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Error Bounds for a Gauss-Type Quadrature Rule to Evaluate Hypersingular Integrals

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Abstract. In the present paper we consider hypersingular integrals of the following type

$$\oint_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}} w_\alpha(x)dx, \tag{1}$$

where the integral is understood in the Hadamard finite part sense, p is a positive integer, $w_\alpha(x) = e^{-x}x^\alpha$ is a Laguerre weight of parameter $\alpha \geq 0$ and $t > 0$. In [6] we proposed an efficient numerical algorithm for approximating (1), focusing our attention on the computational aspects and on the efficient implementation of the method. Here, we introduce the method discussing the theoretical aspects, by proving the stability and the convergence of the procedure for density functions f s.t. $f^{(p)}$ satisfies a Dini- type condition. For the sake of completeness, we present some numerical tests which support the theoretical estimates.

1. Introduction

Hypersingular integrals

$$\mathcal{H}_p(\mathcal{G}, t) := \oint_a^b \frac{\mathcal{G}(x)}{(x-t)^{p+1}}dx, \qquad p \in \{1, 2, \dots\}, \quad a < t < b,$$

were introduced in a more general context by Hadamard [13] and are defined as the finite part of divergent integrals (shortly FP integrals), i.e.

$$\oint_a^b \frac{dx}{(x-t)^{p+1}} = \lim_{\varepsilon\to 0^+}\left(\int_a^{t-\varepsilon} \frac{dx}{(x-t)^{p+1}}dx + \int_{t+\varepsilon}^b \frac{dx}{(x-t)^{p+1}}dx - \frac{1-(-1)^p}{p\varepsilon^p}\right) = \frac{1}{p}\left(\frac{1}{(a-t)^p} - \frac{1}{(b-t)^p}\right)$$

Many properties fulfilled by finite part integrals over bounded intervals can be found in [22] (see also [10], [24], [16]).

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In the present paper we consider hypersingular integrals over the positive semiaxis

$$\mathcal{H}_p(fw_\alpha, t) := \oint_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}} w_\alpha(x) dx, \qquad t > 0,$$

where p is a positive integer and $w_\alpha(x) = x^\alpha e^{-x}$ is a Laguerre weight with $\alpha \geq 0$. For functions f s.t $f^{(p)}$ satisfies hypotheses of Dini-type, starting from the decomposition

$$\mathcal{H}_p(fw_\alpha, t) \;=\; \int_0^{+\infty} \frac{f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} w_\alpha(x) dx + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \oint_0^{+\infty} \frac{w_\alpha(x)}{(x-t)^{p+1-k}} dx, \tag{2}$$

we prove the existence of $\mathcal{H}_p(fw_\alpha, t)$. In [6] we approximated the first right-hand integral by a subsequence of "truncated" Gauss-Laguerre rules, conveniently chosen to avoid the numerical cancellation arising when a quadrature knot is "close" to t. Nevertheless, we didn't give the proofs of the stability and the convergence of the method, focusing our attention only on the implementation aspects of the procedure. Instead here we discuss mainly the theoretical aspects, by proving the stability and the convergence of the rule introduced there and also a result of more general interest (see Lemma 2.1).

The procedure is completed since the remaining Hadamard transforms of w_α in (2) are known in terms of special functions, and they can be computed with high accuracy by standard routines.

FP integrals are employed, for instance, in the numerical solution of hypersingular integral equations, which are model for many physics and engineering problems (see [22] and the references therein [10, 11, 15, 22, 27]). However the literature devoted to their approximation is richer in the case of bounded intervals (see, for instance, [1, 2, 10, 14, 21–23, 29]). Recently some new different methods have been proposed also in the case of unbounded intervals (see [5–8, 25]).

The paper is organized as follows. Section 2 is devoted to some notations and preliminary results. Section 3 contains the estimate of $\mathcal{H}_p(fw_\alpha, t)$ under suitable assumptions on f and the numerical method accompanied by results about the stability and the rate of convergence of the error. In the successive Section 4 we propose some numerical experiments, in order to show the efficiency of the rule. Finally, Section 5 contains the proofs of the stated results.

2. Basic definitions and properties

Along all the paper the constant C will be used several times, having different meaning in different formulas. Moreover from now on we will write $C \neq C(a, b, \ldots)$ in order to say that C is a positive constant independent of the parameters $a, b, \ldots$, and $C = C(a, b, \ldots)$ to say that C depends on $a, b, \ldots$. Moreover, if $A, B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < C \neq (A, B)$ such that

$$\frac{B}{C} \leq A \leq CB.$$

Finally, $\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m.

2.1. *Function spaces*

With $w_\alpha(x) = x^\alpha e^{-x}, \alpha \geq 0$, we denote by C_{w_α} the following set of functions

$$C_{w_\alpha} = \begin{cases} \left\{ f \in C^0((0, +\infty)) : \lim\limits_{\substack{x \to +\infty \\ x \to 0^+}} (fw_\alpha)(x) = 0 \right\}, & \alpha > 0, \\[2ex] \left\{ f \in C^0([0, +\infty)) : \lim\limits_{x \to +\infty} (fw_\alpha)(x) = 0 \right\}, & \alpha = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_{w_\alpha}} := \|fw_\alpha\|_\infty = \sup_{x\geq 0} \left|(fw_\alpha)(x)\right|,$$

where $C^0(E)$ is the space of the continuous functions on the set E. In the next we will use $\|f\|_E := \sup_{x\in E} |f(x)|$.

For smoother functions, we introduce the Sobolev-type spaces of order $r \in \mathbb{N}$

$$W_r(w_\alpha) = \left\{ f \in C_{w_\alpha} \ : \ f^{(r-1)} \in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r w_\alpha\|_\infty < +\infty \right\},$$

where $AC((0,+\infty))$ denotes the set of all functions which are absolutely continuous on every closed subset of $(0,+\infty)$ and $\varphi(x) = \sqrt{x}$. We equip these spaces with the norm

$$\|f\|_{W_r(w_\alpha)} := \|fw_\alpha\|_\infty + \|f^{(r)}\varphi^r w_\alpha\|_\infty.$$

For any $f \in C_{w_\alpha}$ we consider the following main part of the $k-$th $\varphi-$modulus of smoothness

$$\Omega_\varphi^k(f,u)_{w_\alpha} = \sup_{0<h\leq u} \|w_\alpha \Delta_{h\varphi}^k f\|_{I_{kh}},$$

where $I_{kh} = \left[4k^2h^2, \frac{C}{h^2}\right]$, C is a fixed positive constant, and

$$\Delta_{h\varphi}^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + h\varphi(x)(k-i)\right).$$

The complete $k-$th modulus of smoothness is given by ([4] and also [20])

$$\omega_\varphi^k(f,u)_{w_\alpha} \quad = \quad \Omega_\varphi^k(f,u)_{w_\alpha} + \inf_{P\in\mathbb{P}_{k-1}} \|(f-P)w_\alpha\|_{(0,4k^2u^2)} + \inf_{Q\in\mathbb{P}_{k-1}} \|(f-Q)w_\alpha\|_{\left(\frac{1}{u^2},+\infty\right)}.$$

By means of $\Omega_\varphi^k(f,u)_{w_\alpha}$ we define the Zygmund-type spaces

$$Z_\lambda(w_\alpha) := \left\{ f \in C_{w_\alpha} \ : \ \sup_{u>0} \frac{\Omega_\varphi^k(f,u)_{w_\alpha}}{u^\lambda} < +\infty \right\}$$

of parameter $0 < \lambda < k$, equipped with the norm

$$\|f\|_{Z_\lambda(w_\alpha)} = \|fw_\alpha\|_\infty + \sup_{u>0} \frac{\Omega_\varphi^k(f,u)_{w_\alpha}}{u^\lambda}.$$

We recall that with $r = \lfloor\lambda\rfloor$ it is $W_{r+1}(w_\alpha) \subseteq Z_\lambda(w_\alpha) \subseteq W_r(w_\alpha)$ and, by arguments similar to those used in [8, Lemma 2.1], for $0 < \lambda < 1$ and $p \in \mathbb{N}$, $f^{(p)} \in Z_\lambda(w_\alpha\varphi^p)$ implies $f \in Z_{\lambda+p}(w_\alpha)$ and viceversa.

