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Certain Family of Integral Operators Associated with Multivalent Functions Preserving Subordination and Superordination

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Abstract. In this paper, we obtain subordination, superordination and sandwich-type results regarding to certain family of integral operators defined on the space of multivalent functions in the open unit disk. Also, an application of the subordination and superordination theorems to the Gauss hypergeometric function are considered. These new results generalize some previously well-known sandwich-type theorems.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ be the class of functions analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Denote $\mathcal{H}_0 = \mathcal{H}[0, 1]$ and $\mathcal{H} = \mathcal{H}[1, 1]$. Also, let $\mathcal{P}$ denote the class of functions

$$\mathcal{P} = \{h \in \mathcal{H}[0, 1] : h(z)h'(z) \neq 0,\ z \in \mathbb{U}^* := \mathbb{U} \setminus \{0\}\},$$

and $\mathcal{A}(p)$ be the class of all functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \ (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic in $\mathbb{U}$. We note that $\mathcal{A}(1) = \mathcal{A}$.

Let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\phi\left(\mathfrak{p}(z), z\mathfrak{p}'(z); z\right) \prec h(z), \tag{2}$$

where "$\prec$" stands for subordination (see [10, 13, 20, 21]), then $\mathfrak{p}(z)$ is a solution of (2). The univalent function $\mathfrak{q}(z)$ is called a dominant of the solutions of (2) if $\mathfrak{p}(z) \prec \mathfrak{q}(z)$ for all $\mathfrak{p}(z)$ satisfying (2). A univalent dominant $\tilde{\mathfrak{q}}$ that satisfies $\tilde{\mathfrak{q}} \prec \mathfrak{q}$ for all dominants of (2) is called the best dominant. If $\mathfrak{p}(z)$ and $\phi\left(\mathfrak{p}(z), z\mathfrak{p}'(z); z\right)$ are univalent in $\mathbb{U}$ and if $\mathfrak{p}(z)$ satisfies

$$h(z) \prec \phi\left(\mathfrak{p}(z), z\mathfrak{p}'(z); z\right), \tag{3}$$

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then $\mathfrak{p}(z)$ is a solution of (3). An analytic function $\mathfrak{q}(z)$ is called a subordinant of the solutions of (3) if $\mathfrak{q}(z) \prec \mathfrak{p}(z)$ for all $\mathfrak{p}(z)$ satisfying (3). A univalent subordinant $\tilde{\mathfrak{q}}$ that satisfies $\mathfrak{q} \prec \tilde{\mathfrak{q}}$ for all subordinants of (3) is called the best subordinant (see [13, 14]).

For $f_i(z) \in \mathcal{A}(p)$ $(i = 1, 2, \ldots, n)$, $h(z) \in \mathcal{P}$ and $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ with $\beta \neq 0$, we introduce the integral operator $I^{p,n}_{h;\alpha_i,\beta} : \mathcal{A}(p) \to \mathcal{A}(p)$ as follows

$$I^{p,n}_{h;\alpha_i,\beta}[f_i](z) = \left(\frac{p\sum_{i=1}^{n}\alpha_i}{z^{p\sum_{i=1}^{n}\alpha_i - p\beta}} \int_0^z \left(\prod_{i=1}^{n} f_i^{\alpha_i}(t)\right) h^{-1}(t)h'(t)dt \right)^{\frac{1}{\beta}}, \tag{4}$$

where all powers are principal ones.

For special casese of of the above defined integral operator (see Srivastava et al. [19], Aouf et al. [1], Cho and Bulboacă [5], Miller et al. [15], Bulboacă [2–4], Cho et al. [6], Cho and Srivastava [7] and Owa and Srivastava [17]).

We recall some definitions which we will be used in our paper.

Definition 1.1. *[13] Denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$ where $E(q) = \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\}$ and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further, denote by $\mathcal{Q}(a)$ the subclass of $\mathcal{Q}$ for which $q(0) = a$.*

Definition 1.2. *[13] A function $L(z,t)$ $(z \in \mathbb{U},\ t \geq 0)$ is a subordination chain if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for all $0 \leq s \leq t$.*

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $h \in \mathcal{P}$, $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ with $\beta \neq 0$ such that $\Re\left(p\sum_{i=1}^{n}\alpha_i - 1\right) > 0$, $z \in \mathbb{U}$ and all powers are principal ones.

