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The Positive Solutions of Fractional Differential Equation with Riemann-Stieltjes Integral Boundary Conditions

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Abstract. In this paper, we study a class of fractional differential equations with Riemann-Stieltjes integral boundary conditions. The existence and uniqueness of positive solutions for the boundary value problem are obtained via the use of fixed point theorems on cones in partially ordered Banach spaces. Many of the multi-point and integral boundary value problems studied previously studied are also included in our results.

1. Introduction

Fractional differential equations have got considerable attention and importance during the last decades, mainly due to its demonstrated applications in numerous diverse and widespread fields of engineering, physics, biology, mechanics, and so forth; see [5, 10, 18, 23]. Recently, there has been a significant development in the theory of boundary value problems, see [2–4, 6–9, 11–17, 19, 20, 22, 24, 26–28]. Many researchers are keen to study the existence of positive solutions to integral boundary value problem of nonlinear fractional differential equations.

As a more general concept compared with the classical Riemann integral, Riemann-Stieltjes integral is a very useful tool in many research fields. In probability theory, it can be applied to both continuous and discrete random variables. In physics, partly discrete and partly continuous mass distributions problems can also be handled by using this integral. In the differential equation theory, Riemann-Stieltjes integral boundary value problems contain not only the classical Riemann integral boundary value problems, but also the two-point boundary value problems and the multi-point boundary value problems, see [21]. However, as far as we know, the study for the Riemann-Stieltjes integral boundary value problems of fractional differential equations is relatively scarce and many aspects of this theory are worth exploring.

In [19], Sun and Zhao investigated the following classical Riemann integral boundary value problem of the fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)\mathrm{d}s, \end{cases}$$

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where $2 < \alpha \leq 3$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, and $g, q : (0,1) \to \mathbb{R}_+$ are also continuous functions with $g, q \in L^1(0,1)$. The existence results are proved by using the monotone iteration method.

Li et al. [11] considered the fractional Riemann-Stieltjes integral boundary value problem of the following form

$$\begin{cases} D_{0^+}^{\alpha} u(t) + p(t)f(t,u(t)) + q(t)g(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 h(s)u(s)\mathrm{d}A(s), \end{cases}$$

where $\int_0^1 h(s)u(s)\mathrm{d}A(s)$ denotes the Riemann-Stieltjes integral with a signed measure, in which A is a function of bounded variation. Using the properties of the Green function and the fixed point theory in the cones, the authors obtained some results on the existence of positive solutions.

Motivated by the works mentioned above, in this paper we are concerned with the existence and uniqueness of positive solutions of the Riemann-Liouville fractional differential equation

$$D_{0^+}^{\alpha} u(t) + f(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha-2}u(t), \cdots, D_{0^+}^{\alpha-n+1}u(t)) = 0, \quad t \in (0,1) \tag{1}$$

with the Riemann-Stieltjes integral boundary conditions

$$\begin{cases} D_{0^+}^{\alpha-i}u(0) = 0, \quad i = 3, 4, \cdots, n, \\ D_{0^+}^{\alpha-2}u(0) = \int_0^1 u(s)\mathrm{d}B_1(s), \\ D_{0^+}^{\alpha-1}u(1) = \int_0^1 u(s)\mathrm{d}B_2(s), \end{cases} \tag{2}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\alpha-i}$ are the Riemann-Liouville fractional derivatives, $n-1 < \alpha \leq n$, $n \geq 3$ ($n \in \mathbb{N}$), $f : [0,1] \times \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous, $\mathbb{R}_+ = [0, +\infty)$, $\int_0^1 u(s)\mathrm{d}B_i(s)$ are Riemann-Stieltjes integrals, where B_i ($i = 1, 2$) are nondecreasing functions.

We establish a general framework to find the positive solutions for a class of fractional differential equations, providing an effective way to deal with such problems. By utilizing our method, we do not require operators to be compact or continuous, which are able to weaken the conditions imposed on the nonlinearity. In particular, our results not only guarantee the existence of upper-lower solutions to the problems and a unique fixed point, but also construct an iterative sequence for approximating the fixed point.

2. Preliminaries

For the definitions and related theorems concerning fractional calculus, we refer to the monograph [10].

Lemma 2.1. ([1]) *Assume that B is a nondecreasing function on $[a,b]$. If f, g are Riemann-Stieltjes integrable with respect to B on $[a,b]$, and if $f(x) \leq g(x)$ for all $x \in [a,b]$, then we have*

$$\int_a^b f(x)\mathrm{d}B(x) \leq \int_a^b g(x)\mathrm{d}B(x).$$

Lemma 2.2. ([1]) *Let h be continuous at each point (x,y) of a rectangle $Q = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$. Assume that B is a nondecreasing function on $[a,b]$ and let H be the function defined on $[c,d]$ by the equation*

$$H(y) = \int_a^b h(x,y)\mathrm{d}B(x),$$

then H is continuous on $[c,d]$.

Lemma 2.3. ([10]) *Assume that $u \in C(0,1)\bigcap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1)\bigcap L(0,1)$. Then we have the equation*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_k \in \mathbb{R}$, $k = 1,2,3,\cdots,n$, where n is the smallest integer greater than or equal to α.