Now we state the following result which can be useful in the next and also in other contexts.

Lemma 2.1. *Let $f \in C_{w_\alpha}$ and $P_m \in \mathbb{P}_m$.Then*

$$\int_0^{\frac{1}{\sqrt{m}}} \frac{\omega_\varphi(f-P_m,u)_{w_\alpha}}{u} du \leq C\left(\|(f-P_m)w_\alpha\|_\infty + \int_0^{\frac{1}{\sqrt{m}}} \frac{\omega_\varphi^r(f,u)_{w_\alpha}}{u} du \right),$$

where $r \in \mathbb{N}$ with $r < m$ and $0 < C \neq C(m,f)$.

2.2. Orthogonal polynomials and Truncated Gauss-Laguerre rule

Let $w_\alpha(x) = e^{-x}x^\alpha$ be the Laguerre weight of parameter $\alpha \geq 0$ and let $\{p_m(w_\alpha)\}_m$ be the sequence of the corresponding orthonormal polynomials with positive leading coefficients

$$p_m(w_\alpha, x) = \gamma_m(w_\alpha)x^m + \text{ terms of lower degree}, \quad \gamma_m(w_\alpha) > 0.$$

Denoting by $x_{m,k}, k = 1, \dots, m$, the zeros of $p_m(w_\alpha)$ in increasing order, we recall that (see [28])

$$\frac{C}{m} < x_{m,1} < x_{m,2} < \dots < x_{m,m} < 4m + 2\alpha - Cm^{\frac{1}{3}}.$$

From now on, for any fixed $0 < \theta < 1$, the integer $j := j(m)$ will denote the index of the zero of $p_m(w_\alpha)$ s.t.

$$x_{m,j} = \min_{k=1,2,..,m} \{x_{m,k} \ : \ x_{m,k} \geq 4m\theta\}. \tag{3}$$

Inside the segment $(0, x_{m,j})$ the distance between two consecutive zeros of $p_m(w_\alpha)$ can be estimated as follows

$$\Delta x_{m,k} \sim \Delta x_{m,k-1} \sim \sqrt{\frac{x_{m,k}}{m}}, \qquad \Delta x_{m,k} = x_{m,k+1} - x_{m,k}, \qquad k = 1, 2, \dots, j.$$

Now we recall the so called "truncated" Gauss-Laguerre rule introduced in [17] and based on the first j zeros of $p_m(w_\alpha)$, j defined in (3),

$$\int_0^{+\infty} f(x)w_\alpha(x)dx = \sum_{k=1}^{j} f(x_{m,k})\lambda_{m,k} + R_m(f), \tag{4}$$

where $\{\lambda_{m,k}\}_{k=1}^m$ are the Christoffel numbers w.r.t. w_α and $R_m(f)$ is the remainder term.

3. The main results

Consider

$$\mathcal{H}_p(fw_\alpha, t) := \fint_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}} w_\alpha(x)dx, \quad w_\alpha(x) = x^\alpha e^{-x}, \ \alpha \geq 0,$$

where the integral is defined in the Hadamard sense. Assuming f sufficiently smooth, we use the following decomposition

$$\begin{aligned}
\mathcal{H}_p(fw_\alpha, t) \quad &= \quad \int_0^{+\infty} \frac{f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} w_\alpha(x)dx + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \fint_0^{+\infty} \frac{w_\alpha(x)}{(x-t)^{p+1-k}}dx \\
&=: \quad \mathcal{F}_p(fw_\alpha, t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathcal{H}_{p-k}(w_\alpha, t).
\end{aligned} \tag{5}$$

We prove the existence of the right hand side in (5) for $f^{(p)}$ satisfying a Dini- type condition.

Theorem 3.1. *Let $p \geq 1$, $\alpha \geq 0$. For any function f s.t.*

$$\int_0^1 \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha\varphi^p}}{u} du < \infty, \tag{6}$$

and for any fixed $t > 0$,

$$t^p|\mathcal{H}_p(fw_\alpha, t)| \leq C\left(\int_0^1 \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha\varphi^p}}{u} du + \|f\|_{W_p(w_\alpha)}\right), \qquad 0 < C \neq C(f, t). \tag{7}$$

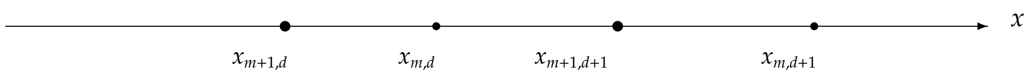
Remark 3.1. *In particular, if $f^{(p)} \in Z_\lambda(w_\alpha \varphi^p)$, by* (7) *we deduce*

$$t^p |\mathcal{H}_p(fw_\alpha, t)| \le C \left(\|f\|_{Z_{p+\lambda}(w_\alpha)} + \|f\|_{W_p(w_\alpha)} \right), \qquad 0 < C \neq C(f,t).$$

For the convenience of the reader, we briefly expose the numerical procedure proposed in [6].

Let $t \in (0, 4m\theta)$ be fixed, $0 < \theta < 1$. Recalling that the zeros $\{x_{m,k}\}_{k=1}^m$ of $p_m(w_\alpha)$ interlace the zeros $\{x_{m+1,k}\}_{k=1}^{m+1}$ of $p_{m+1}(w_\alpha)$, there exists an index $d \in \{1, 2, \dots, m-1\}$ s.t.

$$x_{m,d} \le t \le x_{m,d+1},$$



and the integral $\mathcal{F}_p(fw_\alpha, t)$ is approximated by the "truncated" rule (4) of order m^*, where m^* is selected as follows

$$m^* = \begin{cases} m+1, & \text{if } |x_{m+1,d+1} - t| > \min\{|t - x_{m,d}|, |t - x_{m,d+1}|\}, \\ m, & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} \mathcal{F}_p(fw_\alpha, t) &= \sum_{i=1}^{j} \frac{f(x_{m^*,i}) - \sum_{k=0}^p \frac{f^{(k)}(t)}{k!}(x_{m^*,i} - t)^k}{(x_{m^*,i} - t)^{p+1}} \lambda_{m^*,i} + e_{p,m^*}(fw_\alpha, t) \\ &:= \mathcal{F}_{p,m^*}(fw_\alpha, t) + e_{p,m^*}(fw_\alpha, t), \end{aligned}$$

where $e_{p,m^*}(fw_\alpha, t)$ is the remainder term.

The subsequence $\{\mathcal{F}_{p,m^*}(fw_\alpha, t)\}_{m^*}$ of the Gaussian sequence $\{\mathcal{F}_{p,m}(fw_\alpha, t)\}_{m \in \mathbf{N}}$ satisfies [3]

$$\min_{i=1,\dots,m^*} |x_{m^*,i} - t| = |x_{m^*,d} - t| \ge C \sqrt{\frac{x_{m^*,d}}{m}}, \tag{8}$$

i.e., the minimal distance of any Gaussian knot from t is large enough inside the range $(0, 4m^*\theta)$.

Therefore possible numerical instability arising when a Gaussian knots is "close" to t is avoided. Moreover (8) is crucial in order to prove the stability and the convergence of the procedure, since for $f \in W_p(w_\alpha)$

$$\left| \varphi^p(t) \frac{f(x_{m^*,d}) - \sum_{k=0}^p \frac{f^{(k)}(t)}{k!}(x_{m^*,d} - t)^k}{(x_{m^*,d} - t)^{p+1}} \lambda_{m^*,d} \right| \le C \|f^{(p)} w_\alpha \varphi^p\|_\infty \le C$$

(see the proof of Lemma 5.6).

In conclusion one has

$$\mathcal{H}_p(fw_\alpha, t) = \mathcal{H}_{p,m^*}(fw_\alpha, t) + e_{p,m^*}(fw_\alpha, t), \tag{9}$$

where

$$\mathcal{H}_{p,m^*}(fw_\alpha, t) = \mathcal{F}_{p,m^*}(fw_\alpha, t) + \sum_{k=0}^p \frac{f^{(k)}(t)}{k!} \mathcal{H}_{p-k}(w_\alpha, t). \tag{10}$$

About the stability and the convergence of (10) we prove the following theorems.

