 [2, Lemma 2.9]. For an arbitrary $c \in \Gamma_x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ we have $|x_* - c| \leq r$, $\forall\, x_* \in st - LIM^r x$.

Theorem 1.5 [5, Corollary 3(3)]. Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers then the statistical cluster points set Γ_x of $x = \{x_n\}_{n\in\mathbb{N}}$ is a compact set of $\mathbb{R}$.

Theorem 1.6 [18, Corollary 1]. If $x = \{x_n\}_{n\in\mathbb{N}}$ is a statistically bounded sequence in $\mathbb{R}^m$ (m-dimensional space) then the set Γ_x is non-empty and compact.

A real number c is said to be a f-statistical cluster point [15, Definition 3.1] of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ provided that the f-density of the set $\{k \in \mathbb{N} : |x_k - c| < \varepsilon\}$ is different from zero for every $\varepsilon > 0$. The set of all f-statistical cluster points of the sequence x is denoted by Γ_x^f.

Theorem 1.7 [15, Corollary 3.9]. For any sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in any normed linear space, the set Γ_x^f is closed.

In a natural way, in this paper we combine the approaches of f-statistical convergence, weighted statistical convergence and rough statistical convergence and introduce new and more general summability methods, namely, f-rough weighted statistical convergence, f-weighted statistically bounded and f-weighted statistical cluster point. On a continuation we also define f-rough weighted statistical limit set $W^t st - LIM_f^r x$ and and f-weighted statistical cluster points set $W^t\Gamma_x^f$. We mainly investigate whether the above mentioned Theorems 1.2 to 1.7 are satisfied for f-rough weighted statistical convergence or not? The answer is no. Some new examples are constructed to ensure the deviation of basic Theorems 1.2 to 1.7. So both the sets don't follow the nature of usual extension properties which will be discussed here.

2. Main Results

Recently, Braha [3, Definition 1.1] have defined the concept of weighted statistical convergence as follows: Let $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ be two non-negative real sequences such that

$$P_n = \sum_{k=1}^{n} p_k\ ,\ P_{-1} = p_{-1} = 0 \text{ and } Q_n = \sum_{k=1}^{n} q_k\ ,\ Q_{-1} = q_{-1} = 0\ \ \forall\ \ n \in \mathbb{N}.$$

Convolution of the above sequences we will denote by $R_n = (p * q)_n = \sum_{k=0}^{n} p_k.q_{n-k}$. A sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent to a real number c if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{R_n}\ |\{k \le R_n : p_{n-k}q_k|x_k - c| \ge \varepsilon\}| = 0.$$

In this case we write $S_{N_{p,q}} - \lim_{n\to\infty} x_n = c$.

But the above definition is not well defined in general. This follows from the following remark.

Remark 2.1. Let us consider that $p_n = 1$, $q_n = \frac{1}{n}\ \forall\ n \in \mathbb{N}$ and $x = \{x_n\}_{n\in\mathbb{N}}$ be any bounded sequence of real numbers. It is quite clear that $S_{N_{p,q}} - \lim_{n\to\infty} x_n = c$ where c is any real number, i.e., any bounded sequence of real numbers is weighted statistically convergent to any real number. Hence the definition is not well defined. So the definition of weighted statistical convergence need to be modified.

Again in the year (2014) Ghosal [10, Definition 2.1 and Remark 2.1(i)] (for $\alpha_n = 1, \beta_n = n\ \forall\ n \in \mathbb{N}$ and $\gamma = 1$) weighted $\alpha\beta$-statistical convergence of order γ coincides with the concept of weighted statistical convergence as follows: Let $t = \{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = \sum_{k=1}^{n} t_k\ \forall\ n \in \mathbb{N}$. Then the sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent to c if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n} |\{k \le T_n : t_k|x_k - c| \ge \varepsilon\}| = 0.$$

Remark 2.2. If the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is properly divergent to $+\infty$ (for example $t_n = n \ \forall\ n \in \mathbb{N}$) then $\liminf_{n\to\infty} t_n$ can't exists. So the definition of weighted $\alpha\beta$-statistical convergence can't consider the case when weighted sequence is properly divergent to $+\infty$. For this reason we can generalize the definition of weighted statistical convergence by using the unbounded modulus function.

Definition 2.3. Let f be an unbounded modulus function and $t = \{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $t_n > \delta \ \forall\ n \in \mathbb{N}$ (where δ is a positive real number) and $T_n = \sum_{k=1}^{n} t_k \ \forall\ n \in \mathbb{N}$. Then the sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be f-weighted statistically convergent to c if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - c| \ge \varepsilon\}|) = 0.$$

In this case we write $x_n \xrightarrow{W^t st^f} c$.

It can be very easy to proved that $x_n \xrightarrow{W^t st^f} c$ and $x_n \xrightarrow{W^t st^f} d$ then $c = d$.

Throughout the paper we assume that $t = \{t_n\}_{n\in\mathbb{N}}$ is a weighted sequence of real numbers such that $t_n > \delta \ \forall\ n \in \mathbb{N}$ (where δ is a positive real number), $T_n = \sum_{k=1}^{n} t_k \ \forall\ n \in \mathbb{N}$ and f be an unbounded modulus function.

Now we introduce the definition of f-rough weighted statistical convergence as follows:

Definition 2.4. Let r be a non-negative real number and $t = \{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. Then the sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be f-rough weighted statistically convergent to x_* w.r.t the roughness of degree r if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - x_*| \ge r + \varepsilon\}|) = 0$$

and we write $x_n \xrightarrow[r]{W^t st^f} x_*$. The set $W^t st - LIM^r_f x = \{x_* \in \mathbb{R} : x_n \xrightarrow[r]{W^t st^f} x_*\}$ is called the f-rough weighted statistical limit set of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ with degree of roughness r. The sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be f-rough weighted statistically convergent with degree of roughness r provided that $W^t st - LIM^r_f x \neq \emptyset$.

Theorem 2.5. The set $W^t st - LIM^r_f x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is convex.

Proof. Proof is similar to the Theorem 2.7 [2], so omitted. □

In general, for a weighted sequence $\{t_n\}_{n\in\mathbb{N}}$, the f-rough weighted statistical limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$ may not be unique, infact, it may be infinite for some roughness of degree $r > 0$. We will show this in our next example.

Example 2.6. Let us define the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ and the weighted sequence $t = \{t_n\}_{n\in\mathbb{N}}$ in the following manner.

$$x_n = \begin{cases} (-1)^n, & \text{if} \quad n \neq m^2 \ \forall\ m \in \mathbb{N}, \\ n^2, & \text{otherwise}, \end{cases}$$

and

$$t_n = \begin{cases} 1 + 1/n, & \text{if } n \neq m^2 \ \forall\ m \in \mathbb{N}, \\ n, & \text{otherwise.} \end{cases}$$

Let $r \geq 4$ and $f(x) = \sqrt{x}$. Then for any $0 < \varepsilon < 1$,

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - 0| \geq r + \varepsilon\}|) = 0 \text{ and } \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - 1| \geq r + \varepsilon\}|) = 0.$$

This shows that 0 and 1 are the f-rough weighted statistical limits of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$.