We denote the constants

$$w_i = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} \mathrm{d}B_i(t), \qquad m_i = \frac{1}{\Gamma(\alpha-1)} \int_0^1 t^{\alpha-2} \mathrm{d}B_i(t), \qquad i = 1,2; \tag{3}$$

$$\Lambda = (1-m_1)(1-w_2) - m_2 w_1. \tag{4}$$

Throughout this paper, we always assume that

(H1) $m_1 = \frac{1}{\Gamma(\alpha-1)} \int_0^1 t^{\alpha-2} \mathrm{d}B_1(t) \le 1$;

(H2) $\Lambda = (1-m_1)(1-w_2) - m_2 w_1 > 0$.

We will use the space $E = \{u : u \in C[0,1],\ D_{0^+}^{\alpha-j} u \in C[0,1],\ j = 1,2,\cdots,n-1\}$ and endowed the norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)| + \sum\limits_{j=1}^{n-1} \max\limits_{t\in[0,1]} |D_{0^+}^{\alpha-j} u(t)|$. Obviously, E is a Banach space. Let $P = \{u \in E : u(t) \ge 0,\ D_{0^+}^{\alpha-j} u(t) \ge 0,\ j = 1,2,\cdots,n-1,\ t \in [0,1]\}$, then $P \subset E$ is a normal cone.

Lemma 2.4. *Let $\xi \in C[0,1]$ and $n-1 < \alpha \le n$, then the linear fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \xi(t) = 0, \quad t \in (0,1), \\ D_{0^+}^{\alpha-i} u(0) = 0, \quad i = 3,4,\cdots,n, \\ D_{0^+}^{\alpha-2} u(0) = \int_0^1 u(s) \mathrm{d}B_1(s), \\ D_{0^+}^{\alpha-1} u(1) = \int_0^1 u(s) \mathrm{d}B_2(s) \end{cases} \tag{5}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s)\xi(s) \mathrm{d}s, \tag{6}$$

where

$$G(t,s) = G_0(t,s) + \frac{m_2 q_1(s) + (1-m_1) q_2(s)}{\Lambda \Gamma(\alpha)} t^{\alpha-1} + \frac{(1-w_2) q_1(s) + w_1 q_2(s)}{\Lambda \Gamma(\alpha-1)} t^{\alpha-2}, \tag{7}$$

and

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}, & 0 \le t \le s \le 1 \end{cases} \tag{8}$$

and

$$q_i(s) = \int_0^1 G_0(t,s) \mathrm{d}B_i(t), \quad i = 1,2. \tag{9}$$

Proof By Lemma 2.3, we can transform the equation $D_{0^+}^{\alpha} u(t) + \xi(t) = 0$ to an equivalent integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi(s) \mathrm{d}s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_k \in \mathbb{R},\ k = 1, 2, \cdots, n$. Furthermore,

$$D_{0^+}^{\alpha-1}u(t) = -I_{0^+}^{1}\xi(t) + c_1\Gamma(\alpha), \quad D_{0^+}^{\alpha-2}u(t) = -I_{0^+}^{2}\xi(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \tag{10}$$

and

$$D_{0^+}^{\alpha-i}u(t) = -I_{0^+}^{i}\xi(t) + \sum_{k=1}^{i} c_k \frac{\Gamma(\alpha-k+1)}{\Gamma(i-k+1)} t^{i-k}, \quad i = 3, 4, \cdots, n. \tag{11}$$

From $D_{0^+}^{\alpha-i}u(0) = 0,\ i = 3, 4, \cdots, n$, we know that $c_3 = c_4 = \cdots = c_n = 0$.

And by the boundary conditions $D_{0^+}^{\alpha-2}u(0) = \int_0^1 u(s)\mathrm{d}B_1(s)$ and $D_{0^+}^{\alpha-1}u(1) = \int_0^1 u(s)\mathrm{d}B_2(s)$, we can get

$$c_1 = \frac{1}{\Gamma(\alpha)}\Big(\int_0^1 \xi(s)\mathrm{d}s + \int_0^1 u(s)\mathrm{d}B_2(s)\Big),$$

$$c_2 = \frac{1}{\Gamma(\alpha-1)}\int_0^1 u(s)\mathrm{d}B_1(s).$$

Then

$$\begin{aligned} u(t) =& -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\xi(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^1 \xi(s)\mathrm{d}s + \int_0^1 u(s)\mathrm{d}B_2(s)\Big) + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 u(s)\mathrm{d}B_1(s) \\ =& \frac{1}{\Gamma(\alpha)}\int_0^t \Big(t^{\alpha-1} - (t-s)^{\alpha-1}\Big)\xi(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}\xi(s)\mathrm{d}s \\ &+ \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 u(s)\mathrm{d}B_2(s) + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 u(s)\mathrm{d}B_1(s), \end{aligned}$$

which is

$$u(t) = \int_0^1 G_0(t,s)\xi(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 u(s)\mathrm{d}B_2(s) + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 u(s)\mathrm{d}B_1(s). \tag{12}$$