Theorem 3.2. *For any* $t \in (0, 4m\theta_1)$ *fixed, with* $0 < \theta_1 < \theta < 1$*, and for any function* f *satisfying* (6)

$$t^p |\mathcal{H}_{p,m^*}(f w_\alpha, t)| \leq C\left(\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha \varphi^p}}{u} du + \|f\|_{W_p(w_\alpha)}\right),$$

where $0 < C \neq C(m, f, t)$.

Theorem 3.3. *Let* $0 < \lambda < 1$*. For any* $f \in Z_{\lambda+p}(w_\alpha)$ *and* $t \in (0, 4m\theta_1)$ *fixed, with* $0 < \theta_1 < \theta < 1$,

$$t^p |\mathcal{H}_p(f w_\alpha, t) - \mathcal{H}_{p,m^*}(f w_\alpha, t)| \leq C\left(\frac{\log m}{\sqrt{m^\lambda}} \|f\|_{Z_{p+\lambda}(w_\alpha)} + e^{-Am} \|f\|_{W_p(w_\alpha)}\right),$$

where $0 < C \neq C(m, f, t)$ *and* $0 < A \neq A(m, f, t)$.

Corollary 3.1. *If* $f \in Z_{\lambda+p+q}(w_\alpha)$*,* $q \geq 0$*, then we get*

$$t^p |\mathcal{H}_p(f w_\alpha, t) - \mathcal{H}_{p,m^*}(f w_\alpha, t)| \leq C \frac{\|f\|_{Z_{\lambda+p+q}(w_\alpha)}}{\sqrt{m^{\lambda+q}}} \log m \tag{11}$$

and if $f \in W_{p+q}(w_\alpha)$*,* $q \geq 1$*, then we obtain*

$$t^p |\mathcal{H}_p(f w_\alpha, t) - \mathcal{H}_{p,m^*}(f w_\alpha, t)| \leq C \frac{\|f\|_{W_{p+q}(w_\alpha)}}{\sqrt{m^q}} \log m. \tag{12}$$

We conclude the section by highlighting that in (9) "choice of the subsequence" and "truncation" both play a key role from a computational point of view: "truncation" allows to avoid overflow phenomena arising when f exponentially growths, "choice of the subsequence" prevents numerical cancellation. Moreover, from the theoretical point of view, they are both necessary ingredients in the analysis of the stability and the convergence of the method.

4. Numerical experiments

In this section we give some numerical tests obtained by approximating $\mathcal{H}_p(f w_\alpha, t)$ by $\{\mathcal{H}_{p,m^*}(f w_\alpha, t)\}_m$. Since the exact values of the integrals are unknown, we will retain as exact the values computed with $m = 1000$ and we will set

$$\bar{e}_{p,m^*}(f w_\alpha, t) = |\mathcal{H}_{p,m^*}(f w_\alpha, t) - \mathcal{H}_{p,1024}(f w_\alpha, t)|. \tag{13}$$

All the computations have been performed in double-machine precision ($eps \sim 2.22044e-16$) and in the tables the symbol "-" means that the machine accuracy has been achieved.

Moreover, we will use the following definition of the truncation index (see [9])

$$j = \min_{k=1,\dots,m^*} \{k \,:\, \lambda_{m^*,k} < eps\} \tag{14}$$

taking into account that $\lambda_{m^*,k} \sim \Delta x_{m^*,k} w_\alpha(x_{m^*,k})$. The above definition is equivalent to (3) in the sense that there exists a $\theta \in (0, 1)$ s.t. $x_{m^*,j-1} < 4m^*\theta < x_{m^*,j}$, where j is defined in (14). To have an idea of the percentage of the knots involved in the truncation process, depending on the choice of θ, see [26].

In Tables 1-3 we have displayed the absolute error as defined in (13), the order m^* of the Gauss-Laguerre rule and the corresponding j as defined in (14).

Finally, details on the computation of $\mathcal{H}_{p-k}(w_\alpha)$ in (10) can be found in [6].

Example 4.1. Consider the following integral

$$\mathcal{H}_1(fw_{\frac{1}{4}},t) = \int_0^{+\infty} \frac{\log(x+6)}{(x^2+36)^2(x-t)^2} x^{\frac{1}{4}} e^{-x} dx, \quad f(x) = \frac{\log(x+6)}{(x^2+36)^2}, \quad \alpha = \frac{1}{4}, \quad p = 1.$$

Since the function f is very smooth, according to estimate (12), the convergence is very fast and, as shown in Table 1, for different choices of t at least 14 exact decimal digits are attained with only 18 function computations. The worst results is obtained for $t = 3.2e-6$ due to the the unboundedness of $\mathcal{H}_1(fw_{\frac{1}{4}},t)$ as $t \to 0$.

j	$\bar{e}_{1,m^*}(fw_{\frac{1}{4}}, 3.2e-6)$	$\bar{e}_{1,m^*}(fw_{\frac{1}{4}}, \frac{5}{2})$	$\bar{e}_{1,m^*}(fw_{\frac{1}{4}}, 4)$
6 ($m^* = 6$)	$2.0853e-9$	$8.7646e-10$	$3.6672e-10$
12 ($m^* = 12$)	$1.8307e-12$	$1.4764e-12$	$5.0814e-13$
18 ($m^* = 22$)	$1.1111e-15$	–	–

Table 1: Example 4.1

Example 4.2. Consider the integral

$$\mathcal{H}_2(fw_{\frac{1}{2}},t) = \int_0^{+\infty} \frac{\arctan^{3.4} x}{(x-t)^3} \sqrt{x} e^{-x} dx, \quad f(x) = \arctan^{3.4} x, \quad \alpha = \frac{1}{2}, \quad p = 2.$$

Since $f \in Z_{7.8}(w_{\frac{1}{2}})$, by (11) the error behaves like $\log m/m^{2.9}$. Thus, for instance, for $m^* = 801$ one can expect 7 exact decimal digits, at most. Anyway, as Table 2 shows, better numerical results can be achieved when t is far from 0, where $\mathcal{H}_2(fw_{\frac{1}{2}},t)$ is unbounded and, in addition, $f^{(4)}$ is discontinuous. In all the cases, we used only 119 samples of f for computing $\mathcal{H}_{2,801}(fw_{\frac{1}{2}},t)$.

In order to show that theoretical and numerical errors agree, in Figure 1 we show the behavior of the ratio

$$\overline{err}_m = \frac{\max_{i=1,\dots,40} \bar{e}_{2,m^*}(fw_{\frac{1}{2}}, y_i)}{\frac{\log m}{m^{2.9}}}, \quad \{y_i\}_{i=1,\dots,40} \in [0,10],$$

as m increases. The graph shows that $\overline{err}_m$ is almost constant (it varies in $(2.2, 6)$), confirming in such a way the theoretical error estimate.

j	$\bar{e}_{2,m^*}(fw_{\frac{1}{2}}, 0.25e-5)$	$\bar{e}_{2,m^*}(fw_{\frac{1}{2}}, \frac{1}{2})$	$\bar{e}_{2,m^*}(fw_{\frac{1}{2}}, \frac{5}{2})$
42 ($m^* = 101$)	$1.5695e-5$	$1.0164e-9$	$1.0120e-11$
60 ($m^* = 201$)	$4.0570e-6$	$3.8774e-11$	$2.9851e-13$
85 ($m^* = 401$)	$9.3800e-7$	$4.1807e-11$	$1.7408e-13$
103 ($m^* = 600$)	$3.2620e-7$	$1.3292e-12$	$4.9876e-14$
119 ($m^* = 801$)	$1.0485e-7$	$4.6074e-15$	$3.7248e-14$

Table 2: Example 4.2

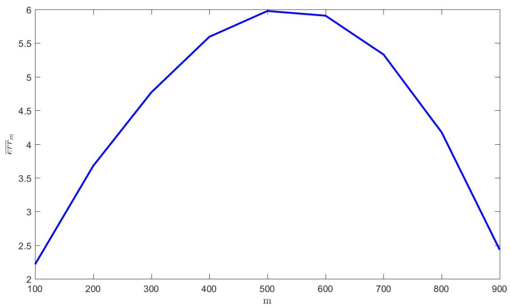


Figure 1: Example 4.2: Graph of $\overline{err}_m$

Example 4.3. Consider the integral

$$\mathcal{H}_3(fw_0,t) = \int_0^{+\infty} \frac{\left|x-\frac{1}{4}\right|^{9.4}}{(x-t)^4} e^{-x}dx, \quad f(x) = \left|x-\frac{1}{4}\right|^{9.4}, \quad \alpha = 0, \quad p = 3.$$

