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On the Jacobson and Simple Radicals of Semigroups

Emil Ilić-Georgijević[a]

[a]*Faculty of Civil Engineering, University of Sarajevo, Bosnia and Herzegovina*

Abstract. In this paper we study radicals of a semigroup which is the union of a family of its subsets, indexed by a nonempty set, such that the intersection of two distinct subsets is contained in the set of left zeroes of that semigroup.

1. Introduction

Radicals of semigroups have been investigated in many articles (see for instance [29]). They represent semigroup analogs of radicals of rings. Theory of radicals of rings is an active area of research (see [6]), and in particular, the theory of radicals of graded rings has been actively investigated and has various important applications (see [21, 22, 27, 28, 31]). Recently (see [11–14, 20, 21, 23]), some results are obtained regarding radicals and graded radicals of rings graded in the following sense.

Definition 1.1 ([21, 23]). *Let R be a ring, and Δ a partial groupoid, that is, a set with a partial binary operation. Also, let $\{R_\delta\}_{\delta\in\Delta}$ be a family of additive subgroups of R. We say that $R = \bigoplus_{\delta\in\Delta} R_\delta$ is* Δ-graded *and R* induces Δ *(or R is a* Δ-graded ring inducing Δ*) if the following two conditions hold:*

i) $R_\xi R_\eta \subseteq R_{\xi\eta}$ *whenever $\xi\eta$ is defined;*

ii) $R_\xi R_\eta \neq 0$ *implies that the product $\xi\eta$ is defined.*

Δ-graded rings inducing Δ will be simply called graded rings in the rest of the paper. The set $\bigcup_{\delta\in\Delta} R_\delta$ is called the homogeneous part of a graded ring $R = \bigoplus_{\delta\in\Delta} R_\delta$. A notion equivalent to that in Definition 1.1 has been studied in [2, 9, 25] from a different point of view. If we restrict addition and multiplication to the homogeneous part of a graded ring, we obtain a partial structure called an *anneid* [2, 9, 25], and an origin of this point of view to grading goes back to [24]. For a survey on anneids, one may consult [26, 33].
The homogeneous part of a graded ring is obviously a semigroup with respect to multiplication. This motivates us to introduce the following type of a semigroup (see also the notion of a *quasi-graded ring* from [2, 25]). However, we decide to work with semigroups which are generally without zero, and if S is a semigroup, by $O(S)$ we denote the set of all left zeroes of S, according to notation from [10].

Definition 1.2. *Let S be a semigroup, and Δ a partial groupoid, that is, a set with a partial binary operation. Also, let $\{S_\delta\}_{\delta\in\Delta}$ be a family of subsets, called* components, *of S. We say that $S = \bigcup_{\delta\in\Delta} S_\delta$ is* homogeneous semigroup inducing Δ *(shortly,* homogeneous semigroup*) if the following three conditions hold:*

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Email address: emil.ilic.georgijevic@gmail.com (Emil Ilić-Georgijević)

i) $S_\xi \cap S_\eta \subseteq O(S)$ *for all distinct* $\xi, \eta \in \Delta$;

ii) $S_\xi S_\eta \subseteq S_{\xi\eta}$ *whenever* $\xi\eta$ *is defined;*

iii) $S_\xi S_\eta \nsubseteq O(S)$ *implies that the product* $\xi\eta$ *is defined.*