Integrating both sides of (12) from 0 to 1 by $B_i(t)$ respectively, we obtain

$$\begin{cases} \int_0^1 u(t)\mathrm{d}B_1(t) = \int_0^1 q_1(s)\xi(s)\mathrm{d}s + w_1\int_0^1 u(s)\mathrm{d}B_2(s) + m_1\int_0^1 u(s)\mathrm{d}B_1(s), \\ \int_0^1 u(t)\mathrm{d}B_2(t) = \int_0^1 q_2(s)\xi(s)\mathrm{d}s + w_2\int_0^1 u(s)\mathrm{d}B_2(s) + m_2\int_0^1 u(s)\mathrm{d}B_1(s). \end{cases}$$

Then we can get

$$\begin{cases} \int_0^1 u(t)\mathrm{d}B_1(t) = \frac{1}{\Lambda}\Big((1-w_2)\int_0^1 q_1(s)\xi(s)\mathrm{d}s + w_1\int_0^1 q_2(s)\xi(s)\mathrm{d}s\Big), \\ \int_0^1 u(t)\mathrm{d}B_2(t) = \frac{1}{\Lambda}\Big((1-m_1)\int_0^1 q_2(s)\xi(s)\mathrm{d}s + m_2\int_0^1 q_1(s)\xi(s)\mathrm{d}s\Big). \end{cases} \tag{13}$$

Substituting (13) into (12), we can get

$$\begin{aligned} u(t) =& \int_0^1 G_0(t,s)\xi(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Lambda\Gamma(\alpha)}\Big((1-m_1)\int_0^1 q_2(s)\xi(s)\mathrm{d}s + m_2\int_0^1 q_1(s)\xi(s)\mathrm{d}s\Big) \\ &+ \frac{t^{\alpha-2}}{\Lambda\Gamma(\alpha-1)}\Big((1-w_2)\int_0^1 q_1(s)\xi(s)\mathrm{d}s + w_1\int_0^1 q_2(s)\xi(s)\mathrm{d}s\Big) \\ =& \int_0^1 G(t,s)\xi(s)\mathrm{d}s. \end{aligned}$$

The proof is completed.

□

Lemma 2.5. *The function $G_0(t,s)$ defined by (8) has the following properties:*
(1) *$G_0(t,s)$ is a continuous function and $G_0(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$;*
(2) $\frac{t^{\alpha-1}(1-s)^{\alpha-2}s}{\Gamma(\alpha)} \leq G_0(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *for $(t,s) \in [0,1] \times [0,1]$.*

Proof By the expression of $G_0(t,s)$, it is clear that $G_0(t,s)$ is continuous and $G_0(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$.

For $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned}\Gamma(\alpha)G_0(t,s) =& t^{\alpha-1} - (t-s)^{\alpha-1} \geq t^{\alpha-1} - (t-ts)^{\alpha-1}\\ \geq& (t-ts)^{\alpha-2}\Big((t-(t-ts)\Big)\\ \geq& t^{\alpha-1}(1-s)^{\alpha-2}s.\end{aligned}$$

On the other hand, we have

$$\Gamma(\alpha)G_0(t,s) = t^{\alpha-1} - (t-s)^{\alpha-1} \leq t^{\alpha-1}.$$

For $0 \leq t \leq s \leq 1$, we can easily get

$$t^{\alpha-1}(1-s)^{\alpha-2}s \leq \Gamma(\alpha)G_0(t,s) = t^{\alpha-1}.$$

The proof is completed. □

Lemma 2.6. *The function $G(t,s)$ defined by (7) has the following properties:*
(1) *$G(t,s)$ is continuous;*
(2) $\frac{(1-s)^{\alpha-2}s}{\Lambda\Gamma(\alpha)}\Big((1-m_1)t^{\alpha-1} + w_1 t^{\alpha-2}\Big) \leq G(t,s) \leq \frac{1}{\Lambda\Gamma(\alpha-1)}\Big((1-m_1)t^{\alpha-1} + w_1 t^{\alpha-2}\Big)$ *for $(t,s) \in [0,1] \times [0,1]$.*

Proof Since $G_0(t,s)$ is continuous, we can easily get $q_i(s)$ is continuous by Lemma 2.2. Therefore, $G(t,s)$ is continuous.

By Lemma 2.5, we can show that

$$\frac{t^{\alpha-1}(1-s)^{\alpha-2}s}{\Gamma(\alpha)} \leq G_0(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t,s \in [0,1].$$

In view of Lemma 2.1, we have

$$\begin{aligned}q_i(s) =& \int_0^1 G_0(t,s)\mathrm{d}B_i(t) \geq \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}s}{\Gamma(\alpha)}\mathrm{d}B_i(t)\\ =& (1-s)^{\alpha-2}sw_i\end{aligned}$$

and

$$q_i(s) = \int_0^1 G_0(t,s)\mathrm{d}B_i(t) \leq \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}B_i(t) = w_i.$$

So, we have

$$\begin{aligned}G(t,s) =& G_0(t,s) + \frac{m_2q_1(s) + (1-m_1)q_2(s)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-w_2)q_1(s) + w_1q_2(s)}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ \geq& \frac{(1-s)^{\alpha-2}s}{\Gamma(\alpha)}t^{\alpha-1} + \frac{m_2q_1(s) + (1-m_1)q_2(s)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-w_2)q_1(s) + w_1q_2(s)}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ \geq& \frac{(1-s)^{\alpha-2}s\Big(\Lambda + m_2w_1 + (1-m_1)w_2\Big)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-s)^{\alpha-2}s\Big((1-w_2)w_1 + w_1w_2\Big)}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ =& \frac{(1-s)^{\alpha-2}s(1-m_1)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-s)^{\alpha-2}sw_1}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ \geq& \frac{(1-s)^{\alpha-2}s}{\Lambda\Gamma(\alpha)}\Big((1-m_1)t^{\alpha-1} + w_1t^{\alpha-2}\Big).\end{aligned}$$