In this case the function f belongs to $Z_{9.4}(w_0)$ and, according to the estimate (11), the error behaves like $\log m/m^{3.2}$. As Table 3 shows, the worst results are achieved when t approaches the point $\frac{1}{4}$ where $f^{(10)}$ is singular. In Table 4, we report the estimated order of convergence

$$EOC_m = \frac{\log(\widetilde{err}_m/\widetilde{err}_{2m})}{\log 2},$$

where

$$\widetilde{err}_m = \max_{i=1,\dots,40} \bar{e}_{3,m^*}(fw_0, y_i), \quad \{y_i\}_{i=1,\dots,40} \in [0,10],$$

for increasing values of m, and in Figure 2 we display its graphical behavior. As one can see the numerical convergence order agrees with the theoretical one.

j	$\bar{e}_{3,m^*}(fw_0, 0.025)$	$\bar{e}_{3,m^*}(fw_0, 0.2499999999)$	$\bar{e}_{3,m^*}(fw_0, 1)$
53 ($m^* = 100$)	$7.0854e-10$	$2.5554e-8$	$8.4070e-11$
75 ($m^* = 201$)	$2.9609e-10$	$2.5907e-9$	$9.1565e-13$
106 ($m^* = 400$)	$1.0921e-11$	$1.1180e-10$	$8.7260e-14$
130 ($m^* = 600$)	$1.3145e-12$	$9.5558e-11$	–
150 ($m^* = 801$)	$1.6816e-13$	$3.4475e-12$	–

Table 3: Example 4.3

m	EOC_m
8	3.5843
16	3.8797
32	2.8256
64	3.2699
128	3.3143
256	3.2159
512	3.2914

Table 4: Example 4.3

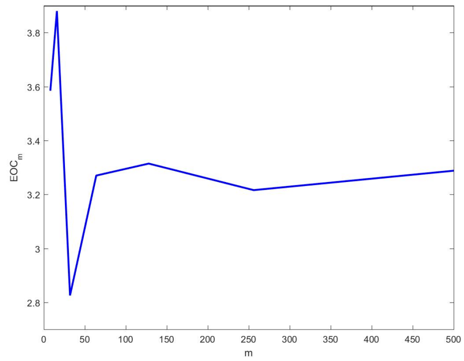


Figure 2: Example 4.3: Graph of EOC_m

5. The proofs

Denoting by

$$E_m(f)_{w_\alpha} = \inf_{P \in \mathbb{P}_m} \|(f-P)w_\alpha\|_\infty$$

the error of best polynomial approximation in C_{w_α}, the following Stechkin and Jackson inequalities hold true (see [4] and also [20])

$$\omega_\varphi^k(f,t)_{w_\alpha} \leq Ct^k \sum_{i=0}^{\lfloor \frac{1}{t^2} \rfloor} (1+i)^{\frac{k}{2}-1} E_i(f)_{w_\alpha}, \quad 0 < C \neq C(m,f), \tag{15}$$

$$E_m(f)_{w_\alpha} \leq C\omega_\varphi^k \left(f, \frac{1}{\sqrt{m}}\right)_{w_\alpha}, \quad 0 < C \neq C(m,f). \tag{16}$$

Proof of Lemma 2.1 Let $P_m \in \mathbb{P}_m$. We have

$$\int_0^{\frac{1}{\sqrt{m}}} \frac{\omega_\varphi(f-P_m,u)_{w_\alpha}}{u} du = \sum_{j=m}^{\infty} \int_{\frac{1}{\sqrt{j+1}}}^{\frac{1}{\sqrt{j}}} \frac{\omega_\varphi(f-P_m,u)_{w_\alpha}}{u} du \leq C \sum_{j=m}^{\infty} \frac{\omega_\varphi\left(f-P_m, \frac{1}{\sqrt{j}}\right)_{w_\alpha}}{j}$$

and, using (15), we get

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{m}}} \frac{\omega_\varphi(f-P_m,u)_{w_\alpha}}{u}du &\le C\sum_{j=m}^{\infty}\frac{1}{j^{\frac{3}{2}}}\sum_{i=0}^{j}\frac{E_i(f-P_m)_{w_\alpha}}{\sqrt{1+i}}\\
&= C\sum_{j=m}^{\infty}\frac{1}{j^{\frac{3}{2}}}\left[\sum_{i=0}^{m-1}\frac{E_i(f-P_m)_{w_\alpha}}{\sqrt{1+i}}+\sum_{i=m}^{j}\frac{E_i(f-P_m)_{w_\alpha}}{\sqrt{1+i}}\right]\\
&\le C\|(f-P_m)w_\alpha\|_\infty\left(\sum_{j=m}^{\infty}\frac{1}{j^{\frac{3}{2}}}\right)\left(\sum_{i=0}^{m-1}\frac{1}{\sqrt{1+i}}\right)\\
&+ C\sum_{j=m}^{\infty}\frac{1}{j^{\frac{3}{2}}}\sum_{i=m}^{j}\frac{E_i(f)_{w_\alpha}}{\sqrt{1+i}}\\
&\le C\|(f-P_m)w_\alpha\|_\infty + C\sum_{j=m}^{\infty}\frac{1}{j^{\frac{3}{2}}}\sum_{i=m}^{j}\frac{E_i(f)_{w_\alpha}}{\sqrt{i}}.
\end{aligned}$$

Concerning the second addendum, applying (16), we have

$$\begin{aligned}
\sum_{j=m}^{\infty}\frac{1}{j^{\frac{3}{2}}}\sum_{i=m}^{j}\frac{E_i(f)_{w_\alpha}}{\sqrt{i}} &= \sum_{i=m}^{\infty}\frac{E_i(f)_{w_\alpha}}{\sqrt{i}}\sum_{j=i}^{\infty}\frac{1}{j^{\frac{3}{2}}} \le C\sum_{i=m}^{\infty}\frac{E_i(f)_{w_\alpha}}{i}\\
&\le C\sum_{i=m}^{\infty}\int_{\frac{1}{\sqrt{i+1}}}^{\frac{1}{\sqrt{i}}}\frac{\omega_\varphi^r(f,t)_{w_\alpha}}{u}du = C\int_0^{\frac{1}{\sqrt{m}}}\frac{\omega_\varphi^r(f,u)_{w_\alpha}}{u}du.
\end{aligned}$$

Thus the proof is complete. □

In order to prove Theorems 3.1, 3.2 and 3.3, we premise some lemmas.

Lemma 5.1. *Let $\alpha \ge 0$, $p \ge 1$ and $t > 0$. If $f \in C_{w_\alpha}$ we get*

$$\left|\int_{|x-t|\ge 1}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx\right| \le C\|fw_\alpha\|,$$

where $0 < C \ne C(t,f)$.

Proof. Setting $x - t = ut$, for any $t > 0$, we have

$$\left|\int_{t+1}^{\infty}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx\right| \le C\frac{\|fw_\alpha\|_\infty}{t^p}\int_{\frac{1}{t}}^{+\infty}\frac{du}{u^{p+1}} \le C\|fw_\alpha\|_\infty$$

and, therefore, the lemma follows for $0 < t < 1$. In the case $t > 1$ we have to estimate also the term

$$\left|\int_0^{t-1}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx\right| \le C\frac{\|fw_\alpha\|_\infty}{t^p} \le C\|fw_\alpha\|_\infty.$$

□

Lemma 5.2. *[8, Lemma 6.1] For $\alpha \ge 0$, we have for $0 < t \le 1$*

$$\left|\fint_{|x-t|<1}\frac{w_\alpha(x)}{(x-t)^{p+1}}dx\right| \le Cw_\alpha(t)\begin{cases}t^{-p} & p \ge 1\\ \log t^{-1} & p = 0\end{cases},\quad 0 < C \ne C(t).$$

Lemma 5.3. *For any function* $f \in W_p(w_\alpha)$,

$$\left|\sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \fint_{|x-t|<1} \frac{w_\alpha(x)}{(x-t)^{p-k+1}} dx\right| \leq \frac{C}{t^p} \|f\|_{W_p(w_\alpha)},$$

where $0 < C \neq C(f)$.