Since the set $W^t st - LIM^r_f x$ is convex so $[0, 1] \subset W^t st - LIM^r_f x$. This shows that the set containing f-rough weighted statistical limits of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is infinite. Note that for this sequence $x = \{x_n\}_{n\in\mathbb{N}}$, weighted statistical limit set is empty.

Theorem 2.7. The f-rough weighted statistical limit set $W^t st - LIM^r_f x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is singleton or empty if the weighted sequence $t = \{t_n\}_{n\in\mathbb{N}}$ is f-statistically unbounded.

Proof. Let $W^t st - LIM^r_f x = \emptyset$, then the theorem is obvious. So assuming $W^t st - LIM^r_f x \neq \emptyset$. Let there exists two points $x_* \neq y_*$ such that $x_*, y_* \in W^t st - LIM^r_f x$. Take $\varepsilon = \frac{|x_* - y_*|}{2} > 0$.

Case 1: Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is properly divergent to $+\infty$. Then for any positive real number G there exist $n_0 \in \mathbb{N}$ such that $t_n > G\ \forall\ n \geq n_0$.

Then

$$1 = \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k \geq \frac{2r + 2\varepsilon}{|x_* - y_*|}\}|) = \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_* - y_*| \geq 2r + 2\varepsilon\}|)$$

$$\leq \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - x_*| \geq r + \varepsilon\}|) + \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - y_*| \geq r + \varepsilon\}|) = 0.$$

Which is a contradiction.

Case 2: Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is unbounded but not properly divergent to $+\infty$. Then there exists two infinite subsets of $\mathbb{N}$ say K and L such that $K \cup L = \mathbb{N}$, $K \cap L = \emptyset$ and $\{t_n\}_{n\in K}$ is a unbounded subsequence and $\{t_n\}_{n\in L}$ is a bounded subsequence of $\{t_n\}_{n\in\mathbb{N}}$.

Subcase 2(i): Let the f-density of K is zero i.e., $d_f(K) = 0$. Since $\{t_n\}_{n\in L}$ is a bounded subsequence of $\{t_n\}_{n\in\mathbb{N}}$ then there exists a positive real number G such that $t_n < G\ \forall\ n \in L$. So $\{n \in \mathbb{N} : t_n \geq G\} = K \setminus \{\text{a finite subset of } \mathbb{N}\}$. Since the f-density of the set K is zero. So $d_f(\{k \in \mathbb{N} : t_k \geq G\}) = 0$ i.e.,

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |t_k| \geq G\}|) = 0,$$

which contradict that $\{t_n\}_{n\in\mathbb{N}}$ is f-statistically unbounded.

Subcase 2(ii): Let the f-density of K is nonzero, i.e., $d_f(K) \neq 0$. Since T_n (where $n \in \mathbb{N}$) is a real number then there exists a natural number m such that $m \leq T_n < m + 1$. Since f is an increasing function so we have

$$f(m) \leq f(T_n) \leq f(m + 1).$$

Then

$$|\{k \leq T_n : k \in K\}| + 1 \geq |\{k \leq m + 1 : k \in K\}|,$$

$$\Rightarrow f(|\{k \le T_n : k \in K\}| + 1) \ge f(|\{k \le m+1 : k \in K\}|),$$
$$\Rightarrow f(|\{k \le T_n : k \in K\}|) + f(1) \ge f(|\{k \le m+1 : k \in K\}|),$$
$$\Rightarrow \frac{1}{f(T_n)} f(|\{k \le T_n : k \in K\}|) + \frac{f(1)}{f(T_n)} \ge \frac{1}{f(m+1)} f(|\{k \le m+1 : k \in K\}|).$$

Let $n \to \infty$ then $m \to \infty$, so we get $\lim\limits_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : k \in K\}|) \neq 0$.

Then

$$\begin{aligned}
0 \neq \lim_{n\to\infty} \frac{1}{f(T_n)} & f(|\{k \le T_n : k \in K \setminus \{\text{a finite subset of } \mathbb{N}\}\}|) \\
&\le \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k \ge \frac{2r+2\varepsilon}{|x_* - y_*|}\}|) \\
&= \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_* - y_*| \ge 2r + 2\varepsilon\}|) \\
\le \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - x_*| \ge r + \varepsilon\}|) &+ \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - y_*| \ge r + \varepsilon\}|) = 0,
\end{aligned}$$

which is a contradiction. Hence the result. □

If the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is f-statistically bounded then there exists a positive real number M such that

$$\lim_{m\to\infty} \frac{1}{f(m)} f(|\{k \le m : t_k \ge M\}|) = 0.$$

Since T_n (where $n \in \mathbb{N}$) is a real number then there exists a natural number m such that $m \le T_n < m+1$. Since f is an increasing function so we have

$$f(m) \le f(T_n) \le f(m+1).$$

Now we have

$$|\{k \le T_n : t_k \ge M\}| \le |\{k \le m : t_k \ge M\}| + 1,$$
$$\Rightarrow f(|\{k \le T_n : t_k \ge M\}|) \le f(|\{k \le m : t_k \ge M\}|) + f(1),$$
$$\Rightarrow \frac{1}{f(T_n)} f(|\{k \le T_n : t_k \ge M\}|) \le \frac{1}{f(m)} f(|\{k \le m : t_k \ge M\}|) + \frac{f(1)}{f(m)}.$$

If $m \to \infty$ then $n \to \infty$. Then

$$\Rightarrow \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k \ge M\}|) = 0.$$

So, $\lim\limits_{n\to\infty} \dfrac{1}{f(T_n)} f(|\{k \le T_n : k \in A\}|) = 1$, where $A = \{k \in \mathbb{N} : t_k < M\}$ (as we know that if $K \subset \mathbb{N}$ and $d_f(K) = 0$ then $d_f(\mathbb{N} \setminus K) = 1$). Then the subsequence $\{t_n\}_{n\in A}$ of the sequence $\{t_n\}_{n\in\mathbb{N}}$, is bounded so the limit inferior exists. Throughout the paper $\liminf\limits_{n\in A} t_n$ denote the limit inferior of the sequence $\{t_n\}_{n\in A}$ when the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is f-statistically bounded.

Theorem 2.8. For a sequence $x = \{x_n\}_{n\in\mathbb{N}}$, we have

$$\operatorname{diam}(W^t st - LIM_f^r x) \le \begin{cases} \frac{2r}{\liminf\limits_{n\in A} t_n}, & \text{if the weighted sequence is } f\text{-statistically bounded,} \\ 0, & \text{if the weighted sequence is } f\text{-statistically unbounded.} \end{cases}$$

In general it has no smaller bound than $\frac{2r}{\liminf\limits_{n\in A} t_n}$ if the weighted sequence is f-statistically bounded.

Proof. Case 1: If possible let the diam$(W^t st - LIM_f^r x) > \frac{2r}{\liminf_{n\in A} t_n} > 0$, then there exists a positive real number $\alpha \in (0, \liminf_{n\in A} t_n)$ such that diam$(W^t st - LIM_f^r x) > \frac{2r}{\alpha} > \frac{2r}{\liminf_{n\in A} t_n}$.