If $s \in S \setminus O(S)$, then we define the *degree* $\delta(s)$ of s to be the unique $\xi \in \Delta$ such that $s \in S_\xi$. If $s \in O(S)$, then we assign a degree $\delta(s)$ which acts on Δ as a left zero. The set of all such degrees is denoted by $O(\Delta)$ and by Δ^* we denote $\Delta \setminus O(\Delta)$. If there exists $\delta \in \Delta$ such that $a, b \in S_\delta$, then we write $a\#b$, in consistency with designation of *addability* [2, 9, 25] of homogeneous elements of a graded ring.
Since homogeneous semigroups naturally arise from graded rings, it is easy to construct various examples of such semigroups. For examples of graded rings, see [11, 12, 21, 23].
Clearly, the notion of a homogeneous semigroup generalizes the notion of a 0-band of semigroups and the notion of a band of semigroups. For results on radicals of 0-bands of semigroups, see [15, 18], and for radicals of bands of semigroups, see [16, 17]. The aim of this paper is to introduce and study *regular Jacobson radicals* and *regular simple radicals* of homogeneous semigroups. If $S = \bigcup_{\delta\in\Delta} S_\delta$ is a homogeneous semigroup, we will see that under the assumption that Δ is cancellative, that is, if S is a *regular homogeneous semigroup*, the regular Jacobson radical and the regular simple radical of S coincide with their classical counterparts. The main results of this paper describe the Jacobson radical and the simple radical of a regular homogeneous semigroup $S = \bigcup_{\delta\in\Delta} S_\delta$ via the Jacobson radical and the simple radical of a component of S which corresponds to an idempotent element from Δ^*. These represent generalizations of results obtained in [8, 9] for the *graded Jacobson radical* of a graded ring to the case of homogeneous semigroups. For all the notions related to semigroup theory which are not included in this article, we refer to [3, 4] and [29], and for the notions related to graded rings, see [13, 21, 23].

2. Regular Jacobson and simple radicals

One of the approaches to radicals of semigroups is through the appropriate classes of acts (see [29]) by analogy with a description of radicals of rings by the appropriate classes of modules (see [1] and also [6]). Acts are in the focus of contemporary research as well. In particular, there is much activity in describing monoids through acts over monoids, and for more recent results on this, we refer to [7]. If M is an S-act, we put $FM = \{x \in M \mid xa = x\ (a \in S)\}$, the set of zeroes of M (see [10]). The following notion represents a generalization of a graded module over a graded ring from [2, 9, 25].

Definition 2.1. *Let* $S = \bigcup_{\delta\in\Delta} S_\delta$ *be a homogeneous semigroup. A* homogeneous S-act *is an* S*-act* $M = \bigcup_{d\in D} M_d$*, where* $\{M_d\}_{d\in D}$ *is a family of its subsets, such that* $M_s \cap M_t \subseteq FM$ *for all distinct* $s, t \in D$*, and where* D *is a nonempty set. If* $x_1, x_2 \in M$ *then we write* $x_1\#x_2$ *if there exists* $d \in D$ *such that* $x_1, x_2 \in M_d$.

Let $M = \bigcup_{d\in D} M_d$ be a homogeneous S-act, where $S = \bigcup_{\delta\in\Delta} S_\delta$ is a homogeneous semigroup. Then, obviously, for all $\delta \in \Delta$ and all $d \in D$ there exists $t \in D$ such that $M_d S_\delta \subseteq M_t$. Also, if N is an S-subact of M, then N is a homogeneous S-act $\bigcup_{d\in D} N_d$, where $N_d = M_d \cap N\ (d \in D)$.

Definition 2.2. *A homogeneous* S*-act* M *is said to be* regular *if* $xa\#xb$, $xa, xb \notin FM$*, implies* $a\#b$ *for all* $x \in M$, $a, b \in S$.

Definition 2.3. *A mapping* $f : M \to M'$ *between two homogeneous* S*-acts* M *and* M' *is called a* homomorphism of homogeneous acts *if it is a homomorphism between* M *and* M' *observed as* S*-acts and if* $x\#y$, $x, y \in M$*, implies* $f(x)\#f(y)$.

Remark 2.4. *Let us notice that any trivial classical homomorphism is also a homomorphism in the sense of Definition 2.3.*