On the other hand, we can obtain that

$$\begin{aligned}G(t,s) =&G_0(t,s) + \frac{m_2q_1(s) + (1-m_1)q_2(s)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-w_2)q_1(s) + w_1q_2(s)}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ \leq&\frac{1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{m_2q_1(s) + (1-m_1)q_2(s)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-w_2)q_1(s) + w_1q_2(s)}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ \leq&\frac{\Lambda + m_2w_1 + (1-m_1)w_2}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{(1-w_2)w_1 + w_1w_2}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ =&\frac{(1-m_1)}{\Lambda\Gamma(\alpha)}t^{\alpha-1} + \frac{w_1}{\Lambda\Gamma(\alpha-1)}t^{\alpha-2}\\ \leq&\frac{1}{\Lambda\Gamma(\alpha-1)}\Big((1-m_1)t^{\alpha-1} + w_1t^{\alpha-2}\Big).\end{aligned}$$

The proof is completed.

□

Lemma 2.7. *Let $\xi \in C([0,1],\mathbb{R}_+)$, then the unique solution $u = u(t)$ of boundary value problem (5) has the following properties:*

$$u(t) \geq 0, \quad D_{0^+}^{\alpha-1}u(t) \geq 0, \quad D_{0^+}^{\alpha-2}u(t) \geq 0, \quad \cdots, \quad D_{0^+}^{\alpha-n+1}u(t) \geq 0.$$

Proof By Lemma 2.6 it is easy to verify that function $G(t,s) \geq 0$, $(t,s) \in [0,1]\times[0,1]$. From (6) and let $\xi \in C([0,1],\mathbb{R}_+)$ we can get that $u(t) \geq 0$.

Next, we show that $D_{0^+}^{\alpha-j}u(t) \geq 0$, $j = 1,2,\cdots,n-1$.

From (10) and (11) we have

$$D_{0^+}^{\alpha-1}u(t) = -I_{0^+}^{1}\xi(t) + c_1\Gamma(\alpha) = \int_t^1 \xi(s)\mathrm{d}s + \int_0^1 u(s)\mathrm{d}B_2(s) \geq 0,$$

$$\begin{aligned}D_{0^+}^{\alpha-j}u(t) =& -I_{0^+}^{j}\xi(t) + \sum_{k=1}^{j} c_k\frac{\Gamma(\alpha-k+1)}{\Gamma(j-k+1)}t^{j-k} = -I_{0^+}^{j}\xi(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(j)}t^{j-1} + c_2\frac{\Gamma(\alpha-1)}{\Gamma(j-1)}t^{j-2}\\ =&\frac{1}{\Gamma(j)}\int_0^t\Big(t^{j-1} - (t-s)^{j-1}\Big)\xi(s)\mathrm{d}s + \frac{t^{j-1}}{\Gamma(j)}\int_t^1 \xi(s)\mathrm{d}s + \frac{t^{j-1}}{\Gamma(j)}\int_0^1 u(s)\mathrm{d}B_2(s) + \frac{t^{j-2}}{\Gamma(j-1)}\int_0^1 u(s)\mathrm{d}B_1(s)\\ \geq&0.\end{aligned}$$

The proof is completed.

□

Here we present some basic concepts in order Banach spaces, which can be found in [25] and will be used later.

Suppose that E is a Banach space which is partially ordered by a cone $P \subset E$, that is,

$$x, y \in E, \quad x \preceq y \Leftrightarrow y - x \in P.$$

We define the order interval $[x_1, x_2] = \{x \in E \mid x_1 \preceq x \preceq x_2\}$ for all $x_1, x_2 \in E$.

The zero element of E is denoted by θ. P is called normal if there exists a constant $N > 0$ such that $\theta \preceq x \preceq y$ implies $\|x\| \leq N\|y\|$. The operator $A : E \to E$ is said to be increasing whenever $x \preceq y$ implies $Ax \preceq Ay$. For all $x, y \in E$, the symbol $x \sim y$ means that there are $\lambda > 0$ and $\mu > 0$ such that

$$\lambda y \preceq x \preceq \mu y.$$

Finally, given $h > \theta$, we denote P_h by the set

$$P_h = \{x \in E \mid x \sim h\}.$$

It is easy to see that $P_h \subset P$.

Let P be a normal cone in real Banach space E, and (a, b) be an interval. We say an operator $A : P \to P$ is τ-φ-concave, if there exist two positive-valued functions $\tau(t), \varphi(t)$ on interval (a, b) such that $\tau : (a, b) \to (0, 1)$ is a surjection, $\varphi(t) \geq \tau(t)$ for all $t \in (a, b)$ and $A\Big(\tau(t)x\Big) \geq \varphi(t)Ax$ for all $x \in P$, $t \in (a, b)$.