Proof. By Lemma 5.2 and taking into account

$$\sum_{j=0}^{r} a_j \|f^{(j)}\varphi^{(j)} w_\alpha\|_\infty \leq C\left(\|f w_\alpha\|_\infty + \|f^{(r)}\varphi^{(r)} w_\alpha\|_\infty\right), \tag{17}$$

it follows that

$$\begin{aligned}\left|\sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \fint_{|x-t|<1} \frac{w_\alpha(x)}{(x-t)^{p-k+1}} dx\right| &\leq C\left(\sum_{k=0}^{p-1} \frac{\|f^{(k)}\varphi^k w_\alpha\|_\infty}{t^{p-\frac{k}{2}}} + \log t^{-1} \frac{\|f^{(p)}\varphi^p w_\alpha\|_\infty}{t^{\frac{p}{2}}}\right) \\ &\leq \frac{C}{t^p} \sum_{k=0}^{p} \|f^{(k)}\varphi^k w_\alpha\|_\infty \leq \frac{C}{t^p} \|f\|_{W_p(w_\alpha)}. \qquad \square\end{aligned}$$

Denoting by

$$R_p(f, x, t) := f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} (x-t)^k,$$

the Taylor's remainder term, we recall its Peano form

$$R_p(f, x, t) = \frac{1}{(p-1)!} \int_t^x [f^{(p)}(\tau) - f^{(p)}(t)](x-\tau)^{p-1} d\tau. \tag{18}$$

Lemma 5.4. *Let* $\alpha \geq 0$, $p \geq 1$ *and* $t > 0$. *If*

$$\int_0^1 \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha \varphi^p}}{u} du < \infty,$$

then

$$\left|\int_{|x-t|<1} \frac{R_p(f, x, t)}{(x-t)^{p+1}} w_\alpha(x) dx\right| \leq C\left(\frac{1}{\sqrt{t^p}} \int_0^1 \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha \varphi^p}}{u} du + \frac{1}{t^p} \|f\|_{W_p(w_\alpha)}\right),$$

where $0 < C \neq C(t, f)$.

Proof. We first assume $0 < t < 1$ and use the following decomposition

$$\begin{aligned}\int_{|x-t|<1} \frac{R_p(f, x, t)}{(x-t)^{p+1}} w_\alpha(x) dx &= \int_0^{2t} \frac{R_p(f, x, t)}{(x-t)^{p+1}} w_\alpha(x) dx + \int_{2t}^{t+1} \frac{f(x)}{(x-t)^{p+1}} w_\alpha(x) dx \\ &- \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \int_{2t}^{t+1} \frac{w_\alpha(x)}{(x-t)^{p-k+1}} dx \\ &=: I_1(t) + I_2(t) + I_3(t).\end{aligned} \tag{19}$$

Using (18) we get

$$\begin{aligned}I_1(t) &= \frac{1}{(p-1)!} \int_0^t \left[\int_x^t [f^{(p)}(t) - f^{(p)}(\tau)](\tau - x)^{p-1} d\tau\right] \frac{w_\alpha(x)}{(t-x)^{p+1}} dx \\ &+ \frac{1}{(p-1)!} \int_t^{2t} \left[\int_t^x [f^{(p)}(\tau) - f^{(p)}(t)](x-\tau)^{p-1} d\tau\right] \frac{w_\alpha(x)}{(x-t)^{p+1}} dx\end{aligned}$$

and, by the changes of variables $x = t - u\sqrt{t}$, $\tau = t - z\sqrt{t}$ in the first integral and $x = t + u\sqrt{t}$, $\tau = t + z\sqrt{t}$ in the second integral, we get

$$\begin{aligned} I_1(t) &= \frac{1}{(p-1)!}\int_0^{\sqrt{t}}\left[\int_0^u [f^{(p)}(t) - f^{(p)}(t - z\sqrt{t})](u-z)^{p-1}dz\right]\frac{w_\alpha(t - u\sqrt{t})}{u^{p+1}}du \\ &+ \frac{1}{(p-1)!}\int_0^{\sqrt{t}}\left[\int_0^u [f^{(p)}(t + z\sqrt{t}) - f^{(p)}(t)](u-z)^{p-1}dz\right]\frac{w_\alpha(t + u\sqrt{t})}{u^{p+1}}du. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |I_1(t)| &\le C\int_0^{\sqrt{t}}\frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha\varphi^p}}{u}\left[\frac{w_\alpha(t - u\sqrt{t})}{w_\alpha(t)\varphi^p(t)} + \frac{w_\alpha(t + u\sqrt{t})}{w_\alpha(t)\varphi^p(t)}\right]du \\ &\le \frac{C}{\varphi^p(t)}\int_0^1\frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha\varphi^p}}{u}du. \end{aligned} \tag{20}$$

Moreover, since $x - t \ge \frac{x}{2}$ we get

$$|I_2(t)| \le C\|fw_\alpha\|_\infty\int_{2t}^{t+1}\frac{dx}{x^{p+1}} \le \frac{C}{t^p}\|fw_\alpha\|_\infty \tag{21}$$

and using Lemma 5.3 we obtain

$$|I_3(t)| \le \frac{C}{t^p}\|f\|_{W_p(w_\alpha)}. \tag{22}$$

Combining (20), (21), (22) with (19), the thesis follows for $0 < t < 1$. In the case $t \ge 1$, by similar arguments used in the previous case, we get

$$\left|\int_{|x-t|<1}\frac{R_p(f, x, t)}{(x-t)^{p+1}}w_\alpha(x)dx\right| = \left|\int_{t-1}^{t+1}\frac{R_p(f, x, t)}{(x-t)^{p+1}}w_\alpha(x)dx\right| \le \frac{C}{\varphi^p(t)}\int_0^1\frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha\varphi^p}}{u}du$$

and the lemma is completely proved.

□

Proof of Theorem 3.1 By

$$\begin{aligned} \left|\int_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx\right| &\le \left|\int_{|x-t|\ge 1}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx\right| + \left|\int_{|x-t|<1}\frac{R_p(f, x, t)}{(x-t)^{p+1}}w_\alpha(x)dx\right| \\ &+ \sum_{k=0}^{p}\frac{|f^{(k)}(t)|}{k!}\left|\int_{|x-t|<1}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx\right|, \end{aligned}$$

the theorem easily follows by using Lemmas 5.1, 5.4 and 5.3.

□

In what follows, for the sake of simplicity, will write $\{x_k\}_{k=1}^{m*}$ instead of $\{x_{m^*,k}\}_{k=1}^{m^*}$ to denote the zeros of $p_{m^*}(w_\alpha)$ and $\{\lambda_k\}_{k=1}^{m*}$ instead of $\{\lambda_{m^*,k}\}_{k=1}^{m^*}$.

Lemma 5.5. *Let $\alpha \ge 0$, $p \ge 1$ and $t > 0$. If $f \in C_{w_\alpha}$ we obtain*

$$\left|\sum_{|x_i-t|\ge 1}\frac{f(x_i)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\right| \le C\|fw_\alpha\|_\infty,$$

where $0 < C \ne C(t, f)$.

Proof. Assume $0 < t < 1$. Recalling that

$$\lambda_{m,i}(w_\alpha) \sim \Delta x_{m,i} w_\alpha(x_{m,i}), \qquad i = 1, 2, \dots m, \tag{23}$$

we have

$$\begin{aligned}
\left|\sum_{|x_i-t|\geq 1} \frac{f(x_i)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\right| &\leq C\|fw_\alpha\|_\infty \sum_{t+1<x_i<x_j} \frac{\Delta x_i}{(x_i-t)^{p+1}} \leq C\|fw_\alpha\|_\infty \int_{t+1}^{+\infty} \frac{dx}{(x-t)^{p+1}} \\
&\leq \frac{C}{t^p}\|fw_\alpha\|_\infty \int_{\frac{1}{t}}^{+\infty} \frac{du}{u^{p+1}} \leq C\|fw_\alpha\|_\infty.
\end{aligned}$$

The case $t > 1$ follows by similar arguments.