Let μ be a congruence on a homogeneous S-act $M = \bigcup_{d\in D} M_d$. By $[x]_\mu$, or just by $[x]$ when there is no threat of confusion, we denote the congruence class which contains the element $x \in M$ and M/μ denotes the set

of all congruence classes of M with respect to μ. It is easy to see that M/μ is itself a homogeneous S-act $\bigcup_{d\in D}(M/\mu)_d$ with $(M/\mu)_d = M_d/\mu \cap M_d \times M_d$ $(d \in D)$ if we put $[x]_\mu a = [xa]_\mu$ for all $x \in M$ and $a \in S$, by analogy with the case of factor structures obtained from a graded ring [2, 9, 21, 25] (see also [19]). If we write $[x]_\mu\#[y]_\mu$, it means that representatives of these classes come from the same component of M.
We may consider a homogeneous semigroup $S = \bigcup_{\delta\in\Delta} S_\delta$ as a homogeneous S-act under right multiplication with $FS = O(S)$. A homogeneous semigroup S is *right regular* if it is regular as a homogeneous S-act. It is clear how to define a *left regular* homogeneous semigroup, and a homogeneous semigroup is said to be *regular* if it is both left and right regular, that is, if Δ is a cancellative partial groupoid. Let us notice that if μ is a right congruence on S, then any classical homomorphism of S/μ is a homomorphism of S/μ in the sense of Definition 2.3.
We proceed with definitions of the *regular Jacobson radical* and the *regular simple radical* of a homogeneous semigroup by analogy with the notion of the *graded Jacobson radical* of a graded ring from [8, 9].
The notion of an *irreducible homogeneous S-act* coincides with the classical one. A homogeneous S-act M is said to be *totally irreducible* if $MS \nsubseteq FM$ and if M has no non-trivial homomorphisms in the sense of Definition 2.3. Let M be a homogeneous S-act and let us define a congruence $\delta_M \subseteq S \times S$ by $(a_1, a_2) \in \delta_M$ if and only if for all $x \in M$ we have $xa_1 = xa_2$.

Definition 2.5. *The congruence $J^r(S) = \bigcap_{M\in I} \delta_M$, where I is the set of all irreducible regular homogeneous S-acts is called the* regular Jacobson radical *of S. The congruence $P^r(S) = \bigcap_{M\in I} \delta_M$, where I is the set of all totally irreducible regular homogeneous S-acts is called the* regular simple radical *of S. In both cases, if $I = \emptyset$, for the radical we take the universal congruence.*

Remark 2.6. *If S is a homogeneous semigroup, then let us notice that the classical Jacobson (classical simple) radical of S, denoted by $J(S)$ $(P(S))$, is contained in the regular Jacobson (regular simple) radical $J^r(S)$ $(P^r(S))$ of S. Also, the classical Jacobson and the classical simple radical of a homogeneous semigroup correspond to the notion of the* large Jacobson radical of a graded ring *from [8].*

Let us recall (see for instance [29]), a right congruence μ on a semigroup S is called *modular* if it is distinct from the universal congruence and if there exists an element $e \in S$ such that $[e]a = [a] \in S/\mu$ for all $a \in S$. Such an element e is called a *left unity modulo μ*.
From [8, 9] we know that every graded left unity modulo a proper homogeneous modular right ideal of a regular graded ring has the same degree, which is a nonzero idempotent element of the grading set. Here we have the following result.

Lemma 2.7. *Let $S = \bigcup_{\delta\in\Delta} S_\delta$ be a regular homogeneous semigroup and μ a modular right congruence on S. Then all the left unities modulo μ have the same degree which is an idempotent element of Δ^*. This degree is called the* degree of a modular right congruence μ.

Proof. Let e and f be left unities modulo μ of S. Then $[e]a = [f]a = [a] \in S/\mu$ for all $a \in S$. Since μ is distinct from the universal congruence, there exists $a \in S$ such that $[ea] \notin F(S/\mu)$. Now, since S is regular, we obtain $e\#f$. Hence, all the left unities modulo μ come from the same component of S, say S_ϵ, and moreover $\epsilon^2 = \epsilon$ since $e^2\#e$. Note that $e^2 \notin O(S)$, again since μ is distinct from the universal congruence. □

Since classical homomorphism theorems obviously transfer to the case of homogeneous acts, it is easy to verify the following lemma.

Lemma 2.8. *A homogeneous S-act M is strictly cyclic (totally irreducible) if and only if $M \cong S/\mu$, where μ is a modular right congruence (maximal modular right congruence) on S.*

Following the proof of the fact that the classical Jacobson (classical simple) radical of a semigroup S coincides with the intersection of all modular right congruences μ on S for which S/μ is irreducible (of all maximal modular right congruences on S) (Corollary 6.15 and Corollary 6.16 in [10]), one may easily verify the following theorem.