Then there exists $y, z \in W^t st - LIM_f^r x$ such that $|y - z| > \frac{2r}{\alpha}$. Take $\varepsilon \in (0, \frac{\alpha|y-z|}{2} - r)$. Since $y, z \in W^t st - LIM_f^r x$ then

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - y| \ge r + \varepsilon\}|) = 0 \text{ and } \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - z| \ge r + \varepsilon\}|) = 0.$$

Since the weighted sequence is f-statistically bounded then there exists a positive real number M such that $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : k \in \mathbb{N} \setminus A\}|) = 0$, where $A = \{k \in \mathbb{N} : t_k < M\}$. Also $\alpha < \liminf_{n\in A} t_n$ then there exists a natural number k_0 such that $\alpha < t_n$, $\forall\, n \ge k_0$ and $n \in A$.

Choose $B = A \setminus \{1, 2, 3, ..., k_0 - 1\}$, $C = \{k \in \mathbb{N} : t_k|x_k - y| < r + \varepsilon\}$ and $D = \{k \in \mathbb{N} : t_k|x_k - z| < r + \varepsilon\}$. Let $K = \mathbb{N} \setminus B, L = \mathbb{N} \setminus C$ and $P = \mathbb{N} \setminus D$, so $d_f(K) = d_f(L) = d_f(P) = 0$.

Then

$$|\{k \le T_n : k \in K \cup L\}| \le |\{k \le T_n : k \in K\}| + |\{k \le T_n : k \in L\}|,$$

$$\Rightarrow \frac{1}{f(T_n)} f(|\{k \le T_n : k \in K \cup L\}|) \le \frac{1}{f(T_n)} f(|\{k \le T_n : k \in K\}|) + \frac{1}{f(T_n)} f(|\{k \le T_n : k \in L\}|).$$

So it follows that $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : k \in K \cup L\}|) = 0$.

Similarly, we can prove that $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : k \in K \cup L \cup P\}|) = 0$.

So we have $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : k \in B\cap C\cap D\}|) = 1$ (since $\mathbb{N}\setminus(B\cap C\cap D) = (\mathbb{N}\setminus B)\cup(\mathbb{N}\setminus C)\cup(\mathbb{N}\setminus D) = K\cup L\cup P$).

Now we choose any $p \in B \cap C \cap D$. Then $t_p > \alpha$, $t_p|x_p - y| < r + \varepsilon$ and $t_p|x_p - z| < r + \varepsilon$.

$$\therefore\ \alpha|y - z| \le \alpha|x_p - y| + \alpha|x_p - z| < t_p|x_p - y| + t_p|x_p - z| < 2(r + \varepsilon) < \alpha|y - z|,$$

(since $\varepsilon < \frac{\alpha|y-z|}{2} - r$), which is a contradiction.

Case 2: Let the weighted sequence is f-statistically unbounded. Then from the Theorem 2.7, diam$(W^t st - LIM_f^r x) = 0$.

Now let us prove the second part of the theorem.

Consider a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{W^t st^f} c$ and the weighted sequence $t = \{t_n\}_{n\in\mathbb{N}}$ is bounded which satisfies the condition $t_n < \liminf_{n\in\mathbb{N}} t_n$ $\forall\, n \in \mathbb{N}$ (such a sequence exists for example, let $f(x) = \sqrt{x}$, $\forall\, x \ge 0$,

$$x_n = \begin{cases} 1, \text{ if } n = m^2, \forall\, m \in \mathbb{N}, \\ 0, \text{ otherwise}, \end{cases}$$

$$\text{and } t_n = 2 - \frac{1}{n}\ \forall\, n \in \mathbb{N},$$

then $x_n \xrightarrow{W^t st^f} 0$ and $t_n < 2 = \liminf_{n\in\mathbb{N}} t_n$ $\forall\, n \in \mathbb{N}$).

Then for any $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - c| \ge \varepsilon\}|) = 0$.

Choosing $L = \liminf_{n\in\mathbb{N}} t_n$ and $\overline{B}_{\frac{r}{L}}(c) = \{y \in \mathbb{R} : |y - c| \le \frac{r}{L}\} = [c - \frac{r}{L}, c + \frac{r}{L}]$.

Let $y \in \overline{B}_{\frac{r}{L}}(c)$, then

$$t_k|x_k - y| \le t_k|x_k - c| + t_k|c - y| < \varepsilon + L\frac{r}{L} = r + \varepsilon \ \forall\, k \in \mathbb{N} \setminus \{k \in \mathbb{N} : t_k|x_k - c| \ge \varepsilon\},$$

$$\Rightarrow \ \{k \in \mathbb{N} : t_k|x_k - y| \ge r + \varepsilon\} \subseteq \{k \in \mathbb{N} : t_k|x_k - c| \ge \varepsilon\}.$$

Therefore we get $y \in W^t st - LIM^r_f x$. So $W^t st - LIM^r_f x = \overline{B}_{\frac{r}{L}}(c)$. Since the $\text{diam}(\overline{B}_{\frac{r}{L}}(c)) = \frac{2r}{L}$, this shows that in general, the upper bound $\frac{2r}{\liminf_{n\in\mathbb{N}} t_n}$ of the diameter of the set $W^t st - LIM^r_f x$ can't be decreased anymore. □

In [2, Theorem 2.2] Aytar showed that the diameter of a rough statistical limit set is $\le 2r$ (where r is the roughness of the convergence). For the case of f-rough weighted statistical convergence the diameter of rough weighted statistical limit set may be strictly greater than $2r$. We show this in our next example.

Example 2.9. Define $x = \{x_n\}_{n\in\mathbb{N}}$ and $t = \{t_n\}_{n\in\mathbb{N}}$ by,

$$x_n = \begin{cases} 2 + \frac{1}{n}, & \text{if } n \ne m^2, m \in \mathbb{N}, \\ n^2, & \text{otherwise,} \end{cases}$$

and

$$t_n = \begin{cases} \frac{1}{2} + \frac{1}{n}, & \text{if } n \ne m^2, m \in \mathbb{N}, \\ n, & \text{otherwise.} \end{cases}$$

Let $r = 1$ and $f(x) = \sqrt{x}\ \forall\, x \ge 0$.

Then $\{k \in \mathbb{N} : t_k|x_k - 0| \ge 1 + \varepsilon\} \subseteq \{k \in \mathbb{N} : k = m^2\ \forall\, m \in \mathbb{N}\} \cup D$ (where D is a finite subset of natural number).

So, $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - 0| \ge 1 + \varepsilon\}|) = 0 \Rightarrow\ 0 \in W^t st - LIM^1_f x$.

Similarly, we get $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - 4| \ge 1 + \varepsilon\}|) = 0 \Rightarrow\ 4 \in W^t st - LIM^1_f x$.

From the Theorem 2.5 we get, $[0, 4] \subset W^t st - LIM^1_f x$.

Now take any $x_* > 0$. Then for any $0 < \varepsilon < \frac{x_*}{3}$ we have,

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k + x_*| \ge 1 + \varepsilon\}|) = 1,$$

(where $t_k|x_k + x_*| = (\frac{1}{2} + \frac{1}{k})(2 + \frac{1}{k} + x_*) = (1 + \frac{x_*}{2}) + (\frac{3}{2k} + \frac{x_*}{k} + \frac{1}{k^2}) > 1 + \varepsilon\ \forall\, k \ne m^2$ and $k \ge k_0$, for some positive integer k_0 and $m \in \mathbb{N}$). This implies $-x_* \notin W^t st - LIM^1_f x$.