□

Lemma 5.6. *Let $\alpha \geq 0$, $p \geq 1$, $t > 0$ fixed, and d defined in (8). For any function f satisfying* (6), *we have*

$$\left|\sum_{i=d-1}^{d+1} \frac{f(x_{m^*,i}) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x_{m^*,i}-t)^k}{(x_{m^*,i}-t)^{p+1}} \lambda_{m^*,i}\right| \leq C\frac{1}{\varphi^p(t)} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi(f^{(p)},u)_{w_\alpha\varphi^p}}{u} du,$$

where $0 < C \neq C(t, f)$.

Proof. We assume $t > x_d$, $d \geq 1$. By (18)

$$\begin{aligned}
|R_p(f, x_d, t)| &= \left|\frac{1}{(p-1)!}\int_{x_d}^t [f^{(p)}(\tau) - f^{(p)}(t)](x_d-\tau)^{p-1} d\tau\right| \\
&\leq \frac{\sqrt{t}}{(p-1)!} \int_0^{\frac{t-x_d}{\sqrt{t}}} |f^{(p)}(t) - f^{(p)}(t - z\sqrt{t})|(t - x_d - z\sqrt{t})^{p-1} dz \\
&= \frac{\sqrt{t}}{(p-1)! w_\alpha(t)\varphi^p(t)} \int_0^{\frac{t-x_d}{\sqrt{t}}} |\Delta_{z\varphi} f^{(p)}(t) w_\alpha(t)\varphi^p(t)|(t - x_d - z\sqrt{t})^{p-1} dz.
\end{aligned}$$

Then, since by (8) it is $t - x_d \sim \frac{\sqrt{x_d}}{\sqrt{m}} \sim \frac{\sqrt{t}}{\sqrt{m}}$, we obtain

$$\begin{aligned}
\left|R_p(f, x_d, t)\right| &\leq C\frac{\sqrt{t}}{w_\alpha(t)\varphi^p(t)} \int_0^{\frac{(t-x_d)}{\sqrt{t}}} \sup_{0<z<\frac{1}{\sqrt{m}}} \|(\Delta_{z\varphi} f^{(p)}) w_\alpha \varphi^p\|_\infty (t - x_d - z\sqrt{t})^{p-1} dz \\
&\leq \frac{C}{w_\alpha(t)\varphi^p(t)} \Omega_\varphi\left(f^{(p)}, \frac{1}{\sqrt{m}}\right)_{w_\alpha\varphi^p} (t - x_d)^p.
\end{aligned}$$

Therefore by (23) we have

$$\frac{\left|R_p(f, x_d, t)\right| \lambda_d}{(t-x_d)^{p+1}} \leq C\frac{w_\alpha(x_d)}{w_\alpha(t)\varphi^p(t)} \Omega_\varphi\left(f^{(p)}, \frac{1}{\sqrt{m}}\right)_{w_\alpha\varphi^p} \frac{\Delta x_d}{(t-x_d)}.$$

Moreover, using [4, Lemma 4.1] $w_\alpha(x_d) \sim w_\alpha(t)$, $\frac{\Delta x_d}{(t-x_d)} \sim 1$ and taking into account

$$\Omega_\varphi\left(f^{(p)}, \frac{1}{\sqrt{m}}\right)_{w_\alpha\varphi^p} \leq C\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi\left(f^{(p)}, t\right)_{w_\alpha\varphi^p}}{t} dt,$$

we obtain

$$\frac{\left|R_p(f,x_d,t)\right|\lambda_d}{(t-x_d)^{p+1}} \le \frac{C}{\varphi^p(t)}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi\left(f^{(p)},t\right)_{w_\alpha\varphi^p}}{t}dt.$$

Since by similar arguments

$$\begin{aligned}\frac{\left|R_p(f,x_{d+1},t)\right|\lambda_{d+1}}{(x_{d+1}-t)^{p+1}} &\le \frac{C}{\varphi^p(t)}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi\left(f^{(p)},t\right)_{w_\alpha\varphi^p}}{t}dt\\ \frac{\left|R_p(f,x_{d-1},t)\right|\lambda_{d-1}}{(x_{d-1}-t)^{p+1}} &\le \frac{C}{\varphi^p(t)}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi\left(f^{(p)},t\right)_{w_\alpha\varphi^p}}{t}dt,\end{aligned}$$

the thesis follows. We omit the case $t < x_d$, since it follows by similar arguments.

□

Lemma 5.7. *Let* $\alpha \ge 0$, $p \ge 1$ *and* $t > 0$. *If*

$$\int_0^1 \frac{\Omega_\varphi(f^{(p)},u)_{w_\alpha\varphi^p}}{u}du < \infty,$$

then

$$\left|\sum_{|x_i-t|<1}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\right| \le C\left(\frac{1}{\sqrt{t^p}}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi(f^{(p)},u)_{w_\alpha\varphi^p}}{u}du + \frac{1}{t^p}\|f\|_{W_p(w_\alpha)}\right),$$

where $0 < C \ne C(t,f)$.

Proof. Assume $0 < t \le 1$. We have

$$\begin{aligned}\sum_{0\le x_i\le t+1}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha) &= \sum_{\substack{0\le x_i\le t\\ i\ne d-1,d,d+1}}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha) + \sum_{\substack{t\le x_i\le 2t\\ i\ne d-1,d,d+1}}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\\ &+ \sum_{2t\le x_i\le t+1}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha) + \sum_{i=d-1}^{d+1}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\\ &=: A(t)+B(t)+C(t)+D(t).\end{aligned}\tag{24}$$

We first estimate $A(t)$. By (18) we have

$$\begin{aligned}|R_p(f,x_i,t)| &= \frac{1}{(p-1)!}\left|\int_{x_i}^t [f^{(p)}(\tau)-f^{(p)}(t)](x_i-\tau)^{p-1}d\tau\right|\\ &\le \frac{\sqrt{t}}{(p-1)!}\int_0^{\frac{t-x_i}{\sqrt{t}}}|f^{(p)}(t)-f^{(p)}(t-z\sqrt{t})|(t-x_i-z\sqrt{t})^{p-1}dz\\ &\le \frac{C}{w_\alpha(t)\varphi^p(t)}\Omega_\varphi\left(f^{(p)},\frac{t-x_i}{\sqrt{t}}\right)_{w_\alpha\varphi^p}(t-x_i)^p.\end{aligned}$$

Therefore by (23), we get

$$|A(t)| \le \frac{C}{\varphi^p(t)}\sum_{0\le x_i\le t}\frac{w_\alpha(x_i)}{w_\alpha(t)}\Omega_\varphi\left(f^{(p)},\frac{t-x_i}{\sqrt{t}}\right)_{w_\alpha\varphi^p}\frac{\Delta x_i}{(t-x_i)}$$

and, taking into account that [3] $\frac{w_\alpha(x_i)}{w_\alpha(t)} \le C$, $\frac{\Delta x_i}{(x_i-t)} \le C$, $\frac{t-x_i}{\sqrt{t}} \le \sqrt{t} \le C\sqrt{x_d} \sim \frac{1}{\sqrt{m}}$ and that

$$\Omega_\varphi\left(f^{(p)}, \frac{1}{\sqrt{m}}\right)_{w_\alpha\varphi^p} \quad \le \quad C\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi\left(f^{(p)}, t\right)_{w_\alpha\varphi^p}}{t}dt,$$

we obtain

$$|A(t)| \le \frac{C}{\varphi^p(t)}\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi\left(f^{(p)}, t\right)_{w_\alpha\varphi^p}}{t}dt. \tag{25}$$

Similarly proceeding we get

$$|B(t)| \le \frac{C}{\varphi^p(t)}\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi\left(f^{(p)}, t\right)_{w_\alpha\varphi^p}}{t}dt. \tag{26}$$