Theorem 2.9. *Let S be a homogeneous semigroup. Then the regular Jacobson (regular simple) radical of S coincides with the intersection of all (maximal) modular right congruences μ on S for which S/μ is an irreducible (totally irreducible) regular homogeneous S-act.*

Corollary 2.10. *If S is a regular homogeneous semigroup, then the regular Jacobson (regular simple) radical of S coincides with the classical Jacobson (classical simple) radical of S.*

Let $R = \bigoplus_{\delta\in\Delta} R_\delta$ be a graded ring for which Δ is a cancellative partial groupoid (a regular graded ring), and let ϵ be a nonzero idempotent element from Δ. We know from [8, 9] that then there exists a one-to-one correspondence between the maximal homogeneous modular right ideals of R of degree ϵ and the maximal modular right ideals of R_ϵ given by $I \mapsto I \cap R_\epsilon$ (the inverse mapping given by $I \mapsto \bigoplus_{\delta\in\Delta}\{x \in R_\delta \mid xR \cap R_\epsilon \subseteq I\}$). (For similar observations regarding 2-graded rings, see [32].) This correspondence implies that the classical Jacobson radical of the ring R_ϵ coincides with the intersection of the ring R_ϵ and the graded Jacobson radical of R (see [8, 9]). The following theorems represent generalizations of the above mentioned results to homogeneous semigroups. In the proof of the first of them we as usual write S^1 to designate a semigroup which is obtained from a semigroup S by adjoining an identity element 1.

Theorem 2.11. *Let $S = \bigcup_{\delta\in\Delta} S_\delta$ be a regular homogeneous semigroup, $|O(S)| \leq 1$, and $\epsilon \in \Delta^*$ an idempotent element. Then there exists a one-to-one correspondence between (maximal) modular right congruences on S of degree ϵ and (maximal) modular right congruences on S_ϵ.*

Proof. Let μ be a modular right congruence on the semigroup S_ϵ, and e a left unity modulo μ. Let us define a relation $\hat{\mu} \subseteq S \times S$ by $(a_1, a_2) \in \hat{\mu}$ if and only if $(a_1, a_2)S^1 \cap S_\epsilon \times S_\epsilon \subseteq \mu$, where $(a_1, a_2)S^1$ denotes the set $\{(a_1 s, a_2 s) \mid s \in S^1\}$. Clearly, $\hat{\mu}$ is a right congruence on S. We claim that $\hat{\mu} \cap S_\epsilon \times S_\epsilon = \mu$. Let $(a_1, a_2) \in \mu$ and let $s \in S$ be such that $(a_1 s, a_2 s) \in S_\epsilon \times S_\epsilon$, and $(a_1 s, a_2 s) \notin O(S) \times O(S)$. Then, since S is regular, we obtain that $\delta(s) = \epsilon$. Therefore $(a_1 s, a_2 s) \in \mu$. Obviously, $(a_1, a_2)1 \in \mu$ for every $(a_1, a_2) \in \mu$. Hence $\mu \subseteq \hat{\mu} \cap S_\epsilon \times S_\epsilon$. Now, let $(a_1, a_2) \in \hat{\mu} \cap S_\epsilon \times S_\epsilon$. Since e is a left unity modulo μ, we have $(ea_1, a_1), (ea_2, a_2) \in \mu \subseteq \hat{\mu}$. Therefore $(ea_1, ea_2) \in \hat{\mu}$ which implies $(ea_1, ea_2) \in \mu$ since $ea_1, ea_2 \in S_\epsilon$. Now, since $(ea_1, a_1) \in \mu$ and $(ea_2, a_2) \in \mu$, we have $(a_1, a_2) \in \mu$. Hence $\hat{\mu} \cap S_\epsilon \times S_\epsilon \subseteq \mu$. Let us now check whether e is a left unity modulo $\hat{\mu}$. So, let $a \in S$ and let $b \in S^1$ be such that $eab, ab \in S_\epsilon$. Since e is a left unity modulo μ, we have $(eab, ab) \in \mu$. Hence $(ea, a) \in \hat{\mu}$. Thus $\hat{\mu}$ is a modular right congruence on S of degree ϵ.
If μ is a maximal modular right congruence on S_ϵ, then we claim that $\hat{\mu}$ is a maximal right congruence on S. Since Corollary 6.4 in [5] also holds for right congruences on semigroups, there exists a maximal right congruence λ on S such that $\lambda \supseteq \hat{\mu}$ and $\lambda \cap S_\epsilon \times S_\epsilon = \hat{\mu} \cap S_\epsilon \times S_\epsilon = \mu$. If $(a_1, a_2) \in \lambda$, then $(a_1, a_2)S^1 \cap S_\epsilon \times S_\epsilon \subseteq \lambda \cap S_\epsilon \times S_\epsilon = \mu$. Therefore $(a_1, a_2) \in \hat{\mu}$. Hence $\hat{\mu} = \lambda$ is a maximal right congruence.
On the other hand, if μ is a (maximal) modular right congruence on S of degree ϵ, then it is an easy exercise to prove that the relation $\mu \cap S_\epsilon \times S_\epsilon$ is a (maximal) modular right congruence on S_ϵ. □