Similarly, we can show that, $(4 + x_*) \notin W^t st - LIM^1_f x\ \forall\, x_* > 0$.

This shows that $W^t st - LIM^1_f x = [0, 4]$.

So, $\text{diam}(W^t st - LIM^1_f x) = 4 > 2r$ (since $r = 1$). Hence the result.

Theorem 2.10. The set $W^t st - LIM^r_f x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is closed.

Proof. Case 1: Let $\{t_n\}_{n\in\mathbb{N}}$ be f-statistically bounded and $W^t st - LIM^r_f x = \emptyset$, then it is trivial.

So assume that $W^t st - LIM^r_f x \neq \phi$ and let $p_* \in \overline{W^t st - LIM^r_f x}$. Then there exists a sequence $p = \{p_n\}_{n\in\mathbb{N}}$ in $W^t st - LIM^r_f x$ such that $p_n \to p_*$ as $n \to \infty$. We have to show that $p_* \in W^t st - LIM^r_f x$.

Since $\{t_n\}_{n\in\mathbb{N}}$ is f-statistically bounded, then $\lim\limits_{n\to\infty} \dfrac{1}{f(T_n)} f(|\{k \leq T_n : t_k \geq M\}|) = 0$.
Also $p_n \to p_*$ as $n \to \infty$. Then for any $\varepsilon > 0$, there exists k_0 such that $|p_k - p_*| < \frac{\varepsilon}{2M}$ $\forall\ k \geq k_0$.

Then from the triangle inequality, $t_k|x_k - p_*| \leq t_k|x_k - p_{k_0}| + t_k|p_{k_0} - p_*|$ $\forall\ k \in \mathbb{N}$.

We have $\{k \in \mathbb{N} : t_k|x_k - p_{k_0}| < r + \frac{\varepsilon}{2}\} \cap \{k \in \mathbb{N} : t_k < M\} \subseteq \{k \in \mathbb{N} : t_k|x_k - p_*| < r + \varepsilon|\}$,

$\Rightarrow \{k \in \mathbb{N} : t_k|x_k - p_*| \geq r + \varepsilon|\} \subseteq \{k \in \mathbb{N} : t_k|x_k - p_{k_0}| \geq r + \frac{\varepsilon}{2}\} \cup \{k \in \mathbb{N} : t_k \geq M\}$,

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - p_*| \geq r + \varepsilon\}|) = 0$.

So, $p_* \in W^t st - LIM^r_f x$. Hence $W^t st - LIM^r_f x$ is closed.

Case 2: Let $\{t_n\}_{n\in\mathbb{N}}$ is f-statistically unbounded. Then by Theorem 2.7, $W^t st - LIM^r_f x$ is either singleton set or empty. Hence it is closed. □

Remark 2.11. The set $W^t st - LIM^r_f x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is path connected, compact, totally bounded, complete and separable. Also the set $W^t st - LIM^r_f x$ contains more than one point then the set is uncountable and is an interval.

Now we introduce the definitions of weighted bounded and f-weighted statistically bounded of a sequence of real numbers as follows:

Definition 2.12. Let $t = \{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. Then the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be weighted bounded if there exists a positive real number G such that $t_k|x_k| < G$ $\forall\ n \in \mathbb{N}$. The set of all weighted bounded sequences is denoted by $W^t(B)$.

Definition 2.13. Let $t = \{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. Then the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be f-weighted statistically bounded if there exists a positive real number G such that

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k| \geq G\}|) = 0.$$

We denote the set of all f-weighted statistically bounded sequence by $W^t S^f(B)$.

Remark 2.14. (i) If a sequence $\{x_n\}_{n\in\mathbb{N}}$ is weighted bounded then it is bounded. But the converse is not true. For example, choose $x_n = \frac{1}{n}$ and $t_n = n^2$ $\forall\ n \in \mathbb{N}$. Then it is quiet clear that $\{x_n\}_{n\in\mathbb{N}}$ is bounded but not weighted bounded.

(ii) It is obvious that $W^t(B) \subseteq W^t S^f(B)$. But it is possible to find out a sequence $\{x_n\}_{n\in\mathbb{N}}$ which is f-statistically bounded but not f-weighted statistically bounded. For this we consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ define in Remark 2.14.(i).

Theorem 2.15. Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfies the condition $\lim\limits_{n\to\infty} \frac{f(t_{n+1})}{f(T_n)} = 0$ and $\{x_n\}_{n\in\mathbb{N}}$ be a f-weighted statistically bounded sequence of real numbers then $\{x_n\}_{n\in\mathbb{N}}$ is f-statistically bounded.

Proof. Let $\delta > 0$ and $\{x_n\}_{n\in\mathbb{N}}$ be a f-weighted statistically bounded sequence. So there exists a positive real number G such that

$$\lim_{n\to\infty}\frac{1}{f(T_n)}f(|\{k \le T_n : t_k|x_k| \ge G\delta\}|) = 0.$$

For $m \in \mathbb{N}$, then there exists a positive integer n such that $T_n < m \le T_{n+1} \Rightarrow f(T_n) \le f(m) \le f(T_{n+1})$. Then

$$|\{k \le m : |x_k| \ge G\}| \le |\{k \le T_{n+1} : t_k|x_k| \ge G\delta\}| = |\{k \le T_n : t_k|x_k| \ge G\delta\}| + t_{n+1},$$

$$\Rightarrow f(|\{k \le m : |x_k| \ge G\}|) \le f(|\{k \le T_n : t_k|x_k| \ge G\delta\}|) + f(t_{n+1}),$$

$$\Rightarrow \frac{1}{f(m)}f(|\{k \le m : |x_k| \ge G\}|) \le \frac{1}{f(T_n)}f(|\{k \le T_n : t_k|x_k| \ge G\delta\}|) + \frac{f(t_{n+1})}{f(T_n)}.$$

Let $m \to \infty$ then $n \to \infty$ and R.H.S tends to 0, it follows that

$$\lim_{m\to\infty}\frac{1}{f(m)}f(|\{k \le m : |x_k| \ge G\}|) = 0$$

and consequently the sequence $\{x_n\}_{n\in\mathbb{N}}$ is f-statistically bounded. □

Is this type of weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfies the condition $\lim_{n\to\infty}\frac{f(t_{n+1})}{f(T_n)} = 0$ exists ? Answer is yes. For example, choose $t_n = n\ \forall\ n \in \mathbb{N}$ and $f(x) = x^p, 0 < p \le 1$ and $x \ge 0$.

Theorem 2.16 (Decomposition Theorem). If $x = \{x_n\}_{n\in\mathbb{N}}$ is a f-weighted statistically bounded sequence of real numbers then there exists a weighted bounded sequence $y = \{y_n\}_{n\in\mathbb{N}}$ and f-weighted statistically null sequence $z = \{z_n\}_{n\in\mathbb{N}}$ such that $x = y + z$. However this decomposition is not unique.

