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Optimization of Lagrange Problem with Higher Order Differential Inclusions and Endpoint Constraints

Elimhan N. Mahmudov[a]

[a]*Department of Mathematics, Istanbul Technical University, Istanbul, Turkey,*
Azerbaijan National Academy of Sciences Institute of Control Systems, Baku, Azerbaijan.

Abstract. In the paper minimization of a Lagrange type cost functional over the feasible set of solutions of higher order differential inclusions with endpoint constraints is studied. Our aim is to derive sufficient conditions of optimality for m th-order convex and non-convex differential inclusions. The sufficient conditions of optimality containing the Euler-Lagrange and Hamiltonian type inclusions as a result of endpoint constraints are accompanied by so-called "endpoint" conditions. Here the basic apparatus of locally adjoint mappings is suggested. An application from the calculus of variations is presented and the corresponding Euler-Poisson equation is derived. Moreover, some higher order linear optimal control problems with quadratic cost functional are considered and the corresponding Weierstrass-Pontryagin maximum principle is constructed. Also at the end of the paper some characteristic features of the obtained result are illustrated by example with second order linear differential inclusions.

1. Introduction

There is a great number of papers dealing with optimal control theory of ordinary and partial differential inclusions (see [1]-[7],[9],[10]-[12],[17],[18],[21],[22] and their references) and has numerous applications in both science and engineering. The problems accompanied with the higher order ordinary differential inclusions are more complicated due to the higher order derivatives. In fact, the difficulty is rather to construct adjoint Euler-Lagrange inclusions and the endpoint conditions. A convenient procedure for eliminating this complication in optimal control theory involving higher order derivatives in general is a formal reduction of these problems by substitution to the system of first order differential inclusions or equations.

That's why optimization of higher order differential inclusions (HODIs) has not been studied in the literature and there have been quite a few number of papers devoted mainly to existence and viability problems for second order differential inclusions (see, for example [3],[8] and their references).

In the paper [3], a class of boundary value problems of nonlinear nth-order differential equations and inclusions with nonlocal and integral boundary conditions is studied. New existence results are obtained by means of some fixed point theorems. Examples are given for the illustration of the results.

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Email address: elimhan22@yahoo.com (Elimhan N. Mahmudov)

In the paper [8], is proved a theorem on the existence of solutions for a second order differential inclusion governed by the Clarke subdifferential of a Lipschitzian function and by a mixed semicontinuous perturbation.

Optimization of higher order differential inclusions was first developed by Mahmudov [13], [15].

The paper[13] is devoted to a second order polyhedral optimization described by ordinary discrete and DFIs. The stated second order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and in terms of the polyhedral Euler-Lagrange inclusions, necessary and sufficient conditions of optimality are derived. Derivation of the sufficient conditions for the second order polyhedral DFIs is based on the discrete-approximation method.

The paper [15] is mainly concerned with the necessary and sufficient conditions of optimality for Cauchy problem (with fixed initial and free endpoint constraints) of arbitrary higher order discrete and third-order differential inclusions. Applying optimality conditions of problems with geometric constraints, for third-order discrete inclusions optimality conditions are formulated.

In the paper [6], two kinds of system of fuzzy differential inclusions are introduced and studied. An existence of the solutions for one system of fuzzy differential inclusions is proved by using continuous selection theorem. An existence of the solutions for another system of fuzzy differential inclusions is also proved by employing the fixed point theorem in the generalized metric space. The results presented in this paper improve and extend some known results concerned with the multivalued Cauchy problem and fuzzy differential inclusions.

In the paper [1], the fractional optimal control problem for differential system is considered. The fractional time derivative is considered in Riemann-Liouville sense. Constraints on controls are imposed. Necessary and sufficient optimality conditions for the fractional Dirichlet and Neumann problems with the quadratic performance functional are derived. Some examples are analyzed in details.

In our present paper, we discuss a special kind of optimization problem with HODIs and endpoint constraints. In fact, the difficulty in the problems with HODIs is rather to construct the Euler-Lagrange type higher order adjoint inclusions and the suitable endpoint conditions. Obviously, if $m = 1$ Mahmudov's and Euler-Lagrange adjoint inclusions coincide. The paper is organized as follows:

In Section 2 we review the necessary facts and supplementary results from the monograph of Mahmudov [17] and papers [14], [16]; Hamiltonian function and argmaximum sets of a set-valued mapping, together with the locally adjoint mapping (LAM) are introduced and for problem with HODIs and endpoint constraints (P_H) governed by set-valued mapping are formulated. The book [17] presents basic concepts and principles of mathematical programming in terms of set-valued analysis and develops a comprehensive optimality theory of problems described by ordinary and partial differential inclusions of parabolic, hyperbolic and elliptic types.

In Section 3 we consider the optimality problem (P_H) for m th-order differential inclusions with endpoint constraints. Construction of convex and nonsmooth analysis are used for the higher order Euler-Lagrange and Hamiltonian inclusions and endpoint conditions at the endpoints $t = 0$ and $t = 1$. For a closed set-valued mapping F the higher order Euler-Lagrange conditions can be rewritten in term of Hamiltonian function in much more convenient form. The general connection between argmaximum set and subdifferential of Hamiltonian plays an important role. We point out that the main difficulty to apply the techniques of [15] is essentially due to the generality of the Lagrange's cost functional and endpoint constraints. Derivation of optimality conditions, which are based on associated discrete-approximate problems with higher order difference operators is omitted.

In Section 4 we are able to use the results in Section 3 to get Euler-Poisson equation for variation of calculus theory. Finally, is considered the higher order "linear" optimal control problem, where the integrand in the Lagrange type functional is some quadratic form with a symmetric positively semidefinite matrix. For this problem are established higher order Euler-Lagrange type adjont differential equation, the endpoint conditions and Weierstrass-Pontryagin maximum principle. Apparently, by using the functional analysis approach in the convex problems, necessity of these conditions for optimality can be justified. Proving necessary conditions is, however, subject of separate discussion and, hence, omitted.