Lemma 2.8. ([25]) *Let E be a Banach space, P be a normal cone in E, and $A : P \to P$ be an increasing and τ-φ-concave operator. Suppose that there exists $\theta \neq h \in P$ such that $Ah \in P_h$. Then there are $v_0, w_0 \in P_h$ and $\delta \in (0, 1)$ such that $\delta w_0 \leq v_0 \leq w_0$ and $v_0 \leq Av_0 \leq Aw_0 \leq w_0$, the operator A has a unique fixed point $u^* \in [v_0, w_0]$, and for any initial $u_0 \in P_h$, constructing successively the sequence $\{u_n\}$ with $u_n = Au_{n-1}$, we have $\|u_n - u^*\| \to 0$ as $n \to \infty$.*

3. Main result

This section is mainly devoted to prove the existence and uniqueness of positive solutions for the Riemann-Stieltjes integral boundary value problems of fractional differential equation (1) - (2).

Theorem 3.1. *Assume that (H1) and (H2) hold, and suppose that:*

(i) $f \in C([0, 1] \times \mathbb{R}_+^n, \mathbb{R}_+)$ *and* $m\Big(\{t \in [0, 1] : f(t, 0, 0, 0, \cdots, 0) \neq 0\}\Big) > 0$, *where for some measurable set* Ω, $m(\Omega)$ *denotes the Lebesgue measure of* Ω;

(ii) $f(t, x_0, x_1, \cdots, x_{n-1}) \leq f(t, y_0, y_1, \cdots, y_{n-1})$ *if and only if* $x_i \leq y_i$, $x_i, y_i \in \mathbb{R}_+$, $i = 0, 1, \cdots, n - 1$;

(iii) There exist two functions τ, φ *which are positive-valued on* $(0, 1)$ *and* $\tau : (0, 1) \to (0, 1)$ *is a surjective function such that* $\varphi(\gamma) \geq \tau(\gamma)$ *and* $f\Big(t, \tau(\gamma)x_0, \tau(\gamma)x_1, \tau(\gamma)x_2, \cdots, \tau(\gamma)x_{n-1}\Big) \geq \varphi(\gamma) f(t, x_0, x_1, x_2, \cdots, x_{n-1})$ *for all* $t, \gamma \in (0, 1)$.

Then

(I) The fractional Riemann-Stieltjes integral boundary value problems (1) - (2) *has a unique positive solution* $u^* = u^*(t)$ *in* $[v_0, w_0]$ *where* $v_0, w_0 \in P_h$ *with* $h(t) = (1 - m_1)t^{\alpha-1} + w_1 t^{\alpha-2}$, *there exists a constant* $\delta \in (0, 1)$ *such that* $\delta w_0 \leq v_0 \leq w_0$ *and*

$$v_0(t) \leq \int_0^1 G(t, s) f\Big(s, v_0(s), D_{0^+}^{\alpha-1} v_0(s), D_{0^+}^{\alpha-2} v_0(s), \cdots, D_{0^+}^{\alpha-n+1} v_0(s)\Big) ds, \quad t \in [0, 1],$$

$$w_0(t) \geq \int_0^1 G(t, s) f\Big(s, w_0(s), D_{0^+}^{\alpha-1} w_0(s), D_{0^+}^{\alpha-2} w_0(s), \cdots, D_{0^+}^{\alpha-n+1} w_0(s)\Big) ds, \quad t \in [0, 1].$$

(II) For any initial $u_0 \in P_h$, *constructing the sequence* $\{u_n\}$ *converges to* u^*, *which is that* $\|u_n - u^*\| \to 0$ *as* $n \to \infty$, *where*

$$u_{n+1}(t) = \int_0^1 G(t, s) f\Big(s, u_n(s), D_{0^+}^{\alpha-1} u_n(s), D_{0^+}^{\alpha-2} u_n(s), \cdots, D_{0^+}^{\alpha-n+1} u_n(s)\Big) ds, \quad n = 0, 1, 2, \cdots.$$

Proof In view of Lemma 2.4, the problem (1) - (2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s) f\Big(s, u(s), D_{0^+}^{\alpha-1} u(s), D_{0^+}^{\alpha-2} u(s), \cdots, D_{0^+}^{\alpha-n+1} u(s)\Big) ds.$$

Define the operator $A : P \to E$ by

$$Au(t) = \int_0^1 G(t, s) f\Big(s, u(s), D_{0^+}^{\alpha-1} u(s), D_{0^+}^{\alpha-2} u(s), \cdots, D_{0^+}^{\alpha-n+1} u(s)\Big) ds, \quad t \in [0, 1].$$

For any $u \in P$ and $f \in C([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+)$, it follows $Au \in P$ from Lemma 2.7.

Clearly, $u = u(t)$ is a positive solution for the problem (1) - (2) if and only if u is a fixed point of A on P.