Proof. Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a f-weighted statistically bounded sequence of real numbers. Then there exists a $G > 0$ such that

$$\lim_{n\to\infty}\frac{1}{f(T_n)}f(|\{k \le T_n : t_k|x_k| \ge G\}|) = 0.$$

Let $P = \{k \in \mathbb{N} : t_k|x_k| > G\}$. Define,

$$y_n = \begin{cases} x_n, & \text{if } \ n \in \mathbb{N}\setminus P, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$z_n = \begin{cases} 0, & \text{if } \ n \in \mathbb{N}\setminus P, \\ x_n, & \text{otherwise.} \end{cases}$$

Then $x = y+z$, where y is a weighted bounded sequence and z is a f-weighted statistically null sequence. So $W^tS^f(B) \subseteq W^t(B) + W^tS_0^f$ (where $W^tS_0^f$ is the set of all f-weighted statistically null sequences).

Now if we take any $y \in W^t(B)$ and $z \in W^tS_0^f$. Then there exist $G > 0$, such that $t_k|y_k| \le G\ \forall\ k \in \mathbb{N}$ and for every $\varepsilon > 0$, we have $\lim_{n\to\infty}\frac{1}{f(T_n)}f(|\{k \le T_n : t_k|z_k| \ge \varepsilon\}|) = 0$.

Since $t_k|y_k + z_k| \le t_k|y_k| + t_k|z_k| \le G + t_k|z_k|$. Then $\{k \in \mathbb{N} : t_k|y_k + z_k| \ge G + \varepsilon\} \subseteq \{k \in \mathbb{N} : t_k|z_k| \ge \varepsilon\}$. This shows $(y + z)$ is a f-weighted statistically bounded sequence of real numbers.

Hence $W^tS^f(B) = W^t(B) + W^tS_0^f$.

2nd Part: Now let $c = \{c_n\}_{n\in\mathbb{N}}$ be a real sequence with finitely many nonzero terms.

Let $\{c_1, c_2, c_3, ..., c_m\}$ are the nonzero terms of $c = \{c_n\}_{n\in\mathbb{N}}$ and $G_1 = \max\{t_1c_1, t_2c_2, t_3c_3, ..., t_mc_m\}$.

Define,

$$p_n = \begin{cases} (x_n - c_n), & \text{if} \quad n \in \mathbb{N} \setminus P, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$q_n = \begin{cases} c_n, & \text{if} \quad n \in \mathbb{N} \setminus P, \\ x_n, & \text{otherwise.} \end{cases}$$

Then $|t_kp_k| = |t_k(x_k - c_k)| \leq |t_kx_k| + |t_kc_k| \leq G + G_1 = G_2 \ \forall \ k \in \mathbb{N}$. Hence $p = \{p_n\}_{n\in\mathbb{N}}$ is weighted bounded sequence of real numbers.

For any $\varepsilon > 0, \{k \in \mathbb{N} : t_k|q_k| \geq \varepsilon\} \subseteq P \cup P_0$ (where P_0 is a finite set). So $q = \{q_n\}_{n\in\mathbb{N}}$ is f-weighted statistically null sequence and we have $x = p + q$. Hence the decomposition is not unique. □

Theorem 2.17. If a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is f-weighted statistically bounded then there exist a non-negative real number r such that $W^tst - LIM_f^r x \neq \varnothing$.

Proof. Let $\varepsilon > 0$. Since the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is f-weighted statistically bounded, then there exists a positive real number G such that

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k| \geq G\}|) = 0.$$

Let $K = \{k \in \mathbb{N} : t_k|x_k| \geq G\}$. Define $r = \inf\{t_k|x_k| : k \in K\}$. Then we have $r \geq G$.

So,

$$\{k \in \mathbb{N} : t_k|x_k| \geq r + \varepsilon\} \subseteq K \ \Rightarrow \ \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k| \geq r + \varepsilon\}|) = 0.$$

Then $0 \in W^tst - LIM_f^r x$. Hence $W^tst - LIM_f^r x \neq \varnothing$. □

But the converse is not true and to show that we will give an example. Again Aytar [2, Theorem 2.4] had shown that a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is statistically bounded if there exists a non-negative real number r such that $st - LIM^r x \neq \varnothing$. For the case of rough weighted statistical convergence we show that f-rough weighted statistical limit set is non-empty where as the sequence is not f-weighted statistically bounded. We show this in our next example.

Example 2.18. Take the sequence

$$x_n = \begin{cases} 5, \text{ if } n = m^2 \ \forall \ m \in \mathbb{N}, \\ 1 - \frac{1}{n^2}, \text{ otherwise.} \end{cases}$$

Let the weighted sequence is defined by $t_n = n \ \forall \ n \in \mathbb{N}$. and take $f(x) = x^p, x \geq 0$ and $0 < p \leq 1$.

Then for any $r, \varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - 1| \geq r + \varepsilon\}|) \leq \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - 1| \geq \varepsilon\}|)$$

$$= \lim_{n\to\infty} \frac{2^p}{(n(n+1))^p} f(|\{k \le \frac{n(n+1)}{2} : k|x_k - 1| \ge \varepsilon\}|)$$
$$\le \lim_{n\to\infty} \frac{2^p}{(n(n+1))^p} f(\sqrt{\frac{n(n+1)}{2}}) = \lim_{n\to\infty} \frac{2^{\frac{p}{2}}}{(n(n+1))^{\frac{p}{2}}} = 0.$$

So the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is f-rough weighted statistically convergent to 1. This implies $1 \in W^t st - LIM^r_f x$.

But for any real $G > 0$, we have $\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k| \ge G\}|) = 1$. So the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is not f-weighted statistically bounded.

Now we introduce the definition of f-weighted statistical cluster point of a sequence of real numbers as follows:

Definition 2.19. Let $t = \{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence and $c \in \mathbb{R}$ is called a f-weighted statistical cluster point of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \le T_n : t_k|x_k - c| < \varepsilon\}|) \ne 0.$$

We denote the set of all f-weighted statistical cluster points of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ by $W^t\Gamma^f_x$.

In [2, Lemma 2.9] Aytar had shown that for an arbitrary $c \in \Gamma_x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ we have $|x_* - c| \le r$, $\forall\ x_* \in st - LIM^r x$. Rather for the case of f-weighted statistical cluster points and the set of f-rough weighted statistical limit points, $|x_* - c|$ may be $> r$ for some $x_* \in W^t st - LIM^r_f x$ and $c \in W^t\Gamma^f_x$. We show this in our next example.

Example 2.20. Consider the sequences $x = \{x_n\}_{n\in\mathbb{N}}$, $t = \{t_n\}_{n\in\mathbb{N}}$ and $f(x) = \sqrt{x}\ \forall\ x \ge 0$ defined in Example 2.9. Then we get $W^t\Gamma^f_x = \{2\}$. It follows that $W^t st - LIM^1_f x = [0, 4]$. Choose $x_* = 0, c = 2$ then $|x_* - c| = 2 > r$ (since $r = 1$).

Now we give some important relations between the set of f-weighted statistical cluster points and the set of f-rough weighted statistical limit points of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$.