2. Notations and preliminaries

Let $\langle x, y\rangle$ be an inner product of elements $x, y \in \mathbb{R}^n$. Let suppose that $F : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from $(\mathbb{R}^n)^m = \underbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}_{m}$ into the set of subsets of $\mathbb{R}^n$. Then a set-valued mapping $F : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ is convex if its $gphF = \{(x, v, v_m) : v_m \in F(x, v)\}, v = (v_1, ..., v_{m-1}), x, v_i \in \mathbb{R}^n, i = 1, 2, ..., m$ is a convex subset of $(\mathbb{R}^n)^{m+1}$. It is convex-valued if $F(x, v)$ is a convex set for each $(x, v) \in domF = \{(x, v) : F(x, v) \neq \emptyset\}$. A set-valued mapping $F : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous at (x^0, v^0) if for any neighbourhood U of zero in $\mathbb{R}^n$ there exists a neighborhood V of zero in $(\mathbb{R}^n)^m$ such that

$$F(x, v) \subseteq F(x^0, v^0) + U, \quad \forall (x, v) \in (x^0, v^0) + V.$$

Hamiltonian function and argmaximum set for a set-valued mapping F is defined as follows

$$H_F(x, v, v_m^*) = \sup_{v_m} \left\{\langle v_m, v_m^*\rangle : v_m \in F(x, v)\right\}, v_m^* \in \mathbb{R}^n,$$
$$F(x, v; v_m^*) = \left\{v_m \in F(x, v) : \langle v_m, v_m^*\rangle = H_F(x, v, v_m^*)\right\},$$

respectively. For a convex F we set $H_F(x, v, v_m^*) = -\infty$ if $F(x, v) = \emptyset$. In other terms, $H_F(x, v, v_m^*)$ is the support function to the set $F(x, v)$, evaluated at v_m^*.

For the reader's convenience let us mention the following definition from the book of Mahmudov [17].

Definition 2.1. *A convex cone $K_A(z_0)$, $z = (x, v_1, v_2)$ is called the cone of tangent directions at a point $z_0 = (x^0, v_1^0, v_2^0) \in A (A \subset \mathbb{R}^{(m+1)n})$ if from $\bar{z} = (\bar{x}, \bar{v}_1, \bar{v}_2) \in K_A(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set A, i.e., there exists a function $\varphi : \mathbb{R}^1 \to \mathbb{R}^{(m+1)n}$ satisfying $z_0 + \lambda\bar{z} + \varphi(\lambda) \in A$ for sufficiently small $\lambda > 0$, where $\lambda^{-1}\varphi(\lambda) \to 0$, as $\lambda \downarrow 0$. The closure of A is denoted by clA.*

It should be pointed out that the cone $K_A(z_0)$ is not uniquely defined . In any case the wider a cone of tangent directions we have the essentially necessary condition for a minimum.

It is known [17] that for convex sets, for sets described by equality and inequality constraints with differentiability property, etc., the cone of tangent directions can be computed easily. Clearly, for a convex set A at a point $z_0 = (x^0, v_1^0, v_2^0) \in A$ we have $\varphi(\lambda) \equiv 0$ and the cone of tangent directions is defined as follows

$$K_A(z_0) = \left\{(\bar{x}, \bar{v}_1, \bar{v}_2) : \bar{x} = \alpha(x - x^0), \;\; \bar{v}_1 = \alpha(v_1 - v_1^0), \;\; \bar{v}_2 = \alpha(v_2 - v_2^0), \; (x, v_1, v_2) \in A, \alpha > 0\right\}.$$

If I is a finite index set, f_i, $i \in I$ are continuously differentiable functions and the gradient vectors $f_i'(z_0), i \in I$ are linearly independent, then by Proposition 3.7 [17] a cone of tangent directions $K_A(z_0)$, $z_0 \in A$ to the set $A = \{z : f_i(z) = 0, i \in I\}$ is computed by the formula $K_A(z_0) = \left\{\bar{z} : \langle\bar{z}, f_i'(z_0)\rangle = 0, i \in I\right\}$.

In general, for a mapping F a set-valued mapping $F^* : \mathbb{R}^n \rightrightarrows \left(\mathbb{R}^n\right)^m$ defined by

$$F^*\left(v_m^*; (x^0, v^0, v_m^0)\right) := \left\{(x^*, v^*) : (x^*, v^*, -v_m^*) \in K^*_{gphF}(x^0, v^0, v_m^0)\right\}$$

is called a locally adjoint set-valued mapping (LAM) to F at a point $(x^0, v^0, v_m^0) \in gphF$, where $K^*_{gphF}(x^0, v^0, v_m^0)$ is the dual to a cone of tangent vectors $K_{gphF}(x^0, v^0, v_m^0)$. In what follows another way to define LAMs in the "non-convex" case is the next one

$$F^*\left(v_m^*; (x^0, v^0, v_m^0)\right) := \left\{(x^*, v^*) : H_F(x, v, -v_m^*) - H_F(x^0, v^0, v_m^0)\right.$$
$$\left.\leq \langle x^*, x - x^0\rangle + \langle v^*, v - v^0\rangle, \forall (x, v) \in \left(\mathbb{R}^n\right)^m\right\}, \; v_m^0 \in F(x^0, v^0; v_m^*),$$

which is called the LAM to non-convex mapping F at a point $(x^0, v^0, v_m^0) \in gphF$. The main advantage of this definition is its simplicity. Clearly, for the convex mapping F the Hamiltonian function $H_F(\cdot, v_m^*)$ is concave and by Theorem 2.1 [17] the latter definition of LAM coincide with the previous definition of LAM, that is

$$F^*\Big(v_m^*; (x^0, v^0, v_m^0)\Big) := \partial_{(x,v)} H_F(x^0, v^0, v_m^*), v_m^0 \in F(x^0, v^0, v_m^*),$$

where by convention $\partial_{(x,v)} H_F(x^0, v^0, v_m^*) = -\partial_{(x,v)}\Big[H_F(x^0, v^0, v_m^*)\Big]$