Theorem 2.12. *Let $S = \bigcup_{\delta\in\Delta} S_\delta$ be a regular homogeneous semigroup, $|O(S)| \leq 1$, and $\epsilon \in \Delta^*$ an idempotent element. Then $P(S_\epsilon) = P(S) \cap S_\epsilon \times S_\epsilon$.*

Proof. Let $(a_1, a_2) \in P(S_\epsilon)$. We claim that $(a_1, a_2) \in P(S)$. In the proof we generalize the technique used in the proof of the corresponding inclusion regarding Jacobson radicals for the case of graded rings in [9]. Suppose that $(a_1, a_2) \notin P(S)$. Then, according to Corollary 2.10 and the proof of Theorem 2.11, there exists x such that $(a_1 x, a_2 x) \in S_\xi \times S_\xi$, but $(a_1 x, a_2 x) \notin P(S_\xi)$, where $\xi \in \Delta^*$ is an idempotent element. Since we assume that S is regular, we have that $\delta = \delta(x)$, ξ and ϵ are mutually distinct. If T is any semigroup and $A = \{s \in T \mid TsT \subseteq O(T)\}$, then A is an ideal of T. Moreover, $A^3 \subseteq O(T)$. Therefore, if $J(A)$ is the classical Jacobson radical of A, we have that $J(A) = A \times A$ according to Proposition 2 in [29]. We know that the Jacobson radical of a semigroup is ideally hereditary (see Lemma 3.5 in [30]), and so $A \times A \subseteq J(T)$. However, if $P(T)$ denotes the classical simple radical of T, then, as we know, $J(T) \subseteq P(T)$ (see for instance [29]). In our case we look at the semigroup S_ξ and conclude that there exists $y \in S_\xi$ such that ya_1x is not a left zero of S_ξ or that there exists $z \in S_\xi$ such that za_2x is not a left zero of S_ξ. Then we have $\xi(\epsilon\delta) = \xi^2 = \xi = \epsilon\delta \notin O(\Delta)$. Since S is by assumption regular, it follows that $\xi\epsilon = \epsilon$, that is, $\xi = \epsilon$, a contradiction. Therefore, $S_\epsilon \times S_\epsilon \cap P(S) \supseteq P(S_\epsilon)$.

Conversely, we know that $P(S)$ coincides with the intersection of all maximal modular right congruences μ on S. Denote the set of all maximal modular right congruences on S by C. Then, for every $\epsilon^2 = \epsilon \in \Delta^*$, we have that $S_\epsilon \times S_\epsilon \cap P(S) = S_\epsilon \times S_\epsilon \cap \bigcap_{\mu \in C} \mu$ is, according to Theorem 2.11, a subset of $P(S_\epsilon)$. Hence, $S_\epsilon \times S_\epsilon \cap P(S) \subseteq P(S_\epsilon)$. □

In order to prove an analogous result for the Jacobson radical, we add more assumptions on a regular homogeneous semigroup which make it an irreducible act over itself.