Theorem 2.21. For an arbitrary $c \in W^t\Gamma^f_x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$, we have

$$|x_* - c| \le \begin{cases} \frac{r}{\liminf_{n\in A} t_n}, & \text{if the weighted sequence is } f\text{-statistically bounded,} \\ \frac{r}{\inf_{n\in\mathbb{N}} t_n}, & \text{if the weighted sequence is } f\text{-statistically unbounded,} \end{cases}$$

$\forall\ x_* \in W^t st - LIM^r_f x$.

Proof. Case 1: Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be f-statistically bounded. If possible, let there exist a point $c \in W^t\Gamma^f_x$ and $x_* \in W^t st - LIM^r_f x$ such that $|x_* - c| > \frac{r}{\liminf_{n\in A} t_n} > 0$.

This implies $\frac{(\liminf_{n\in A} t_n)|x_* - c| - r}{3} > 0$. Then there exists a positive real number $\alpha \in (0, \liminf_{n\in A} t_n)$ such that

$$\frac{(\liminf_{n\in A} t_n)|x_* - c| - r}{3} > \frac{\alpha|x_* - c| - r}{3} > 0.$$

Define $\varepsilon = \frac{\alpha|x_*-c|-r}{3} > 0$. Since $\alpha < \liminf_{n\in A} t_n$ so there exist $k_0 \in \mathbb{N}$ such that $t_n > \alpha \ \forall\ n \geq k_0$ and $n \in A$ where $A = \{k \in \mathbb{N} : t_k < M\}$ and $d_f(\mathbb{N} \setminus A) = 0$.

Let $B_0 = A \setminus \{1, 2, ..., k_0 - 1\}$, then $d_f(\mathbb{N} \setminus B_0) = 0$. Now as $c \in W^t\Gamma_x^f$ so for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - c| < \varepsilon\}|) \neq 0.$$

Take $B = \{k \in \mathbb{N} : t_k|x_k - c| < \varepsilon\}$. Then four subcases arises.

<u>*Subcase 1(i):*</u> If $B \cap B_0 = \emptyset$ then $B \subseteq \mathbb{N} \setminus B_0$. Since $d_f(\mathbb{N} \setminus B_0) = 0$ then $d_f(B) = 0$. Which is a contradiction so this subcase can never happen.

<u>*Subcase 1(ii):*</u> If $B \subseteq B_0$, then $d_f(B \cap B_0) = d_f(B) \neq 0$.

<u>*Subcase 1(iii):*</u> If $B_0 \subseteq B$, then $d_f(B \cap B_0) = d_f(B_0) = 1$.

<u>*Subcase 1(iv):*</u> If $B \cap B_0 \neq \emptyset$, $B \setminus B_0 \neq \emptyset$ and $B_0 \setminus B \neq \emptyset$ then $B \setminus (B \cap B_0) \subseteq \mathbb{N} \setminus B_0$. This implies $d_f(B \setminus (B \cap B_0)) = 0$ (since $d_f(\mathbb{N} \setminus B_0) = 0$).

Again,

$$\{k \leq T_n : k \in B\} = \{k \leq T_n : k \in B \setminus (B \cap B_0)\} \cup \{k \leq T_n : k \in B \cap B_0\},$$
$$\Rightarrow f(|\{k \leq T_n : k \in B\}|) \leq f(|\{k \leq T_n : k \in B \setminus (B \cap B_0)\}|) + f(|\{k \leq T_n : k \in B \cap B_0\}|).$$

Then

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in B\}|)$$
$$\leq \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in B \setminus (B \cap B_0)\}|) + \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in B \cap B_0\}|),$$
$$\Rightarrow \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in B \cap B_0\}|) \neq 0.$$

This shows that for all existing cases $d_f(B \cap B_0) \neq 0$. So there exists natural number $k \in B \cap B_0$.

Then

$$t_k|x_k - x_*| \geq t_k|x_* - c| - t_k|x_k - c| > 3\varepsilon + r - \varepsilon = r + 2\varepsilon > r + \varepsilon,$$
$$\Rightarrow \{k \in \mathbb{N} : t_k|x_k - x_*| > r + \varepsilon\} \supseteq \{k \in \mathbb{N} : t_k|x_k - c| < \varepsilon\} \cap B_0,$$
$$\Rightarrow \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - x_*| \geq r + \varepsilon\}|) \neq 0.$$

This contradicts the fact that $x_* \in W^t st - LIM_f^r x$.

<u>*Case 2:*</u> Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be f-statistically unbounded. If possible, let exists a point $c \in W^t\Gamma_x^f$ and $x_* \in W^t st - LIM_f^r x$ such that $|x_* - c| > \frac{r}{\inf_{n\in\mathbb{N}} t_n}$.

Let $\varepsilon = \frac{\zeta|x_*-c|-r}{2}$, where $\zeta = \inf_{n\in\mathbb{N}} t_n$.

We know that $t_k|x_* - x_k| \geq t_k|x_* - c| - t_k|x_k - c| \geq \zeta|x_* - c| - t_k|x_k - c| = r + 2\varepsilon - t_k|x_k - c| \ \forall\ k \in \mathbb{N}$.

Then

$$\{k \in \mathbb{N} : t_k|x_* - c| < \varepsilon\} \subset \{k \in \mathbb{N} : t_k|x_k - x_*| \geq r + \varepsilon\}.$$

Since $c \in W\Gamma_x^f$, we have $\lim\limits_{n\to\infty} \dfrac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - c| < \varepsilon\}|) \neq 0$ this implies

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - x_*| \geq r + \varepsilon\}|) \neq 0,$$

which contradicts the fact $x_* \in W^t st - LIM_f^r x$. □

Corollary 2.22. If both the sets $W^t\Gamma_x$ and $W^t st - LIM^r x$ are non-empty then from the Theorem 2.8 and Theorem 2.21, we get the set $W^t\Gamma_x$ is bounded.

Pehlivan et al. [18, Corollary 1], had shown that if a sequence in a finite dimensional normed linear space is statistically bounded then the statistical cluster points set is non-empty. For the case of f-weighted statistical convergence the f-weighted statistical cluster point set may be empty even if the space is finite dimensional and the sequence is statistically bounded.

To prove this important fact, we consider the sequence of real numbers $x_n = \frac{1}{n}\ \forall\ n \in \mathbb{N}$ and $t_n = n^2\ \forall\ n \in \mathbb{N}$. Then the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is statistically bounded but $W^t\Gamma_x^f = \emptyset$.

Theorem 2.23. **(a)** For an arbitrary $c \in W^t\Gamma_x^f$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$, we have

$$W^t st - LIM_f^r x \subseteq \begin{cases} \overline{B}_{\frac{r}{p}}(c), & \text{if the weighted sequence is } f\text{-statistically bounded,} \\ \overline{B}_{\frac{r}{q}}(c), & \text{if the weighted sequence is } f\text{-statistically unbounded,} \end{cases}$$

where $p = \liminf\limits_{n\in A} t_n$ and $q = \inf\limits_{n\in\mathbb{N}} t_n$.