In Section 3 we are concerned with the optimal control problem for HODIs (m th- order) with endpoint constraints:

$$\text{minimize } J[x(\cdot)] = \int_0^1 L(x(t), x'(t) ..., x^{(m-1)}(t), t)dt) \tag{1}$$

$$(P_H) \qquad \frac{d^m x(t)}{dt^m} \in F(x(t), x'(t)\ , \ \dots\ , \ x^{(m-1)}(t), t)\ , \ \text{a.e.} \quad t \in [0,1], \tag{2}$$

$$x(0) \in M_0, \quad x'(0) \in M_1\ , \ \dots\ , \ x^{(m-1)}(0) \in M_{m-1}, \tag{3}$$

$$x(1) \in Q_0\ , \ x'(1) \in Q_1\ , \ \dots\ , \ x^{(m-1)}(1) \in Q_{m-1}.$$

Here $F(\cdot, t) : \Big(\mathbb{R}^n\Big)^m \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, $L(\cdot, t) : \Big(\mathbb{R}^n\Big)^m \to \mathbb{R}^1$, is continuous with respect to the first m components and $M_k, Q_k \subset \mathbb{R}^n, k = 0, 1, ..., m-1$ are nonempty subsets. The problem is to find a trajectory $\tilde{x}(t)$ of the generalized boundary value problem (1)-(3) for the m th -order differential inclusions satisfying (2) almost everywhere (a.e.) on $[0, 1]$ and the endpoint conditions (3) on $[0, 1]$ that minimizes the Lagrange type cost functional $J[x(\cdot)]$. We label this problem as (P_H). A feasible solution $x(\cdot)$ is absolutely continuous function on a time interval $[t_0, t_1]$ together with the higher order derivatives until $m-1$, and $x^m(\cdot) \equiv \frac{d^m x(\cdot)}{dt^m} \in L_1^n([0, 1])$. Note that such class of functions is a Banach space, equipped with the different equivalent norms.

The method of discrete-approximation of (P_H) with higher order differential inclusions has been very effective in the investigation of optimality conditions. Thus the basic idea is to replace the continuous problem (P_H) by the discrete-approximation problem:

$$\text{minimize } J_\delta[x(\cdot)] = \sum_{t=m\delta,...,1-m\delta} \delta L\Big(x(t-(m-1)\delta), \Delta x(t-(m-1)\delta), ..., \Delta^{m-1} x(t-(m-1)\delta), t\Big),$$

$$\Delta^m x(t) \in F\Big(x(t), \Delta x(t), ..., \Delta^{m-1} x(t), t\Big), \quad t = 0, \delta, 2\delta ..., 1-\delta, \tag{4}$$

$$x(0) \in M_0, \quad \Delta x(0) \in M_1\ , \ \dots\ , \ \Delta^{m-1} x(0) \in M_{m-1},$$

$$x(1) \in Q_0\ , \ \Delta x(1) \in Q_1\ , \ \dots\ , \ \Delta^{(m-1)} x(1) \in Q_{m-1},$$

where m th-order difference operator is defined as follows

$$\Delta^m x(t) = \frac{1}{\delta^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} x(t + (m-k))\delta, \binom{m}{k} = \frac{m!}{k!(m-k)!}, t = 0, \delta, ..., 1-\delta. \tag{5}$$

In our studies the general approach to optimization of HODIs is the investigation of the corresponding discrete-approximate problem via set-valued mappings. This analysis reveals the hidden relationship between optimal control methods with HODIs and discrete-approximate problem (4), (5); in the problem (P_H) we apply a generalized discrete-approximate Euler-Lagrange transformation formulas for the Cauchy problem [17], however, a major difficulty consists in minimization of the approximated Lagrange functional

$J_\delta[x(\cdot)]$ in its general form and in checking the endpoint conditions connected with the endpoint constraints $x(0) \in M_0,\quad \Delta x(0) \in M_1, ..., \Delta^{m-1}x(0) \in M_{m-1}; x(1) \in Q_0, \Delta x(1) \in Q_1, ..., \Delta^{m-1}x(1) \in Q_{m-1}$. Then by passing to the limit in necessary and sufficient conditions of optimality for problem (4), (5) as $\delta \to 0$ (at least formally), we can establish the sufficient conditions of optimality for continuous problem (P_H). But in the present paper to avoid a long calculations connected with the discretization method, establishment of optimality and endpoint conditions at the endpoints $t = 0$ and $t = 1$ for the discrete-approximate problem (4), (5) are omitted. In the next section are studied sufficient conditions of optimality for problem (P_H).

3. Sufficient conditions of optimality for (P_H).

In this section we introduce the basic notions and notation to be used in the rest of the paper. At first consider a convex optimization problem, where $F(\cdot, t) : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $L(\cdot, t) : (\mathbb{R}^n)^m \to \mathbb{R}^1$ is continuous and convex with respect to the first m components and $M_k, Q_k \subset \mathbb{R}^n, k = 0, 1, ..., m-1$ are convex subsets.

As a result of approximation method described at the end of Section 2 we establish so-called the m th-order Euler-Lagrange differential inclusion for the convex optimization problem (P_H):

$$(i)\qquad \left((-1)^m \tfrac{d^m x^*(t)}{dt^m} + \tfrac{d\eta^*_{m-1}(t)}{dt}, \eta^*_{m-1}(t) + \tfrac{d\eta^*_{m-2}(t)}{dt}, ..., \eta^*_2(t) + \tfrac{d\eta^*_1(t)}{dt}, \eta^*_1(t)\right)$$
$$\in F^*\Big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m)}(t)), t\Big) - \partial_{(x,v)} L\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\Big), \quad \text{a.e.} \quad t \in [0, 1].$$

It is important to note a subtlety in our definition of endpoint conditions in the convex optimization problem at the endpoints $t = 0$ and $t = 1$,

$$(ii)\quad \eta^*_k(0) + (-1)^{k+1} \tfrac{d^k x^*(0)}{dt^k} \in K^*_{M_{m-k-1}}\left(\tfrac{d^{m-k-1}\tilde{x}(0)}{dt^{m-k-1}}\right), k = 0, 1, ..., m-1 \ \left(\eta^*_0(0) = 0\right),$$

$$(iii)\quad -\eta^*_k(1) + (-1)^{k+1} \tfrac{d^k x^*(1)}{dt^k} \in K^*_{Q_{m-k-1}}\left(\tfrac{d^{m-k-1}\tilde{x}(1)}{dt^{m-k-1}}\right), k = 0, 1, ..., m-1 \ \left(\eta^*_0(1) = 0\right),$$

where $K^*_N(x_0)$ is the dual to a cone of tangent vectors $K_N(x_0), x_0 \in N$.

We refer the reader to [17], [18] for a review of notation and terminology for first order ordinary differential inclusions.

Later on we suppose that $x^*(t), t \in [0, 1]$ is absolutely continuous function with the higher order derivatives until $m-1$ and $\frac{d^m x^*(\cdot)}{dt^m} \in L^n_1\big([0, 1]\big)$. In addition, let $\eta^*_k(t), k = 1, ..., m-1, t \in [0, 1]$ be absolutely continuous and $\frac{d\eta^*_k(\cdot)}{dt} \in L^n_1\big([0, 1]\big), k = 1, ..., m-1$.