Further, it follows from (ii) that the operator A is increasing on P, which is that for $u \leq v \in P$,

$$\begin{aligned}
Au(t) =& \int_0^1 G(t,s)f\Big(s, u(s), D_{0^+}^{\alpha-1}u(s), D_{0^+}^{\alpha-2}u(s), \cdots, D_{0^+}^{\alpha-n+1}u(s)\Big)\mathrm{d}s \\
\leq& \int_0^1 G(t,s)f\Big(s, v(s), D_{0^+}^{\alpha-1}v(s), D_{0^+}^{\alpha-2}v(s), \cdots, D_{0^+}^{\alpha-n+1}v(s)\Big)\mathrm{d}s \\
=& Av(t).
\end{aligned}$$

On the other hand, for any $t, \gamma \in (0,1)$, $u \in P$, from (iii) we can get

$$\begin{aligned}
A\Big(\tau(\gamma)u\Big)(t) =& \int_0^1 G(t,s)f\Big(s, \tau(\gamma)u(s), \tau(\gamma)D_{0^+}^{\alpha-1}u(s), \tau(\gamma)D_{0^+}^{\alpha-2}u(s), \cdots, \tau(\gamma)D_{0^+}^{\alpha-n+1}u(s)\Big)\mathrm{d}s \\
\geq& \int_0^1 G(t,s)\varphi(\gamma)f\Big(s, u(s), D_{0^+}^{\alpha-1}u(s), D_{0^+}^{\alpha-2}u(s), \cdots, D_{0^+}^{\alpha-n+1}u(s)\Big)\mathrm{d}s \\
=& \varphi(\gamma)Au(t),
\end{aligned}$$

which implies that A is a τ-φ-concave operator.

Since $0 \leq w_1 \leq m_1 \leq 1$, then

$$h(t) = (1-m_1)t^{\alpha-1} + w_1 t^{\alpha-2} \in P$$

and

$$h(t) = (1-m_1)t^{\alpha-1} + w_1 t^{\alpha-2} \leq 1 - m_1 + m_1 = 1, \quad D_{0^+}^{\alpha-1}h(t) = (1-m_1)\Gamma(\alpha) \leq \Gamma(\alpha),$$

$$D_{0^+}^{\alpha-2}h(t) = (1-m_1)\Gamma(\alpha)t + w_1\Gamma(\alpha-1) \leq (1-m_1)\Gamma(\alpha) + m_1\Gamma(\alpha) = \Gamma(\alpha),$$

$$D_{0^+}^{\alpha-j}h(t) = \frac{(1-m_1)\Gamma(\alpha)t^{j-1}}{\Gamma(j)} + \frac{w_1\Gamma(\alpha-1)t^{j-2}}{\Gamma(j-1)} \leq (1-m_1)\Gamma(\alpha) + m_1\Gamma(\alpha) = \Gamma(\alpha), \quad j = 3, 4, \cdots, n-1.$$

Using Lemma 2.6 and (ii), we obtain

$$\begin{aligned}
Ah(t) =& \int_0^1 G(t,s)f\Big(s, h(s), D_{0^+}^{\alpha-1}h(s), D_{0^+}^{\alpha-2}h(s), \cdots, D_{0^+}^{\alpha-n+1}h(s)\Big)\mathrm{d}s \\
\leq& \int_0^1 \frac{(1-m_1)t^{\alpha-1} + w_1 t^{\alpha-2}}{\Lambda\Gamma(\alpha-1)} f\Big(s, h(s), D_{0^+}^{\alpha-1}h(s), D_{0^+}^{\alpha-2}h(s), \cdots, D_{0^+}^{\alpha-n+1}h(s)\Big)\mathrm{d}s \\
\leq& \frac{h(t)}{\Lambda\Gamma(\alpha-1)} \int_0^1 f\Big(s, 1, \Gamma(\alpha), \Gamma(\alpha), \cdots, \Gamma(\alpha)\Big)\mathrm{d}s.
\end{aligned}$$

Similarly,

$$\begin{aligned}
Ah(t) =& \int_0^1 G(t,s)f\Big(s, h(s), D_{0^+}^{\alpha-1}h(s), D_{0^+}^{\alpha-2}h(s), \cdots, D_{0^+}^{\alpha-n+1}h(s)\Big)\mathrm{d}s \\
\geq& \int_0^1 \frac{(1-s)^{\alpha-2}s}{\Lambda\Gamma(\alpha)}\Big((1-m_1)t^{\alpha-1} + w_1 t^{\alpha-2}\Big) f\Big(s, h(s), D_{0^+}^{\alpha-1}h(s), D_{0^+}^{\alpha-2}h(s), \cdots, D_{0^+}^{\alpha-n+1}h(s)\Big)\mathrm{d}s \\
\geq& \frac{h(t)}{\Lambda\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} s f(s, 0, 0, 0, \cdots, 0)\mathrm{d}s.
\end{aligned}$$

Denote

$$\lambda_1 = \frac{1}{\Lambda\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} s f(s, 0, 0, 0, \cdots, 0) \mathrm{d}s$$

and

$$\lambda_2 = \frac{1}{\Lambda\Gamma(\alpha-1)} \int_0^1 f\Big(s, 1, \Gamma(\alpha), \Gamma(\alpha), \cdots, \Gamma(\alpha)\Big) \mathrm{d}s.$$

Then

$$\lambda_1 h \leq Ah \leq \lambda_2 h.$$

On the other hand, from (ii) we can easily get

$$f\Big(s, 1, \Gamma(\alpha), \Gamma(\alpha), \cdots, \Gamma(\alpha)\Big) \geq f(s, 0, 0, 0, \cdots, 0) \geq 0.$$

Since $m\Big(\{t \in [0,1] : f(t, 0, 0, 0, \cdots, 0) \neq 0\}\Big) > 0$, we have

$$\lambda_1 > 0, \qquad \lambda_2 > \frac{1}{\Lambda\Gamma(\alpha-1)} \int_0^1 f(s, 0, 0, 0, \cdots, 0) \mathrm{d}s > 0.$$

Thus we proved that $Ah \in P_h$.