(b)

$$W^t st - LIM_f^r x \subseteq \begin{cases} \bigcap\limits_{c\in W\Gamma_x^f} \overline{B}_{\frac{r}{p}}(c) \subseteq \{x_* \in \mathbb{R} : W^t\Gamma_x^f \subseteq \overline{B}_{\frac{r}{p}}(x_*)\}, \\ \text{if the weighted sequence is } f\text{-statistically bounded,} \\ \bigcap\limits_{c\in W\Gamma_x^f} \overline{B}_{\frac{r}{q}}(c) \subseteq \{x_* \in \mathbb{R} : W^t\Gamma_x^f \subseteq \overline{B}_{\frac{r}{q}}(x_*)\}, \\ \text{if the weighted sequence is } f\text{-statistically unbounded.} \end{cases}$$

Proof. (a) From the Theorem 2.21 the results are obvious.

(b) Case 1: Let the weighted sequence is f-statistically bounded. Then from the Theorem 2.23(a), we can write

$$W^t st - LIM_f^r x \subseteq \bigcap_{c\in W^t\Gamma_x^f} \overline{B}_{\frac{r}{p}}(c).$$

Now assume that $y \in \bigcap\limits_{c\in W^t\Gamma_x^f} \overline{B}_{\frac{r}{p}}(c)$. Then we have $|y - c| \leq \frac{r}{p}$, forall $c \in W^t\Gamma_x^f$, which equivalent to $W^t\Gamma_x^f \subseteq \overline{B}_{\frac{r}{p}}(y)$, i.e.,

$$\bigcap_{c\in W^t\Gamma_x^f} \overline{B}_{\frac{r}{p}}(c) \subseteq \{x_* \in \mathbb{R} : W^t\Gamma_x^f \subseteq \overline{B}_{\frac{r}{p}}(x_*)\}.$$

Case 2: Proof is similar to Case 1 so omitted. □

The following examples show that equalities of the above Theorem 2.23 may or may not occur.

Example 2.24.

Case 1: The weighted sequence is f-statistically unbounded.

(i) Let

$$x_n = \begin{cases} 0, & \text{if } n \text{ even integer,} \\ 1, & \text{otherwise,} \end{cases}$$

and $t_n = n \; \forall \; n \in \mathbb{N}$, $f(x) = \log(x+1) \; \forall \; x \geq 0$, $r > 0$.

Then $W^t\Gamma_x^f = \{0,1\}$, $q = 1$, $\overline{B}_{\frac{r}{q}}(0) = [-r, r]$, $\overline{B}_{\frac{r}{q}}(1) = [1-r, 1+r]$ and $W^tst - LIM_f^r x = \emptyset$. So

$$\overline{B}_{\frac{r}{q}}(0) \cap \overline{B}_{\frac{r}{q}}(1) = \begin{cases} [1-r, r], & \text{if } r > \frac{1}{2}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

This shows that $W^tst - LIM_f^r x \subsetneqq \overline{B}_{\frac{r}{q}}(0)$ if $r > \frac{1}{2}$.

(ii) Let

$$x_n = \begin{cases} 1, & \text{if } n = m^2 \; \forall \; m \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and $t_n = n \; \forall \; n \in \mathbb{N}$, $f(x) = \sqrt{x}$ forall $x \geq 0$ and $r = 1$.

Then $W^t\Gamma_x^f = \{0\}$, $q = 1$, $\overline{B}_{\frac{r}{q}}(0) = [-1, 1]$ and $W^tst - LIM_f^r x = \{0\}$. This shows that $W^tst - LIM_f^r x \subsetneqq \overline{B}_{\frac{r}{q}}(0)$.

Case 2: The weighted sequence is f-statistically bounded.

(iii) Let

$$x_n = \begin{cases} 1, & \text{if } n = m^2 \; \forall \; m \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and $t_n = 1 + \frac{1}{n} \; \forall \; n \in \mathbb{N}$, $f(x) = \log(1+x)$ forall $x \geq 0$ and $r = \frac{1}{2}$.

Then $W^t\Gamma_x^f = \{0,1\}$, $p = 1$, $\overline{B}_{\frac{1}{2}}(0) \cap \overline{B}_{\frac{1}{2}}(1) = [-\frac{1}{2}, \frac{1}{2}] \cap [\frac{1}{2}, \frac{3}{2}] = \{\frac{1}{2}\}$ and $W^tst - LIM_f^r x = \{\frac{1}{2}\}$. This shows that $W^tst - LIM_f^r x = \overline{B}_{\frac{1}{2}}(0) \cap \overline{B}_{\frac{1}{2}}(1)$.

(iv) Let

$$x_n = \begin{cases} 2 + \frac{1}{n}, & \text{if } n \neq m^2, m \in \mathbb{N}, \\ n^2, & \text{otherwise,} \end{cases}$$

$$t_n = \begin{cases} \frac{1}{2} + \frac{1}{n}, & \text{if } n \neq m^2, m \in \mathbb{N}, \\ n, & \text{otherwise,} \end{cases}$$

and $r = 1$ and $f(x) = \sqrt{x} \; \forall \; x \geq 0$.

Then $W^t\Gamma_x^f = \{2\}$, $p = \frac{1}{2}$, $\overline{B}_{\frac{r}{p}}(2) = [0,4]$ and $W^t st - LIM_f^r x = [0,4]$ (by the Example 2.9). This shows that $W^t st - LIM_f^r x = \overline{B}_{\frac{r}{p}}(0)$.

Remark 2.25. From the Example 2.24 (i) and (iii), we get the set $W^t\Gamma_x^f$ is not a convex set.

Theorem 2.26. The set $W^t\Gamma_x^f$ is closed if the weighted sequence $t = \{t_n\}_{n\in\mathbb{N}}$ is f-statistically bounded.

Proof. If $W^t\Gamma_x^f = \phi$ then there is nothing to prove. So we assume that $W^t\Gamma_x^f \neq \phi$. Let $x_* \in \overline{W^t\Gamma_x^f}$. Then there exist a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $W^t\Gamma_x^f$ such that $y_n \to x_*$ as $n \to \infty$. We have to show that $x_* \in W^t\Gamma_x^f$. Let $\varepsilon > 0$. Since $\{t_n\}_{n\in\mathbb{N}}$ is f-statistically bounded so there exist $M > 0$ such that $\lim\limits_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in \mathbb{N} \setminus A\}|) = 0$, where $A = \{k \in \mathbb{N} : t_k < M\}$.

Since $y_n \to x_*$ as $n \to \infty$, so there exist $k_0 \in \mathbb{N}$ such that $|y_n - x_*| < \frac{\varepsilon}{2M}$ $\forall$ $n \geq k_0$.

Let $B = \{k \in \mathbb{N} : t_k|x_k - y_{k_0}| < \frac{\varepsilon}{2}\}$. Since $y_{k_0} \in W^t\Gamma_x^f$ so $d_f(B) \neq 0$.

Then four cases arises.

<u>*Case 1:*</u> If $A \cap B = \emptyset$ then $B \subseteq \mathbb{N} \setminus A$ this implies $d_f(B) = 0$ (since $d_f(\mathbb{N} \setminus A) = 0$). Which is a contradiction so this case can never happen.

<u>*Case 2:*</u> If $B \subseteq A$, then $d_f(A \cap B) = d_f(B) \neq 0$.

<u>*Case 3:*</u> If $A \subset B$, then $d_f(A \cap B) = d_f(A) = 1$.