Besides, in terms of argmaximum set we shall offer a condition providing that the LAM F^* is nonempty at a given point:

$$(iv)\quad \frac{d^m \tilde{x}(t)}{dt^m} \in F\Big(\tilde{x}(t), \tilde{x}'(t)\ , \ ...\ , \ \tilde{x}^{(m-1)}(t); x^*(t), t\Big)\ , \ \text{a.e.} \quad t \in [0, 1],$$

It turns out that the following theorem is true.

Theorem 3.1. *Let $L(\cdot, t) : \left(\mathbb{R}^n\right)^m \to \mathbb{R}^1$ be continuous convex function, $F(\cdot, t) : \left(\mathbb{R}^n\right)^m \rightrightarrows \mathbb{R}^n$ be a convex set-valued mapping and $M_k, Q_k \subset \mathbb{R}^n, k = 0, 1, ..., m-1$ be convex subsets. Then for the optimality of the trajectory $\tilde{x}(t)$ in the convex optimization problem (P_H) it is sufficient that there exists a collection of absolutely continuous functions $\eta^*_k(\cdot), k = 1, ..., m-1$ and an absolutely continuous function $x^*(\cdot)$ with the higher order derivatives until $m-1$ satisfying a.e. the m th-order Euler-Lagrange differential inclusion (i), (iv), endpoint conditions (ii) and (iii) at the endpoints $t = 0, t = 1$.*

Proof. Obviously, by Theorem 2.1 [17] the LAM $F^*(v_m^*;(x,v),t) = \partial_{(x,v)}H_F(x,v,v_m^*,t), v_m \in F(x,v;v_m^*,t), v = (v_1,...,v_{m-1})$. On the other hand in the sense of convex analysis by convention convention $-\partial_{(x,v)}L(x,v,t) = \partial_{(x,v)}(-L(x,v,t))$, $(x,v) \in domL(\cdot,t)$. Then taking into account the Moreau-Rockafellar theorem [17], [18], [20] from the condition (i) in term of Hamiltonian function we obtain the m th-order adjoint differential inclusion

$$\begin{aligned}&\left((-1)^m\frac{d^m x^*(t)}{dt^m}+\frac{d\eta^*_{m-1}(t)}{dt},\eta^*_{m-1}(t)+\frac{d\eta^*_{m-2}(t)}{dt},...,\eta^*_2(t)+\frac{d\eta^*_1(t)}{dt},\eta^*_1(t)\right)\\&\in\partial_{(x,v)}\Big[H_F\Big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),x^*(t),t\Big)-L\Big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),t\Big)\Big].\end{aligned}$$

By using of the classical subdifferential definition, we rewrite the last relation in the form

$$\begin{aligned}&H_F\Big(x(t),x'(t),...,x^{(m-1)}(t),x^*(t),t\Big)-H_F\Big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),x^*(t),t\Big)\\&\qquad-L\Big(x(t),x'(t),...,x^{(m-1)}(t),t\Big)+L\Big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),t\Big)\\&\le\Big\langle(-1)^m\frac{d^m x^*(t)}{dt^m}+\frac{d\eta^*_{m-1}(t)}{dt},x(t)-\tilde{x}(t)\Big\rangle+\Big\langle\eta^*_{m-1}(t)+\frac{d\eta^*_{m-2}(t)}{dt},x'(t)-\tilde{x}'(t)\Big\rangle\\&+\Big\langle\eta^*_{m-2}(t)+\frac{d\eta^*_{m-3}(t)}{dt},x''(t)-\tilde{x}''(t)\Big\rangle+\ ...\ +\Big\langle\eta^*_2(t)+\frac{d\eta^*_1(t)}{dt},x^{(m-2)}(t)-\tilde{x}^{(m-2)}(t)\Big\rangle\\&\qquad+\Big\langle\eta^*_1(t),x^{(m-1)}(t)-\tilde{x}^{(m-1)}(t)\Big\rangle,\end{aligned}\tag{6}$$

where we denote by $x(\cdot)$ the feasible solution of the problem (P_H).

It follows from the definition of Hamiltonian function, (6), condition (iv), and the product rule from differentiation that

$$\begin{aligned}&\Big\langle\frac{d^m x(t)}{dt^m},x^*(t)\Big\rangle-\Big\langle\frac{d^m\tilde{x}(t)}{dt^m},x^*(t)\Big\rangle-L\Big(x(t),x'(t),...,x^{(m-1)}(t),t\Big)\\&+L\Big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),t\Big)\le\Big\langle(-1)^m\frac{d^m x^*(t)}{dt^m},x(t)-\tilde{x}(t)\Big\rangle+\frac{d}{dt}\Big\langle\eta^*_{m-1}(t),x(t)-\tilde{x}(t)\Big\rangle\\&\qquad+\frac{d}{dt}\Big\langle\eta^*_{m-2}(t),x'(t)-\tilde{x}'(t)\Big\rangle+...+\frac{d}{dt}\Big\langle\eta^*_1(t),x^{(m-2)}(t)-\tilde{x}^{(m-2)}(t)\Big\rangle.\end{aligned}$$

Now let us rewrite this inequality as follows

$$\begin{aligned}&L\Big(x(t),x'(t),...,x^{(m-1)}(t),t\Big)-L\Big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),t\Big)\\&\ge\left\langle\frac{d^m\Big(x(t)-\tilde{x}(t)\Big)}{dt^m},x^*(t)\right\rangle+\left\langle(-1)^{m+1}\frac{d^m x^*(t)}{dt^m},x(t)-\tilde{x}(t)\right\rangle\\&\qquad-\sum_{k=1}^{m-1}\frac{d}{dt}\left\langle\eta^*_k(t),\frac{d^{m-k-1}(x(t)-\tilde{x}(t))}{dt^{m-k-1}}\right\rangle\end{aligned}\tag{7}$$