Our desired results are obtained in view of Lemma 2.8. The proof is completed. □

Next, we consider the fractional differential equation with homogeneous boundary conditions as a special case when $B_i(t)\,(i = 1, 2)$ are constants which could be zero. We can draw the following deduction.

Corollary 3.2. *The two-point boundary value problem of nonlinear fractional differential equation*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f\Big(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t), \cdots, D_{0^+}^{\alpha-n+1} u(t)\Big) = 0, \quad t \in (0,1), \\ D_{0^+}^{\alpha-i} u(0) = 0, \quad i = 2, 3, 4, \cdots, n, \\ D_{0^+}^{\alpha-1} u(1) = 0 \end{cases} \tag{14}$$

has a unique positive solution $u^* \in P_h$ with $h(t) = t^{\alpha-1}$ if the three conditions $(i) - (iii)$ in Theorem 3.1 are still valid.

Proof Note that boundary value problem (14) is the special case of boundary value problem (1)-(2) when $B_i(t)\,(i = 1, 2)$ are constants, by (3) and (4) we point out that $m_i = w_i = 0 < 1$, $i = 1, 2$ and $\Lambda = 1 > 0$. This means we have verified the hypotheses (H1) and (H2).

In view of Lemma 2.4, we get $q_i(s) \equiv 0$ and $G(t, s) = G_0(t, s)$. The problem (14) is equivalent to the integral equation

$$u(t) = \int_0^1 G_0(t, s) f\Big(s, u(s), D_{0^+}^{\alpha-1} u(s), D_{0^+}^{\alpha-2} u(s), \cdots, D_{0^+}^{\alpha-n+1} u(s)\Big) \mathrm{d}s.$$

Define the operator $T : P \to P$ by

$$Tu(t) = \int_0^1 G_0(t, s) f\Big(s, u(s), D_{0^+}^{\alpha-1} u(s), D_{0^+}^{\alpha-2} u(s), \cdots, D_{0^+}^{\alpha-n+1} u(s)\Big) \mathrm{d}s, \quad t \in [0,1].$$

Obviously, u is a positive solution of the problem (14) if and only if u is a fixed point of T on P.

Similar to the proof in Theorem 3.1, we can obtain that $T : P \to P$ is an increasing and τ-φ-concave operator. So Theorem 3.1 can be applied when $h(t) = t^{\alpha-1}$.

The proof is completed. □

4. Applications

The Riemann-Stieltjes integral boundary conditions cover the multi-point, the classical Riemann integral and the mixed boundary conditions. In this section, we briefly indicate how our results could be applied to some specific boundary value problems.

Application 4.1 Consider the following nonlinear fractional differential equation with the classical Riemann integral boundary conditions

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t) + e^t \sin t + \sqrt{u(t) + D_{0^+}^{\frac{3}{2}}u(t) + D_{0^+}^{\frac{1}{2}}u(t)} = 0, \quad t \in (0,1), \\ D_{0^+}^{-\frac{1}{2}}u(0) = 0, \\ D_{0^+}^{\frac{1}{2}}u(0) = \int_0^1 s^2 u(s)\mathrm{d}s, \\ D_{0^+}^{\frac{3}{2}}u(1) = 2\int_0^1 su(s)\mathrm{d}s. \end{cases} \tag{15}$$

Let $\alpha = \frac{5}{2}$, $n = 3$, $f(t,u,v,w) = e^t \sin t + \sqrt{u+v+w}$ and $B_1(t) = \frac{1}{3}t^3, B_2(t) = t^2, t \in [0,1]$. Then, the boundary value problem (15) is a special case of (1)-(2). Obviously, $B_1(t)$ and $B_2(t)$ are increasing functions on $[0,1]$. $f(t,u,v,w)$ is continuous and $f(t,u_1,v_1,w_1) \le f(t,u_2,v_2,w_2)$ for $u_1 \le u_2, v_1 \le v_2, w_1 \le w_2$, and $f(t,0,0,0) = e^t \sin t$. Thus the condition $m\Big(\{t \in [0,1] : f(t,0,0,0) \neq 0\}\Big) > 0$ holds.

Now, we define $\tau(t) = t$, $\varphi(t) = \sqrt{t}$. Then $\tau : (0,1) \to (0,1)$ is a surjection and $\varphi(t) > \tau(t)$ for $t \in (0,1)$. Hence, for all $t, x \in (0,1)$ and $u, v, w \in \mathbb{R}_+$, we find that

$$\begin{aligned} f\Big(t, \tau(x)u, \tau(x)v, \tau(x)w\Big) &= f(t, xu, xv, xw) = e^t \sin t + \sqrt{xu + xv + xw} \\ &\ge \sqrt{x}(e^t \sin t + \sqrt{u+v+w}) = \varphi(x) f(t,u,v,w). \end{aligned}$$

A simple calculation shows that

$$m_1 = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 t^{\frac{1}{2}}\mathrm{d}B_1(t) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 t^{\frac{5}{2}}\mathrm{d}t \approx 0.3224 < 1.$$

Clearly, the condition (H1) holds. Moreover, through a series of calculations, we obtain

$$m_2 = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 t^{\frac{1}{2}}\mathrm{d}t^2 \approx 0.9027, \quad w_1 = \frac{1}{\Gamma(\frac{5}{2})}\int_0^1 t^{\frac{7}{2}}\mathrm{d}t \approx 0.1672, \quad w_2 = \frac{1}{\Gamma(\frac{5}{2})}\int_0^1 t^{\frac{3}{2}}\mathrm{d}t^2 \approx 0.4299.$$

Then, we can easily get $\Lambda = (1-m_1)(1-w_2) - m_2 w_1 \approx 0.2354 > 0$, which shows that the condition (H2) is proved.