Using similar arguments to Lemma 7 in [19], we obtain the following lemma.

Lemma 2.1. *If $f_i \in \mathcal{A}_{p,h;\alpha_i}$,*

$$\mathcal{A}_{p,h;\alpha_i} = \left\{ f_i(z) \in \mathcal{A}(p) : \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} + 1 + \frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)} \prec R_{p\sum_{i=1}^{n}\alpha_i}(z) \right\},$$

then $I^{p,n}_{h;\alpha_i,\beta}[f_i](z) \in \mathcal{A}(p)$, $z^{-p}I^{p,n}_{h;\alpha_i,\beta}[f_i](z) \neq 0$ and

$$\Re\left(\beta \frac{z\left(I^{p,n}_{h,\alpha_i,\beta}[f_i](z)\right)'}{I^{p,n}_{h,\alpha_i,\beta}[f_i](z)} + p\sum_{i=1}^{n}\alpha_i - p\beta \right) > 0,$$

where $I^{p,n}_{h,\alpha_i,\beta}$ is the integral operator defined by (4).

Theorem 2.2. *Let $f, g \in \mathcal{A}_{p,h;\alpha_i}$, and*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad \left(\phi(z) = z\prod_{i=1}^{n}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i} \frac{zh'(z)}{h(z)}\right), \tag{5}$$

where δ is given by

$$\delta = \frac{1 + |a|^2 - \left|1 - a^2\right|}{4\,\Re\{a\}} \quad \left(a = p\sum_{i=1}^{n}\alpha_i - 1,\ \Re\{a\} > 0\right). \tag{6}$$

Then the subordination condition

$$z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\prec\phi(z), \tag{7}$$

implies that

$$z\left(\frac{I_{h;\alpha_i,\beta}^{p,n}[f_i](z)}{z^p}\right)^{\beta}\prec z\left(\frac{I_{h;\alpha_i,\beta}^{p,n}[g_i](z)}{z^p}\right)^{\beta}, \tag{8}$$

and the function $z\left(\frac{I_{h;\alpha_i,\beta}^{p,n}[g_i](z)}{z^p}\right)^{\beta}$ *is the best dominant.*

Proof. Define the functions $\Psi(z)$ and $\Phi(z)$ in $\mathbb{U}$ by

$$\Psi(z)=z\left(\frac{I_{h;\alpha_i,\beta}^{p,n}[f_i](z)}{z^p}\right)^{\beta}\text{ and }\Phi(z)=z\left(\frac{I_{h;\alpha_i,\beta}^{p,n}[g_i](z)}{z^p}\right)^{\beta}. \tag{9}$$

From Lemma 2.1, it follows that these two functions are well defined. We first show that, if

$$q(z)=1+\frac{z\Phi''(z)}{\Phi'(z)}, \tag{10}$$

then $\Re\{q(z)\}>0$. From (4) and the definitions of $\phi(z)$, $\Phi(z)$, we obtain

$$\left(p\sum_{i=1}^{n}\alpha_i\right)\phi(z)=z\Phi'(z)+\left(p\sum_{i=1}^{n}\alpha_i-1\right)\Phi(z). \tag{11}$$

Hence, it follows that

$$1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{zq'(z)}{q(z)+p\sum_{i=1}^{n}\alpha_i-1}=h(z). \tag{12}$$

It follows from (5) and (12) that

$$\Re\left\{h(z)+p\sum_{i=1}^{n}\alpha_i-1\right\}>0. \tag{13}$$

Moreover, by using the result of [12], we conclude that the differential equation (12) has a solution $q(z)\in\mathcal{H}(\mathbb{U})$ with $h(0)=q(0)=1$. Let

$$\boldsymbol{H}(u,v)=u+\frac{v}{u+p\sum_{i=1}^{n}\alpha_i-1}+\delta.$$

From (12) and (13), we obtain $\Re\{\boldsymbol{H}(q(z);zq'(z))\}>0$. To verify the condition

$$\Re\{\boldsymbol{H}(is;t)\}\le 0\ \left(s\in\mathbb{R};\ t\le-\frac{1+s^2}{2}\right), \tag{14}$$

we proceed as follows:

$$\Re\{\boldsymbol{H}(is;t)\}=\Re\left\{is+\frac{t}{is+a}+\delta\right\}=\delta+\frac{t\,\Re\{a\}}{|is+a|^2}\le-\frac{E_\delta(s)}{2|a+is|^2},$$

where

$$E_{\delta}(s) = (\Re\{a\} - 2\delta)s^2 - 4\delta(\Im\{a\})s + \left(\Re\{a\} - 2\delta|a|^2\right). \tag{15}$$