Integrating the relation (7) implies

$$\int_0^1 \left[L\Big(x(t), x'(t), ..., x^{(m-1)}(t), t\Big) - L\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\Big)\right]dt$$
$$\geq \int_0^1 \left[\left\langle \frac{d^m\Big(x(t)-\tilde{x}(t)\Big)}{dt^m}, x^*(t)\right\rangle + \left\langle (-1)^{m+1}\frac{d^m x^*(t)}{dt^m}, x(t)-\tilde{x}(t)\right\rangle\right]dt$$
$$-\sum_{k=1}^{m-1}\left\langle \eta_k^*(t), \frac{d^{m-k-1}(x(t)-\tilde{x}(t))}{dt^{m-k-1}}\right\rangle\Bigg|_{t=0}^{t=1} = \int_0^1 \left[\left\langle \frac{d^m\Big(x(t)-\tilde{x}(t)\Big)}{dt^m}, x^*(t)\right\rangle\right.$$
$$\left.+\left\langle (-1)^{m+1}\frac{d^m x^*(t)}{dt^m}, x(t)-\tilde{x}(t)\right\rangle\right]dt + \sum_{k=1}^{m-1}\left\langle \eta_k^*(0), \frac{d^{m-k-1}(x(0)-\tilde{x}(0))}{dt^{m-k-1}}\right\rangle$$
$$-\sum_{k=1}^{m-1}\left\langle \eta_k^*(1), \frac{d^{m-k-1}(x(1)-\tilde{x}(1))}{dt^{m-k-1}}\right\rangle \tag{8}$$

Let us denote

$$A(t) = \left\langle \frac{d^m\Big(x(t)-\tilde{x}(t)\Big)}{dt^m}, x^*(t)\right\rangle + \left\langle (-1)^{m+1}\frac{d^m x^*(t)}{dt^m}, x(t)-\tilde{x}(t)\right\rangle \tag{9}$$

and transform the integral of $A(t)$. Consequently in this way, repeating the techniques from [21] by analogy it can be shown the following remarkable integral representation of $A(t)$ (see(9)):

$$\int_0^1 A(t)dt = \sum_{k=0}^{m-1}\left\langle \frac{d^{m-k-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{m-k-1}}, (-1)^{k+1}\frac{d^k x^*(0)}{dt^k}\right\rangle$$
$$-\sum_{k=0}^{m-1}\left\langle \frac{d^{m-k-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{m-k-1}}, (-1)^{k+1}\frac{d^k x^*(1)}{dt^k}\right\rangle. \tag{10}$$

Therefore, substitution (10) into (8) implies

$$\int_0^1 \left[L\Big(x(t), x'(t), ..., x^{(m-1)}(t), t\Big) - L\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\Big)\right]dt$$
$$\geq \sum_{k=0}^{m-1}\left\langle \frac{d^{m-k-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{m-k-1}}, (-1)^{k+1}\frac{d^k x^*(0)}{dt^k}\right\rangle - \sum_{k=0}^{m-1}\left\langle \frac{d^{m-k-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{m-k-1}}, (-1)^{k+1}\frac{d^k x^*(1)}{dt^k}\right\rangle$$
$$+\sum_{k=1}^{m-1}\left\langle \eta_k^*(0), \frac{d^{m-k-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{m-k-1}}\right\rangle - \sum_{k=1}^{m-1}\left\langle \eta_k^*(1), \frac{d^{m-k-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{m-k-1}}\right\rangle$$

Therefore, we have

$$\int_0^1 \Big[L\Big(x(t), x'(t), ..., x^{(m-1)}(t), t\Big) - L\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\Big)\Big]dt$$
$$\geq \sum_{k=0}^{m-1} \left\langle \eta_k^*(0) + (-1)^{k+1}\frac{d^k x^*(0)}{dt^k}, \frac{d^{m-k-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{m-k-1}} \right\rangle$$
$$-\sum_{k=0}^{m-1} \left\langle \eta_k^*(1) + (-1)^{k+1}\frac{d^k x^*(1)}{dt^k}, \frac{d^{m-k-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{m-k-1}}, \right\rangle, \ \Big(\eta_0^*(0) = \eta_0^*(1) = 0\Big). \tag{11}$$

But by the endpoint conditions (ii), (iii) at the endpoints $t = 0$ and $t = 1$

$$\left\langle \eta_k^*(0) + (-1)^{k+1}\frac{d^k x^*(0)}{dt^k}, \frac{d^{m-k-1}\Big(x(0)-\tilde{x}(0)\Big)}{dt^{m-k-1}} \right\rangle \geq 0, \ \forall x^{(k)}(0) \in M_k, \ k = 0, 1, ..., m-1$$
$$-\left\langle \eta_k^*(1) + (-1)^{k+1}\frac{d^k x^*(1)}{dt^k}, \frac{d^{m-k-1}\Big(x(1)-\tilde{x}(1)\Big)}{dt^{m-k-1}} \right\rangle \geq 0, \ \forall x^{(k)}(1) \in Q_k, \ k = 0, 1, ..., m-1. \tag{12}$$

Now taking into account (12), (13) in (11) for all feasible trajectories we have

$$\int_0^1 \Big[L\Big(x(t), x'(t), ..., x^{(m-1)}(t), t\Big) - L\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\Big)\Big]dt \geq 0,$$

that is, $J[x(t)] \geq J[\tilde{x}(t)], \forall x(t), t \in [0,1]$ and $\tilde{x}(t), t \in [0,1]$ is optimal. □

Notice that for a convex and closed set-valued mapping F ($gphF$ is closed) the conditions (i), (iv) of Theorem 3.1 can be replaced by subdifferential of Hamiltonian function with respect to $(x, v), v = (v_1, ..., v_{m-1})$ and v_m^*, respectively. To this end we need the following auxiliary theorem, which is proved under the "pointwise" closedness of F ($F(x, v)$ is closed set for each (x, v)). Evidently, this assumption is weaker than the usual closedness of F ($gphF$ is closed).

Theorem 3.2. *Let $F : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ be convex set-valued mapping and $F(x, v)$, be closed set for each $(x, v) \in domF$ and $H_F(x, v, v_m^*) = \sup_{v_m}\{\langle v_m, v_m^*\rangle : v_m \in F(x, v)\}, v_m^* \in \mathbb{R}^n$. Then*

$$\partial_{v_m^*} H_F(x, v, \bar{v}_m^*) = F(x, v; \bar{v}_m^*), \text{ where } F(x, v; \bar{v}_m^*) = \{v_m \in F(x, v) : \langle v_m, \bar{v}_m^*\rangle = H_F(x, v, \bar{v}_m^*)\}.$$

In particular, if $\bar{v}_m^* = 0$, then $\partial_{v_m^*} H_F(x, v, 0) = F(x, v)$.