By (23) and (17), we have

$$\begin{aligned}
|C(t)| &\le \sum_{2t\le x_i\le t+1} \frac{|f(x_i)|}{(x_i-t)^{p+1}}\lambda_i(w_\alpha) + \sum_{k=0}^{p}\frac{|f^{(k)}(t)|}{k!}\sum_{2t\le x_i\le t+1}\frac{\lambda_i(w_\alpha)}{(x_i-t)^{p-k+1}}\\
&\le C\left(\|fw_\alpha\|_\infty \sum_{2t\le x_i\le t+1}\frac{\Delta x_i}{(x_i-t)^{p-k+1}} + \sum_{k=0}^{p}\frac{\|f^{(k)}w_\alpha\varphi^k\|_\infty}{\varphi^k(t)w_\alpha(t)}\sum_{2t\le x_i\le t+1}\frac{\lambda_i(w_\alpha)}{(x_i-t)^{p-k+1}}\right)\\
&\le C\left(\|fw_\alpha\|_\infty\int_{2t}^{t+1}\frac{dx}{(x-t)^{p+1}} + \sum_{k=0}^{p}\frac{\|f^{(k)}w_\alpha\varphi^k\|_\infty}{\varphi^k(t)}\int_{2t}^{t+1}\frac{dx}{(x-t)^{p-k+1}}\right)\\
&\le C\left(\frac{1}{t^p}\sum_{k=0}^{p-1}\|f^{(k)}w_\alpha\varphi^k\|_\infty + \frac{\log t^{-1}}{\sqrt{t^p}}\|f^{(p)}w_\alpha\varphi^p\|_\infty\right)\\
&\le \frac{C}{t^p}\left(\|fw_\alpha\| + \|f^{(p)}\varphi^p w_\alpha\|_\infty\right).
\end{aligned} \tag{27}$$

Moreover, by Lemma 5.6, we get

$$|D(t)| \le C\frac{1}{\varphi^p(t)}\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha\varphi^p}}{u}du. \tag{28}$$

Finally, combining (25), (26), (27), (28) with (24) the thesis follows for $0 < t < 1$. In the case $t \ge 1$,

$$\sum_{|x_i-t|<1}\frac{R_p(f, x_i, t)}{(x_i-t)^{p+1}}\lambda_i = \sum_{t-1\le x_i\le t+1}\frac{R_p(f, x_i, t)}{(x_i-t)^{p+1}}\lambda_i$$

and the lemma follows by arguments similar to those used in the previous case.

□

Lemma 5.8. *With $\alpha \ge 0$, $p \ge 1$, $k \le p$, for any $t > 0$ we have*

$$|B_{m,k}(t)| := \left|\int_{|x-t|\ge 1}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx - \sum_{|x_i-t|\ge 1}\frac{\lambda_i}{(x_i-t)^{p-k+1}}\right| \quad \le \quad Cw_\alpha(t), \tag{29}$$

where $0 < C \ne C(t, f)$.

Proof. Since $|x-t|, |x_i-t| \geq 1$, we have for any $t>0$

$$\begin{aligned}\left|\int_{t+1}^{\infty}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx-\sum_{t+1<x_i<x_j}\frac{\lambda_i}{(x_i-t)^{p-k+1}}\right| &\leq \int_{t+1}^{\infty} w_\alpha(x)dx+\sum_{t+1<x_i<x_j}\lambda_i\\ &\leq C\int_{t+1}^{\infty} w_\alpha(x)dx \leq C w_\alpha(t).\end{aligned}$$

For $0<t<1$ the proof is complete. In order to complete the proof for $t>1$, we have to estimate

$$\begin{aligned}\int_0^{t-1}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx-\sum_{1\leq x_i\leq t-1}\frac{\lambda_i(w_\alpha)}{(x_i-t)^{p-k+1}} &= \int_0^{x_q}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx-\sum_{1\leq x_i\leq x_q}\frac{\lambda_i(w_\alpha)}{(x_i-t)^{p-k+1}}\\ &+ \int_{x_q}^{t-1}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx, \qquad (30)\end{aligned}$$

where $x_q=\max\{x_k : x_k\leq t-1\}$.

To this end, we recall the Posse-Markov-Stieltjes inequality [12, p.33]

$$\sum_{k=1}^{d-1}\lambda_{m,k}g(x_k)\leq \int_0^{x_d} g(x)w_\alpha(x)dx \leq \sum_{k=1}^{d}\lambda_{m,k}g(x_k), \qquad (31)$$

which holds true for any function g s.t. $g^{(k)}(x)\geq 0, k=0,1,\dots 2m-1, m>1$, for $x\in(0,x_d)$, $d=2,3,\dots,m$. Thus, by (31) with $g(x)=\frac{1}{(t-x)^{p-k+1}}$ and $d=q$, we have

$$0\leq \int_0^{x_q}\frac{w_\alpha(x)}{(t-x)^{p-k+1}}dx-\sum_{i=1}^{q-1}\frac{\lambda_i}{(t-x_i)^{p-k+1}}\leq \frac{\lambda_q}{(t-x_q)^{p-k+1}}\leq C w_\alpha(t), \qquad (32)$$

being $w_\alpha(x_q)\leq C w_\alpha(t)$, $t-x_q\geq 1$, and $\frac{\Delta x_q}{t-x_q}\leq C$.

Finally, since

$$\int_{x_q}^{t+1}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx\leq w_\alpha(x_q)\int_{x_q}^{t+1}\frac{dx}{(x-t)^{p-k+1}}\leq C w_\alpha(t),$$

combining last inequality and (32) with (30), the lemma follows also for $t\geq 1$.

□

Proof of Theorem 3.2 We can write

$$\begin{aligned}|\mathcal{H}_{p,m^*}(fw_\alpha,t)| &= \left|\mathcal{F}_{p,m^*}(fw_\alpha,t)+\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\int_0^{\infty}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx\right|\\ &\leq \left|\sum_{|x_i-t|<1}\frac{R_p(f,x_i,t)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\right|+\left|\sum_{|x_i-t|\geq 1}\frac{f(x_i)}{(x_i-t)^{p+1}}\lambda_i(w_\alpha)\right|\\ &+ \sum_{k=0}^{p}\frac{|f^{(k)}(t)|}{k!}\left|B_{m,k}(t)\right|+\sum_{k=0}^{p}\frac{|f^{(k)}(t)|}{k!}\left|\int_{|x-t|<1}\frac{w_\alpha(x)}{(x-t)^{p-k+1}}dx\right|,\end{aligned}$$

with $B_{m,k}(t)$ defined in (29). Using Lemmas 5.7, 5.5, 5.8, 5.3 and estimate (17), the thesis easily follows.

□

In order to prove Theorem 3.3 we recall the weaker version of the Jackson inequality [4, Corollary 3.6]

$$E_m(f)_{w_\alpha} \leq C\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^k(f,u)_{w_\alpha}}{u}du, \quad 0<C\neq C(m,f), \tag{33}$$

and that, for any $f\in W_r(w_\alpha)$, (see [4] and also [20])

$$E_m(f)_{w_\alpha} \leq \frac{C}{\sqrt{m^r}}E_{m-r}(f^{(r)})_{w_\alpha\varphi^r}, \quad 0<C\neq C(m,f), \tag{34}$$

and, for any $P\in\mathbb{P}_m$, (see [18, Lemma 3.6])

$$\|(f-P)^{(r)}w_\alpha\varphi^r\| \leq C\left[\sqrt{m^r}\|(f-P)w_\alpha\|_\infty + E_{m-r}(f^{(r)})_{w_\alpha\varphi^r}\right], \quad 0<C\neq C(m,f). \tag{35}$$

Moreover, since for functions belonging to $Z_\lambda(w_\alpha), 0<\lambda<1$, we have [4, p. 189]

$$\omega_\varphi^r(f,t)_{w_\alpha\varphi^p} \sim \Omega_\varphi^r(f,t)_{w_\alpha\varphi^p},$$

as a consequence of Lemma 2.1, (33), (34) and (35), under the assumption $f^{(p)}\in Z_\lambda(w_\alpha\varphi^p)$, $0<\lambda<1$ and $p\geq 1$, we deduce

$$E_m(f)_{w_\alpha} \leq \frac{C}{\sqrt{m^p}}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi^r(f^{(p)},u)_{w_\alpha\varphi^p}}{u}du \leq \frac{C}{\sqrt{m^{p+\lambda}}}\|f^{(p)}\|_{Z_\lambda(w_\alpha\varphi^p)}, \tag{36}$$

$$\|(f-P)^{(p)}w_\alpha\varphi^p\|_\infty \leq C\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi^r(f^{(p)},u)_{w_\alpha\varphi^p}}{u}du \leq \frac{C}{\sqrt{m^\lambda}}\|f^{(p)}\|_{Z_\lambda(w_\alpha\varphi^p)}, \tag{37}$$

and

$$\int_0^1 \frac{\Omega_\varphi((f-P)^{(p)},u)_{w_\alpha\varphi^p}}{u}du \leq C\log m\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi^r(f^{(p)},u)_{w_\alpha\varphi^p}}{u}du \leq C\frac{\log m}{\sqrt{m^\lambda}}\|f^{(p)}\|_{Z_\lambda(w_\alpha\varphi^p)}, \tag{38}$$

where in all the cases $0<C\neq C(m,f)$.