The coefficient of s^2 in the quadratic expression $E_{\delta}(s)$ given by (15) is positive or equal to zero and $E_{\delta}(s) \geq 0$. Thus, we see that $\Re\{\boldsymbol{H}(is;t)\} \leq 0$ for all $s \in \mathbb{R}$ and $t \leq -\frac{1+s^2}{2}$. Thus, by using the fact of [11], we conclude that $\Re\{q(z)\} > 0$, that is, that $\Phi(z)$ defined by (9) is convex (univalent) in $\mathbb{U}$. Next, we prove that the subordination condition (7) implies that $\Psi(z) \prec \Phi(z)$, for $\Psi(z)$ and $\Phi(z)$ defined by (9). Without loss of generality, we assume that $\Phi(z)$ is analytic, univalent on $\overline{\mathbb{U}}$ and $\Phi'(\zeta) \neq 0$ $(|\zeta| = 1)$. If not, then we replace $\Psi(z)$ and $\Phi(z)$ by $\Psi(\rho z)$ and $\Phi(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, so we can use them in the proof of our result and the result would follow by letting $\rho \to 1$. Consider the function $L(z,t)$ given by

$$L(z,t) = \left(1 - \frac{1}{p\sum_{i=1}^{n}\alpha_i}\right)\Phi(z) + \frac{(1+t)}{p\sum_{i=1}^{n}\alpha_i} z\Phi'(z) \ \ (0 \leq t < \infty). \tag{16}$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = \left(1 + \frac{t}{p\sum_{i=1}^{n}\alpha_i}\right)\Phi'(0) \neq 0 \ \ (0 \leq t < \infty).$$

This show that $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, satisfy $\lim\limits_{t\to\infty}|a_1(t)| = \infty$ and $a_1(t) \neq 0$. Further, we have

$$\Re\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} = \Re\left\{p\sum\nolimits_{i=1}^{n}\alpha_i - 1 + (1+t)\left(1 + \frac{z\Phi''(z)}{\Phi'(z)}\right)\right\} > 0,$$

since $\Phi(z)$ is convex and $\Re\left(p\sum_{i=1}^{n}\alpha_i - 1\right) > 0$, by using the well-known growth and distortion sharp inequalities for convex functions (see [8]), the second inequality of the result of [18, p. 159] is satisfied and so $L(z,t)$ is a subordination chain. It follows that $\phi(z) = L(z,0)$ and $L(z,0) \prec L(z,t)$, which implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \ \ (0 \leq t < \infty;\ \zeta \in \partial\mathbb{U}). \tag{17}$$

If $\Psi(z)$ is not subordinate to $\Phi(z)$, by using of the result of [13, p. 24] (see also [16]), we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\Psi(z_0) = \Phi(\zeta_0) \ \text{ and } z_0\Psi'(z_0) = (1+t)\zeta_0\Phi'(\zeta_0) \ \ (0 \leq t < \infty). \tag{18}$$

Hence, we have

$$\begin{aligned} L(\zeta_0,t) &= \left(1 - \frac{1}{p\sum_{i=1}^{n}\alpha_i}\right)\Phi(\zeta_0) + \frac{(1+t)}{p\sum_{i=1}^{n}\alpha_i}\zeta_0\Phi'(\zeta_0) \\ &= \left(1 - \frac{1}{p\sum_{i=1}^{n}\alpha_i}\right)\Psi(z_0) + \frac{1}{p\sum_{i=1}^{n}\alpha_i}z_0\Psi'(z_0) \\ &= z\prod\nolimits_{i=1}^{n}\left(\frac{f_i(z_0)}{z_0^p}\right)^{\alpha_i}\frac{z_0h'(z_0)}{h(z_0)} \in \phi(\mathbb{U}). \end{aligned}$$

This contradicts (17). Thus, we deduce that $\Psi \prec \Phi$. Considering $\Psi = \Phi$, we see that the function Φ is the best dominant. □

We now derive the following superordination result.