<u>*Case 4:*</u> If $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ then $B \setminus (A \cap B) \subseteq \mathbb{N} \setminus A$.
This implies $d_f(B \setminus (A \cap B)) = 0$ (since $d_f(\mathbb{N} \setminus A) = 0$).

Then

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in B\}|)$$

$$\leq \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in B \setminus (A \cap B)\}|) + \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in A \cap B\}|),$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : k \in A \cap B\}|) \neq 0.$$

This shows that for all existing cases $d_f(A \cap B) \neq 0$. Now we have the inequality

$$t_k|x_k - x_*| \leq t_k|x_k - y_{k_0}| + t_k|y_{k_0} - x_*|,$$

$$\Rightarrow \{k \in \mathbb{N} : t_k|x_k - y_{k_0}| < \frac{\varepsilon}{2}\} \cap \{k \in \mathbb{N} : t_k < M\} \subseteq \{k \in \mathbb{N} : t_k|x_k - x_*| < \varepsilon\},$$

$$\Rightarrow A \cap B \subseteq \{k \in \mathbb{N} : t_k|x_k - x_*| < \varepsilon\},$$

$$\Rightarrow d_f(\{k \in \mathbb{N} : t_k|x_k - x_*| < \varepsilon\}) \neq 0.$$

This shows that $x_* \in W^t\Gamma_x^f$. This completes the proof. $\square$

Connor et al. [5, Corollary 3(3)] had shown that the statistical cluster points set Γ_x of a bounded sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is a compact subset of $\mathbb{R}$ and Pehlivan et. al. [18, Corollary 1] had shown that if $x = \{x_n\}_{n\in\mathbb{N}}$ is a statistically bounded sequence in $\mathbb{R}^m$ (m-dimensional space) then the set Γ_x is compact. Again Listán-García [15, Corollary 3.9] had shown that, for any sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in any normed linear space, the f-statistical cluster points set Γ_x^f is closed. For the case of f-weighted statistical convergence, the f-weighted statistical cluster points set may not be closed even if the space is $\mathbb{R}$ (finite dimensional) and the sequence is bounded (or, statistically bounded). The following example shows that in general, if the weighted sequence $t = \{t_n\}_{n\in\mathbb{N}}$ is f-statistically unbounded then the set $W^t\Gamma_x^f$ is may not closed (i.e., not compact).

Example 2.27. Let $\mathbb{N} = \displaystyle\bigcup_{j=1}^{\infty}\Delta_j$ be a decomposition of $\mathbb{N}$ (i.e., $\Delta_m \cap \Delta_n = \varnothing$ for $m \neq n$). Assume that $\Delta_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\}\ \forall\ j = 1, 2, 3, \ldots$.

Setting

$$f(x) = x^p\ \forall\ x \geq 0, 0 < p \leq 1,\ \ t_k = k\ \forall\ k \in \mathbb{N},$$
$$x_k = \frac{1}{j} + \frac{1}{k^2}\ \forall\ \ k \in \Delta_j \text{ and } j = 1, 2, 3, \ldots\ .$$

Then for each $j \in \mathbb{N}$, we get

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k - \frac{1}{j}| < \varepsilon\}|) \geq d_f(\Delta_j \setminus \{\text{a finite sub set of } \mathbb{N}\})$$
$$\geq \lim_{n\to\infty} \frac{1}{(T_n)^p}(T_n)^p(\frac{1}{2^j} + \frac{1}{2T_n})^p = \frac{1}{2^{jp}} \neq 0.$$

This shows that $\frac{1}{j} \in W^t\Gamma_x^f\ \forall\ j \in \mathbb{N}$.

Next we assume $k \in \mathbb{N}$ then there exists an integer $j \in \mathbb{N}$ such that $k \in \Delta_j$ some $j \in \mathbb{N}$. This implies k is of the form $k = 2^{j-1}(2s-1)$ where some $s \in \mathbb{N}$.

Now for each $k \in \mathbb{N}$,

$$t_k|x_k| = 2^{j-1}(2s-1)\{\frac{1}{j} + \frac{1}{(2^{j-1}(2s-1))^2}\} = \frac{1}{2}.\frac{2^j}{j}(2s-1) + \frac{1}{2^{j-1}(2s-1)} > \frac{1}{2}$$

(since $2^n > n,\ \forall\ n \geq 1$ and $2s - 1 \geq 1,\ \forall\ s \geq 1$).

So we choose $0 < \varepsilon < \frac{1}{2}$, then

$$\lim_{n\to\infty} \frac{1}{f(T_n)} f(|\{k \leq T_n : t_k|x_k| < \varepsilon\}|) = 0.$$

This implies $0 \notin W^t\Gamma_x^f$. So $W^t\Gamma_x^f$ is not a closed set under f-statistically unbounded weighted sequence.

Listán-García [15, Corollary 3.7] had shown that, if $x = \{x_n\}_{n\in\mathbb{N}}$ is a f-statistically bounded sequence in any normed linear space, then the set Γ_x^f is bounded. But no such example is given that what will effect on the cluster points set Γ_x^f if the sequence is not f-statistically bounded. We show this in our next example.

Example 2.28. Consider a decomposition of $\mathbb{N}$, i.e., $\Delta_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\},\ \forall\ j = 1, 2, 3, \ldots$. Setting $f(x) = x^p\ \forall\ x \geq 0, 0 < p \leq 1$, $t_k = 1\ \forall\ k \in \mathbb{N}$ and $x_k = j + \frac{1}{k^2}\ \forall\ k \in \Delta_j$ and $j = 1, 2, 3, \ldots$.

In this case f-weighted statistical convergence coincides to f-statistical convergence. Then for each $j \in \mathbb{N}$ and $0 < \varepsilon < 1$, we get

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : t_k|x_k - j| < \varepsilon\}|) = d_f(\Delta_j) = \frac{1}{2^{jp}} \neq 0.$$

This shows that $j \in W^t\Gamma_x^f \ \forall\ j \in \mathbb{N}$. So $W^t\Gamma_x^f$ is not a bounded set. This implies the set Γ_x^f may not be bounded.

Theorem 2.29. Let $x = \{x_n\}_{n\in\mathbb{N}}$ and $y = \{y_n\}_{n\in\mathbb{N}}$ be two sequences of real numbers and $t = \{t_n\}_{n\in\mathbb{N}}$ and $s = \{s_n\}_{n\in\mathbb{N}}$ be corresponding weighted sequences such that $d_f(\{k \in \mathbb{N} : x_k \neq y_k\}) = d_f(\{k \in \mathbb{N} : t_k \neq s_k\}) = 0$. Then $W^t\Gamma_x^f = W^s\Gamma_y^f$.

Proof. Let $\ell \in W^t\Gamma_x^f$, $A = \{k \in \mathbb{N} : x_k \neq y_k\}$, $B = \{k \in \mathbb{N} : t_k \neq s_k\}$, $C = \{k \in \mathbb{N} : t_k|x_k - \ell| < \varepsilon\}$ and $D = \{k \in \mathbb{N} : s_k|y_k - \ell| < \varepsilon\}$. Then $C \setminus (A \cup B) \subset D$ and $d_f(A \cup B) = 0$. Hence the result. □

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