Proof of Theorem 3.3 Let $0<\theta<1$ be fixed and let $P\in\mathbb{P}_M$ be the polynomial of best approximation of f in C_{w_α}, with $M=\left\lfloor m\frac{\theta}{1+\theta}\right\rfloor$. With j defined in (3), we have

$$\begin{aligned}\mathcal{H}_p(fw_\alpha,t)-\mathcal{H}_{p,m^*}(fw_\alpha,t) &= \mathcal{H}_p((f-P)w_\alpha,t)-\mathcal{H}_{p,m^*}((f-P)w_\alpha,t)+\sum_{i=j+1}^{m^*}\frac{P(x_i)-\sum_{k=0}^p\frac{P^{(k)}(t)}{k!}(x_i-t)^k}{(x_i-t)^{p+1}}\lambda_{m,i}\\ &=: S_1(t)+S_2(t)+S_3(t).\end{aligned} \tag{39}$$

By Theorems 3.1 and 3.2 we get

$$|S_1(t)|+|S_2(t)| \leq \frac{C}{t^p}\int_0^1\frac{\Omega_\varphi((f-P)^{(p)},u)_{w_\alpha\varphi^p}}{u}du+\frac{C}{t^p}\left\{\|(f-P)w_\alpha\|_\infty+\|(f-P)^{(p)}w_\alpha\varphi^p\|_\infty\right\}$$

and taking into account (36), (37) and (38), we deduce

$$|S_1(t)|+|S_2(t)| \leq \frac{C}{t^p}\frac{\log m}{\sqrt{m^\lambda}}\|f^{(p)}\|_{Z_\lambda(w_\alpha\varphi^p)}. \tag{40}$$

Consider now $S_3(t)$. We have

$$|S_3(t)| \leq \sum_{i=j+1}^{m^*}\frac{|P(x_i)|}{(x_i-t)^{p+1}}\lambda_{m,i}+\sum_{k=0}^p\frac{|P^{(k)}(t)|}{k!}\sum_{i=j+1}^{m^*}\frac{\lambda_{m,i}}{(x_i-t)^{p+1-k}} \tag{41}$$

and, by (23),

$$\sum_{i=j+1}^{m^*} \frac{|P(x_i)|}{(x_i-t)^{p+1}}\lambda_{m,i} \le \|Pw_\alpha\|_{[x_j,+\infty)} \sum_{i=j+1}^{m^*} \frac{\Delta x_i}{(x_i-t)^{p+1}}.$$

Moreover, recalling the following estimate [19, (4) p.590],

$$\max_{x\ge 4(m+1+\alpha)(1+\theta)} |P(x)w_\alpha(x)| \le Ce^{-AM} \max_{0\le x\le 4m\theta} |P(x)w_\alpha(x)|, \tag{42}$$

with C and A positive constants independent of P and m and depending on θ, we deduce

$$\sum_{i=j+1}^{m^*} \frac{|P(x_i)|}{(x_i-t)^{p+1}}\lambda_{m,i} \le Ce^{-Am}\|Pw_\alpha\|_{[0,4m\theta]} \int_{x_{j+1}}^{+\infty} \frac{dx}{(x-t)^{p+1}} \le Ce^{-Am}\|Pw_\alpha\|_\infty. \tag{43}$$

To estimate the second summation in (41), assume at first $t \le \frac{x_{j+1}}{2}$. We have $\frac{x_i-t}{2} > \frac{x_i}{4} > \frac{x_{j+1}}{4} \sim m\theta$ and therefore, by (23),

$$\begin{aligned}
\sum_{k=0}^{p} \frac{|P^{(k)}(t)|}{k!} \sum_{i=j+1}^{m^*} \frac{\lambda_{m,i}}{(x_i-t)^{p+1-k}} &\le C\sum_{k=0}^{p} \frac{|P^{(k)}(t)\varphi^k(t)w_\alpha(t)|}{k!t^{\alpha+\frac{k}{2}}} \sum_{i=j+1}^{m^*} \frac{\Delta x_i e^{-(x_i-t)}x_i^\alpha}{(x_i-t)^{p+1-k}} \\
&\le Ce^{-m\theta}\sum_{k=0}^{p} \frac{\|P^{(k)}\varphi^k w_\alpha\|_{[t,+\infty)}}{t^{\alpha+\frac{k}{2}}} \int_{x_{j+1}}^{4m} \frac{x^\alpha e^{-\frac{x}{4}}}{(x-t)^{p+1-k}}dx \\
&\le Ce^{-m\theta}\sum_{k=0}^{p} \frac{\|P^{(k)}\varphi^k w_\alpha\|_{[t,+\infty)}}{t^{p-\frac{k}{2}}} \int_{2}^{4m} \frac{e^{-\frac{ut}{4}}u^\alpha}{(u-1)^{p+1-k}}du \\
&\le C\frac{e^{-Am}}{t^p}\left\{\|Pw_\alpha\|_\infty + \|P^{(p)}\varphi^p w_\alpha\|_\infty\right\},
\end{aligned} \tag{44}$$

where in the last inequality we used (17). Assume now $t \ge \frac{x_{j+1}}{2}$. Using (23), we get

$$\begin{aligned}
\sum_{k=0}^{p} \frac{|P^{(k)}(t)|}{k!} \sum_{i=j+1}^{m^*} \frac{\lambda_{m,i}}{(x_i-t)^{p+1-k}} &\le C\sum_{k=0}^{p} \frac{\|P^{(k)}\varphi^k w_\alpha\|_{[t,+\infty)}}{k!} \sum_{i=j+1}^{m^*} \frac{\Delta x_i}{t^{\frac{k}{2}}(x_i-t)^{p+1-k}} \\
&\le C\sum_{k=0}^{p} \frac{\|P^{(k)}\varphi^k w_\alpha\|_{[\frac{x_{j+1}}{2},+\infty)}}{t^{\frac{k}{2}}} \sum_{i=j+1}^{m^*} \frac{\Delta x_i}{(x_i-t)^{p+1-k}}.
\end{aligned}$$

Since

$$\sum_{i=j+1}^{m^*} \frac{\Delta x_i}{x_i-t} \le C\log m, \qquad \sum_{i=j+1}^{m^*} \frac{\Delta x_i}{(x_i-t)^{p+1-k}} \le \frac{C}{t^{p-k}}, \quad k \le p-1,$$

taking into account (17) and (42), we can conclude

$$\sum_{k=0}^{p} \frac{|P^{(k)}(t)|}{k!} \sum_{i=j+1}^{m^*} \frac{\lambda_{m,i}}{(x_i-t)^{p+1-k}} \le C\frac{e^{-Am}}{t^p}\left\{\|Pw_\alpha\|_\infty + \|P^{(p)}\varphi^p w_\alpha\|_\infty\right\}.$$

Combining last inequality, (44) and (43) with (41) and taking into account (35), we have

$$|S_3(t)| \le \frac{C}{t^p}e^{-Am}\left\{\|Pw_\alpha\|_\infty + \|P^{(p)}\varphi^p w_\alpha\|_\infty\right\} \le \frac{C}{t^p}e^{-Am}\left\{E_{M-p}(f^{(p)})_{w_\alpha\varphi^p} + \|f\|_{W_p(w_\alpha)}\right\}.$$

The theorem follows combining last inequality and (40) with (39).

□

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