Proof. If $v_m \in F(x, v; v_m^*)$ or, equivalently, if $v_m \in F(x, v)$ and $\langle v_m, \bar{v}_m^*\rangle = H_F(x, v, \bar{v}_m^*)$, then

$$H_F(x, v, v_m^*) - H_F(x, v, \bar{v}_m^*) \geq \langle v_m, v_m^*\rangle - \langle v_m, \bar{v}_m^*\rangle$$

or

$$H_F(x, v, v_m^*) - H_F(x, v, \bar{v}_m^*) \geq \langle v_m, v_m^* - \bar{v}_m^*\rangle,$$

that is $v_m \in \partial_{v_m^*} H_F(x, v, \bar{v}_m^*)$. Suppose now that $\bar{v}_m \in \partial_{v_m^*} H_F(x, v, \bar{v}_m^*)$. At first, we prove that $\bar{v}_m \in F(x, v)$. On the contrary, let $\bar{v}_m \notin F(x, v)$. Then by separation theorems of convex sets (see, for example [17]) there is a vector u such that

$$\sup_{v_m}\Big\{\langle v_m, u\rangle : v_m \in F(x, v)\Big\} < \langle \bar{v}_m, u\rangle. \tag{13}$$

On the other hand, since $\bar{v}_m \in \partial_{v_m^*} H_F(x, v, \bar{v}_m^*)$ and the supremum of difference no less than the difference of supremum's, it follows that

$$\sup_{v_m}\Big\{\langle v_m, v_m^* - \bar{v}_m^*\rangle : v_m \in F(x,v)\Big\} \geq H_F(x,v,v_m^*) - H_F(x,v,\bar{v}_m^*) \geq \langle \bar{v}_m, v_m^* - \bar{v}_m^*\rangle.$$

Recalling that v_m^* is an arbitrary vector, by setting in this inequality $v_m^* = \bar{v}_m^* + u$, we have

$$\sup_{v_m}\Big\{\langle v_m, u\rangle : v_m \in F(x,v)\Big\} \geq \langle \bar{v}_m, u\rangle,$$

which contradicts the inequality (13). It follows that $\bar{v}_m \in F(x,v)$. Now let us rewrite the condition $\bar{v}_m \in \partial_{v_m^*} H_F(x,v,\bar{v}_m^*)$ as follows

$$H_F(x,v,v_m^*) - \langle v_m^*, \bar{v}_m^*\rangle \geq H_F(x,v,\bar{v}_m^*) - \langle \bar{v}_m^*, \bar{v}_m^*\rangle.$$

By setting here $v_m^* = 0$, we have immediately $\langle \bar{v}_m^*, \bar{v}_m^*\rangle \geq H_F(x,v,\bar{v}_m^*)$. On the other hand, $\bar{v}_m \in F(x,v)$ implies that $\langle \bar{v}_m^*, \bar{v}_m^*\rangle \leq H_F(x,v,\bar{v}_m^*)$. As consequence, from the last two inequalities it follows that $\langle \bar{v}_m^*, \bar{v}_m^*\rangle = H_F(x,v,\bar{v}_m^*)$. This equality together with $\bar{v}_m \in F(x,v)$ means that $\bar{v}_m \in F(x,v;\bar{v}_m^*)$. In particular, if $\bar{v}_m^*) = 0$, then $H_F(x,v,0) = \langle v_m, 0\rangle = 0$ for all $v_m \in F(x,v)$ and so $\partial_{v_m^*} H_F(x,v,0) = F(x,v)$. □

Below we prove that the *gphF* of an upper semi-continuous set-valued mapping (not necessarily convex) with closed values is closed.

Lemma 3.3. *Let $F : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous set-valued mapping and $F(x,v)$, be closed set for each $(x,v) \in domF$. Then gphF is closed.*

Proof. We proceed by a contradiction argument; suppose that $(x^k, v^k, v_m^k) \in gphF$ is a convergent sequences and $(x^k, v^k, v_m^k) \to (x^0, v^0, v_m^0)$ but $v_m^0 \notin F(x^0, v^0)$. Then there exists an open set Λ containing $F(x^0, v^0)$ such that $v_m^0 \notin cl\Lambda$ Recalling that the set-valued mapping F is upper semi-continuous it follows that there exists a positive integer k_0 such that $v_m^k \in \Lambda$ for $\forall k > k_0$. Therefore $v_m^0 \in cl\Lambda$, a contradiction. □

Below for a closed set-valued mapping F in term of Hamiltonian function the conditions (i), (iv) of Theorem 3.1 are given in more convenient form.

Corollary 3.4. *For a closed set-valued mapping $F(\cdot, t) : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n$ the conditions (i), (iv) of Theorem 3.1 can be rewritten in term of Hamiltonian function in much more convenient form:*

$$\left((-1)^m \frac{d^m x^*(t)}{dt^m} + \frac{d\eta_{m-1}^*(t)}{dt}, \eta_{m-1}^*(t) + \frac{d\eta_{m-2}^*(t)}{dt}, ..., \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt}, \eta_1^*(t)\right)$$
$$\in \partial_{(x,v)} H_F\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), x^*(t), t\Big) - \partial_{(x,v)} L\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\Big)$$
$$\frac{d^m \tilde{x}(t)}{dt^m} \in \partial_{v_m^*} H_F\Big(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), x^*(t), t\Big), \text{ a.e. } t \in [0,1].$$

Proof. By Theorem 2.1 [17] and Theorem 3.2 we can write

$$F^*\Big(v_m^*; (x, v_1, ..., v_m), t\Big) = \partial_{(x,v)} H_F(x, v_1, ..., v_{m-1}, v_m^*, t),$$
$$F(x, v_1, ..., v_{m-1}; v_m^*, t) = \partial_{v_m^*} H_F(x, v_1, ..., v_{m-1}, v_m^*, t).$$

Then the assertions of corollary are equivalent with the conditions (i), (iv) of Theorem 3.1. □