Theorem 2.3. *Let $f, g \in \mathcal{A}_{p,h;\alpha_i}$, and (5) holds. If $z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}$ is univalent in $\mathbb{U}$ and $z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \in \mathcal{H}[0,1]\cap\mathcal{Q}$, then*

$$\phi(z) \prec z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}, \tag{19}$$

implies that

$$z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}, \tag{20}$$

and the function $z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}$ is the best subordinant.

Proof. Suppose that $\Psi(z)$, $\Phi(z)$ and $q(z)$ are defined by (9) and (10), respectively. As in Theorem 2.2, we have

$$\phi(z) = \left(1 - \frac{1}{p\sum_{i=1}^{n}\alpha_i}\right)\Phi(z) + \frac{1}{p\sum_{i=1}^{n}\alpha_i}z\Phi'(z) = \varphi(G(z), zG'(z))$$

and we obtain $\Re\{q(z)\} > 0$. Next, to obtain the desired result, we show that $\Phi(z) \prec \Psi(z)$. For this, we suppose that

$$L(z,t) = \left(1 - \frac{1}{p\sum_{i=1}^{n}\alpha_i}\right)\Phi(z) + \frac{t}{p\sum_{i=1}^{n}\alpha_i}z\Phi'(z) \quad (0 \le t < \infty).$$

We note that $L(z,t)$ satisfy the conditions $\lim_{t\to\infty}|a_1(t)| = \infty$ and $a_1(t) \neq 0$. Further, we have

$$\Re\left\{\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \Re\left\{p\sum_{i=1}^{n}\alpha_i - 1 + t\left(1 + \frac{z\Phi''(z)}{\Phi'(z)}\right)\right\} > 0,$$

and so $L(z,t)$ is a subordination chain. Therefore, by using the result of [18], we conclude that (19) must imply (20). Moreover, since the differential equation has a univalent solution Φ, it is the best subordinant. □

Combining Theorems 2.2 and 2.3, the following sandwich-type results are derived.

Theorem 2.4. *Let $f, g_j \in \mathcal{A}_{p,h;\alpha_i}$ $(j = 1, 2)$ and*

$$\Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z) = z\prod_{i=1}^{n}\left(\frac{g_{i,j}(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\right).$$

If $z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}$ is univalent in $\mathbb{U}$ and $z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \in \mathcal{H}[0,1]\cap\mathcal{Q}$. Then

$$\phi_1(z) \prec z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \prec \phi_2(z),$$

implies that

$$z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}.$$

Moreover, the functions $z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta}$ and $z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}$ are, respectively, the best subordinant and the best dominant.

We note that the assumption of Theorem 2.4 that the functions

$$z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \text{ and } z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}$$

need to be univalent in $\mathbb{U}$, may be replaced as in the following corollary.

Corollary 2.5. *Let $f, g_j \in \mathcal{A}_{p,h;\alpha_i}$ $(j = 1, 2)$,*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta\ \left(\phi_j(z) = z\prod_{i=1}^{n}\left(\frac{g_{i,j}(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\right),$$

and

$$\Re\left\{1+\frac{z\Theta''(z)}{\Theta'(z)}\right\} > -\delta\ \left(\Theta(z) = z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\right). \tag{21}$$

Then

$$\phi_1(z) \prec z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \prec \phi_2(z),$$

implies that

$$z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}.$$

Proof. To prove Corollary 1, we have to show that condition (21) implies the univalence of $\Theta(z)$ and $\Psi(z) = z\left(\frac{I^{p,n}_{h;\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}$. Since $0 \leq \delta < \frac{1}{2}$, it follows that $\Theta(z)$ is close to convex function in $\mathbb{U}$ (see [9]) and hence $\Theta(z)$ is univalent in $\mathbb{U}$. Also, by using the same techniques as in the proof of Theorem 2.2, we can prove that Ψ is convex (univalent) in $\mathbb{U}$, and so the details may be omitted. Therefore, by applying Theorem 2.4, we obtain the desired result. □

Remark 2.6. *For $p = 1$ in our results, we obtain the results obtained by Aouf et al. [1].*

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