Therefore, all the assumptions in Theorem 3.1 are satisfied. By using Theorem 3.1, we can see that problem (15) has a unique positive solution in P_h with $h(t) = 0.6776t^{\frac{3}{2}} + 0.1672t^{\frac{1}{2}}$.

□

Application 4.2 Consider the fractional multi-point boundary value problem

$$\begin{cases} D_{0^+}^{\frac{7}{2}}u(t) + \Big(u(t)\cdot D_{0^+}^{\frac{5}{2}}u(t)\cdot D_{0^+}^{\frac{3}{2}}u(t)\cdot D_{0^+}^{\frac{1}{2}}u(t)\Big)^{\frac{1}{16}}\ln(2+t) + t^2 = 0, \quad t \in (0,1), \\ D_{0^+}^{\frac{1}{2}}u(0) = D_{0^+}^{-\frac{1}{2}}u(0) = 0, \\ D_{0^+}^{\frac{3}{2}}u(0) = \frac{1}{4}u(\frac{1}{4}) + \frac{1}{8}u(\frac{1}{2}) + \frac{1}{16}u(\frac{3}{4}), \\ D_{0^+}^{\frac{5}{2}}u(1) = \frac{1}{4}u(\frac{1}{2}), \end{cases} \tag{16}$$

where $\alpha = \frac{7}{2}$, $n = 4$, $f(t,u,v,w,z) = (uvwz)^{\frac{1}{16}}\ln(2+t) + t^2$,

$$B_1(t) = \begin{cases} 0, & t \in [0,\frac{1}{4}), \\ \frac{1}{4}, & t \in [\frac{1}{4},\frac{1}{2}), \\ \frac{1}{4}+\frac{1}{8}, & t \in [\frac{1}{2},\frac{3}{4}), \\ \frac{1}{4}+\frac{1}{8}+\frac{1}{16}, & t \in [\frac{3}{4},1], \end{cases} \quad \text{and} \quad B_2(t) = \begin{cases} \frac{1}{4}, & t \in [0,\frac{1}{2}), \\ \frac{1}{2}, & t \in [\frac{1}{2},1]. \end{cases}$$

Easily, $B_1(t)$ and $B_2(t)$ are increasing functions on $[0,1]$. $f(t,u,v,w,z)$ is continuous with $u,v,w,z \in \mathbb{R}_+$ and $f(t,u_1,v_1,w_1,z_1) \le f(t,u_2,v_2,w_2,z_2)$ for $u_1 \le u_2, v_1 \le v_2, w_1 \le w_2, z_1 \le z_2$, and $f(t,0,0,0,0) = t^2$. Thus the condition $m\Big(\{t \in [0,1] : f(t,0,0,0,0) \neq 0\}\Big) > 0$ holds.

Let $\tau(t) = t$, $\varphi(t) = t^{\frac{1}{4}}$, we can get

$$\begin{aligned} f\Big(t,\tau(x)u,\tau(x)v,\tau(x)w,\tau(x)z\Big) =& f(t,xu,xv,xw,xz) = (x^4uvwz)^{\frac{1}{16}}\ln(2+t) + t^2 \\ \ge& x^{\frac{1}{4}}\Big((uvwz)^{\frac{1}{16}}\ln(2+t) + t^2\Big) = \varphi(x)f(t,u,v,w,z). \end{aligned}$$

A simple calculation shows that

$$m_1 = \frac{1}{\Gamma(\frac{5}{2})}\int_0^1 t^{\frac{3}{2}}\mathrm{d}B_1(t) \approx 0.0873 < 1.$$

Clearly, the condition (H1) holds. Moreover, through a series of calculations, we obtain

$$m_2 = \frac{1}{\Gamma(\frac{5}{2})}\int_0^1 t^{\frac{3}{2}}\mathrm{d}B_2(t) \approx 0.0665, \quad w_1 = \frac{1}{\Gamma(\frac{7}{2})}\int_0^1 t^{\frac{5}{2}}\mathrm{d}B_1(t) \approx 0.0182, \quad w_2 = \frac{1}{\Gamma(\frac{7}{2})}\int_0^1 t^{\frac{5}{2}}\mathrm{d}B_2(t) \approx 0.0133.$$

Then we get $\Lambda = (1-m_1)(1-w_2) - m_2w_1 \approx 0.8994 > 0$, which shows that the condition (H2) is proved.

Therefore, all the assumptions in Theorem 3.1 are satisfied. By using Theorem 3.1, we know that problem (16) has a unique positive solution in P_h with $h(t) = 0.9127t^{\frac{5}{2}} + 0.0182t^{\frac{3}{2}}$.

□

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