Corollary 3.5. *Suppose that in the problem (P_H) the conditions*
$F\Big(x(t), x'(t), ..., x^{(m-1)}(t), t\Big) \equiv F\Big(x(t), t\Big)$, $L\Big(x(t), x'(t), ..., x^{(m-1)}(t), t\Big) \equiv L\Big(x(t), t\Big)$, $t \in [0,1]$ *are satisfied. Then the conditions (i)-(iii) of Theorem 3.1 can be simplified as follows*

$$(-1)^m \frac{d^m x^*(t)}{dt^m} \in F^*\Big(x^*(t); (\tilde{x}(t), \tilde{x}^{(m)}(t)), t\Big) - \partial_x L\Big(\tilde{x}(t), t\Big), \text{ a.e. } t \in [0,1],$$
$$(-1)^{k+1} \frac{d^k x^*(0)}{dt^k} \in K^*_{M_{m-k-1}}\left(\frac{d^{m-k-1}\tilde{x}(0)}{dt^{m-k-1}}\right),$$
$$(-1)^k \frac{d^k x^*(1)}{dt^k} \in K^*_{Q_{m-k-1}}\left(\frac{d^{m-k-1}\tilde{x}(1)}{dt^{m-k-1}}\right),\ k = 0,1,...,m-1.$$

Proof. Indeed, since in the present case $F(x, v_1, ..., v_{m-1}, t) \equiv F(x, t)$ we have $v^* = (v_1^*, ..., v_{m-1}^*) = 0$, which implies that in the left hand side of the Euler-Lagrange inclusion (i) the last $m-1$ components identically is equal to zero:

$$\eta_{m-1}^*(t) + \frac{d\eta_{m-2}^*(t)}{dt} \equiv 0, ..., \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} \equiv 0,\ \eta_1^*(t) \equiv 0.$$

Consequently, by sequentially substitution, it follows that $\eta_{m-1}^*(t) \equiv 0$ and so the second term in the first component in (i) is equal to zero, that is $\frac{d\eta_{m-1}^*(t)}{dt} \equiv 0$ identically. Now taking into account that in the endpoint conditions (ii), (iii) $\eta_k^*(0) = \eta_k^*(1) = 0$, we have the desired result. The proof of corollary is completed □

We can state the following theorem concerning optimization of (P_H) in the non-convex case.

Theorem 3.6. *Let (1)-(3) be nonconvex problem, that is $L(\cdot, t) : \left(\mathbb{R}^n\right)m \to \mathbb{R}^1$ be non-convex function, $F(\cdot, t) : \left(\mathbb{R}^n\right)m \rightrightarrows \mathbb{R}^n$ be a non-convex set-valued mapping and $M_k, Q_k \subset \mathbb{R}^n, k = 0, 1, ..., m-1$ be non-convex subsets. Moreover, let $K_{M_k}\left(\tilde{x}^{(k)}(0)\right)$ and $K_{Q_k}\left(\tilde{x}^{(k)}(1)\right)$ $(k = 0, 1, ..., m-1)$ be the cones of tangent directions. Then for the optimality of the trajectory $\tilde{x}(t)$, $t \in [0, 1]$ in the problem (4)-(6) it is sufficient that there exists a collection of absolutely continuous functions $\{x^*(t), \eta_k^*(t), k = 1, ..., m-1\}$, $t \in [0, 1]$ satisfying the conditions:*

$$(a) \quad \left((-1)^m \frac{d^m x^*(t)}{dt^m} + \frac{d\eta_{m-1}^*(t)}{dt} + x^*(t),\ \eta_{m-1}^*(t) + \frac{d\eta_{m-2}^*(t)}{dt} + \frac{dx^*(t)}{dt}, ..., \eta_2^*(t)\right.$$

$$\left. + \frac{d\eta_1^*(t)}{dt} + \frac{d^{m-2}x^*(t)}{dt^{m-2}}, \eta_1^*(t) + \frac{d^{m-1}x^*(t)}{dt^{m-1}}\right) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m)}(t)), t\right),\ a.e.\ t \in [0, 1].$$

$$(b) \quad \eta_k^*(0) + (-1)^{k+1} \frac{d^k x^*(0)}{dt^k} \in K_{M_{m-k-1}}^*\left(\tilde{x}^{(m-k-1)}(0)\right),\ k = 0, 1, ..., m-1,$$

$$-\eta_k^*(1) - (-1)^{k+1} \frac{d^k x^*(1)}{dt^k} \in K_{Q_{m-k-1}}^*\left(\tilde{x}^{(m-k-1)}(1)\right),\ k = 0, 1, ..., m-1,$$

$$(c) \quad L\left(x, v_1, ..., v_{m-1}, t\right) - L\left(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), t\right) \geq \left\langle x^*(t), x - \tilde{x}(t)\right\rangle$$

$$+ \sum_{k=1}^{m-1} \left\langle \frac{d^k x^*(t)}{dt^k}, v_k - \tilde{x}^{(k)}(t)\right\rangle,\ \forall (x, v) \in \left(\mathbb{R}^n\right)^m,\ v = (v_1, ...v_{m-1}),$$

$$(d) \quad \left\langle \frac{d^m \tilde{x}(t)}{dt^m}, x^*(t)\right\rangle = H_F\left(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), x^*(t), t\right),\ a.e.\ t \in [0, 1].$$

Proof. We proceed by analogy with the preceding derivation in the proof of Theorem 3.1:

$$H_F\left(x(t), x'(t), ..., x^{(m-1)}(t), x^*(t), t\right) - H_F\left(\tilde{x}(t), \tilde{x}'(t), ..., \tilde{x}^{(m-1)}(t), x^*(t), t\right)$$
$$\leq \left\langle (-1)^m \frac{d^m x^*(t)}{dt^m} + \frac{d\eta_{m-1}^*(t)}{dt} + x^*(t), x(t) - \tilde{x}(t)\right\rangle + \left\langle \eta_{m-1}^*(t) + \frac{d\eta_{m-2}^*(t)}{dt} + \frac{dx^*(t)}{dt}, x'(t) - \tilde{x}'(t)\right\rangle$$
$$+ ... + \left\langle \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} + \frac{d^{m-2}x^*(t)}{dt^{m-2}}, x^{(m-2)}(t) - \tilde{x}^{(m-2)}(t)\right\rangle$$
$$+ \left\langle \eta_1^*(t) + \frac{d^{m-1}x^*(t)}{dt^{m-1}}, x^{(m-1)}(t) - \tilde{x}^{(m-1)}(t)\right\rangle,$$

whereas

$$\left\langle \frac{d^m x(t)}{dt^m}, x^*(t) \right\rangle - \left\langle \frac{d^m \tilde{x}(t)}{dt^m}, x^*(t) \right\rangle \le \left\langle (-1)^m \frac{d^m x^*(t)}{dt^m} + \frac{d\eta^*_{m-1}(t)}{dt} + x^*(t), x(t) - \tilde{x}(t) \right\rangle$$
$$+\left\langle \eta^*_{m-1}(t) + \frac{d\eta^*_{m-2}(t)}{dt} + \frac{dx^*(t)}{dt}, x'(t) - \tilde{x}'(t) \right\rangle + \dots + \left\langle \eta^*_1(t) + \frac{d^{m-1}x^*(t)}{dt^{m-1}}, x^{(m-1)}(t) - \tilde{x}^{(m-1)}(t) \right\rangle,$$
$$+\left\langle \eta^*_2(t) + \frac{d\eta^*_1(t)}{dt} + \frac{d^{m-2}x^*(t)}{dt^{m-2}}, x^{(m-2)}(t) - \tilde{x}^{(m-2)}(t) \right\rangle.$$