Theorem 2.13. *Let $S = \bigcup_{\delta \in \Delta} S_\delta$ be a regular homogeneous semigroup such that $|O(S)| = 1$. Also, let us assume that $S_\xi \cap S_\eta = O(S)$ for all distinct $\xi, \eta \in \Delta$, and that for every $a \in S \setminus O(S)$ and every $b \in S$ there exists $s \in S$ such that $b = as$. If $\epsilon \in \Delta^*$ is an idempotent element, then $J(S_\epsilon) = J(S) \cap S_\epsilon \times S_\epsilon$.*

Proof. Let us denote the unique left zero of S by z. In Theorem 2.11 we have established a one-to-one correspondence between modular right congruences on S of degree ϵ and modular right congruences on S_ϵ. Let $\hat{\mu}$ be a modular right congruence on S of degree ϵ. Then $S/\hat{\mu}$ is an irreducible homogeneous S-act. Indeed, according to assumptions on S, we have that for all $[x]_{\hat{\mu}} \notin F(S/\hat{\mu})$ and all $[y]_{\hat{\mu}} \in S/\hat{\mu}$, there exists $s \in S$ such that $[y]_{\hat{\mu}} = [xs]_{\hat{\mu}} = [x]_{\hat{\mu}} s$. Also, $(S/\hat{\mu})_\xi \cap (S/\hat{\mu})_\eta = [z]_{\hat{\mu}} \in F(S/\hat{\mu})$ for all distinct $\xi, \eta \in \Delta$. We claim that $F(S/\hat{\mu}) = [z]_{\hat{\mu}}$. Indeed, if $[x]_{\hat{\mu}} \in F(S/\hat{\mu})$, then $[xz]_{\hat{\mu}} = [x]_{\hat{\mu}} z = [x]_{\hat{\mu}}$. On the other hand, $xz \in O(S)$, and therefore $xz = z$. Hence $[x]_{\hat{\mu}} = [z]_{\hat{\mu}}$, and so $F(S/\hat{\mu}) = [z]_{\hat{\mu}}$. Thus $S/\hat{\mu}$ is an irreducible S-act. Let $\mu = \hat{\mu} \cap S_\epsilon \times S_\epsilon$ be the corresponding modular right congruence on S_ϵ. We claim that S_ϵ/μ is an irreducible S_ϵ-act. Let us first prove that $|F(S_\epsilon/\mu)| = 1$, similarly to above. We know that $[z]_\mu \in F(S_\epsilon/\mu)$. Let $[x]_\mu \in F(S_\epsilon/\mu)$. Then $[xz]_\mu = [x]_\mu z = [x]_\mu$, but $z = xz \in O(S)$, and hence $[x]_\mu = [z]_\mu$. Now, let $[x]_\mu \neq [z]_\mu$ and $[y]_\mu$ be arbitrary elements of S_ϵ/μ. We know that there exists $s \in S$ such that $xs = y$. Now, if $y \neq z$, then the regularity of S implies that $s \in S_\epsilon$. On the other hand, if $y = z$, then again there exists $s \in S_\epsilon$ such that $xs = z$, namely $s = z$. Therefore there exists $s \in S_\epsilon$ such that $[y]_\mu = [x]_\mu s$. Hence S_ϵ/μ is an irreducible S_ϵ-act. We have proved that the one-to-one correspondence from Theorem 2.11 is in our case a one-to-one correspondence between modular right congruences $\hat{\mu}$ on S of degree ϵ such that $S/\hat{\mu}$ is irreducible and modular right congruences μ on S_ϵ such that S_ϵ/μ is irreducible. Therefore, the claim of the theorem can now be proved analogously to Theorem 2.12. □

Remark 2.14. *Examples of semigroups which satisfy conditions of the previous theorem exist and are among semigroups obtained from homogeneous parts of graded rings under multiplication. For instance, if R is a division ring, then the homogeneous part of the graded ring $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$ is such an example.*

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