Moreover, observe that for non-convex $L(\cdot,t)$ by the condition (c) for all feasible trajectories $x(\cdot)$ the following inequality is satisfied

$$L\Big(x(t), x'(t), \dots, x^{(m-1)}(t), t\Big) - L\Big(\tilde{x}(t), \tilde{x}'(t), \dots, \tilde{x}^{(m-1)}(t), t\Big)$$
$$\ge \left\langle x^*(t), x(t) - \tilde{x}(t) \right\rangle + \sum_{k=1}^{m-1} \left\langle \frac{d^k x^*(t)}{dt^k}, x^{(k)}(t) - \tilde{x}^{(k)}(t) \right\rangle.$$

Therefore, the relation (7) is justified. In what follows the proof of the second part runs similarly. □

Let us denote $\frac{\partial L}{\partial x^{(k)}} = \frac{\partial L\left(\tilde{x}(t), \tilde{x}'(t), \dots, \tilde{x}^{(m-1)}(t), t\right)}{\partial x^{(k)}}$, $k = 0, \dots, m-1$ and require that

$$L\Big(x, v_1, \dots, v_{m-1}, t\Big) - L\Big(\tilde{x}(t), \tilde{x}'(t), \dots, \tilde{x}^{m-1}(t), t\Big) \ge \left\langle \frac{\partial L}{\partial x}, x - \tilde{x}(t) \right\rangle$$
$$\sum_{k=1}^{m-1} \left\langle \frac{\partial L}{\partial x^{(k)}}, v_k - \tilde{x}^{(k)}(t) \right\rangle, \ \forall (x, v) \in \left(\mathbb{R}^n\right)^m, \ v = (v_1, \dots, v_{m-1}). \tag{14}$$

On the other hand, suppose that the following Euler-Lagrange inclusion is satisfied:

$$\Bigg((-1)^m \frac{d^m x^*(t)}{dt^m} + \frac{d\eta^*_{m-1}(t)}{dt} + \frac{\partial L}{\partial x}, \ \eta^*_{m-1}(t) + \frac{d\eta^*_{m-2}(t)}{dt} + \frac{\partial L}{\partial x'}, \dots, \eta^*_2(t)$$
$$+\frac{d\eta^*_1(t)}{dt} + \frac{\partial L}{\partial x^{(m-2)}}, \eta^*_1(t) + \frac{\partial L}{\partial x^{(m-1)}}\Bigg) \in F^*\Big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \dots, \tilde{x}^{(m)}(t), t\Big). \tag{15}$$

Corollary 3.7. *Under the conditions (14), (15) and conditions (b), (d) of Theorem 3.6 the result of Theorem 3.6 remains true.*

Proof. The formulated corollary can be proved similarly to previous theorems, and so its proof omitted. □

4. Applications of higher order optimization for (P_H) to calculus of variations

Now we apply Theorem 3.6 to get Euler-Poisson (see, for example [9]) sufficient conditions of optimality for the calculus of variations. Let us consider the Variational problem of a functional with a single function, but containing its higher order derivatives. Accordingly, boundary value problem for the calculus of variations is this:

$$\begin{aligned} &\text{minimize } J[x(\cdot)] = \int_0^1 L(x(t), x'(t)..., x^{(m-1)}(t), t)dt) \\ (P_L) \qquad & x(0) = x_0^0,\ x'(0) = x_1^0,\ ...,\ x^{(m-1)}(0) = x_{m-1}^0, \\ & x(1) = x_0^1,\ x'(1) = x_1^1,\ ...,\ x^{(m-1)}(1) = x_{m-1}^1, \end{aligned}$$

where the Lagrangian L is a real-valued function with continuous first partial derivatives, $x(\cdot) \in C^{m-1}\big([0,1]\big)$ and $x_k^0,\ x_k^1,\ k = 0,1,...,m-1$ are fixed vectors. In the presented case $domF(\cdot,t) \equiv \big(\mathbb{R}^n\big)^m$ and $F(\cdot,t) \equiv \mathbb{R}^n$. It follows that $K_{gphF}(x,v,v_m) = \big(\mathbb{R}^n\big)^{m+1}$ and $K^*_{gphF}(x,v,v_m) \equiv \{0,0,0\}$. Therefore

$$F^*\big(v_m^*;(x,v,v_m)\big) = \begin{cases} (0,0) \in \big(\mathbb{R}^n\big)^m, & \text{if } v_m^* = 0, \\ \varnothing\ , & \text{if } v_m^* \neq 0. \end{cases}$$

Then, obviously, $F^* \equiv \{(0,0)\}$ so that $x^* = v_m^* = 0 \in \mathbb{R}^n,\ v^* = 0 \in \big(\mathbb{R}^n\big)^{m-1}$. It means that $x^{*(k)}(t) \equiv 0,\ k = 0,...,m,\ t \in [0,1]$.

Therefore, it follows from (15) that

$$\begin{aligned} &\frac{d\eta_{m-1}^*(t)}{dt} + \frac{\partial L}{\partial x} = 0,\ \eta_{m-1}^*(t) + \frac{d\eta_{m-2}^*(t)}{dt} + \frac{\partial L}{\partial x'} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} + \frac{\partial L}{\partial x^{(m-2)}} = 0,\ \eta_1^*(t) + \frac{\partial L}{\partial x^{(m-1)}} = 0,. \end{aligned} \tag{16}$$

Now, beginning from the last relation by sequential substitution in (16) we derive the Euler-Poisson equation for the problem (P_L):

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x''}\right) - ... + (-1)^{m-1}\frac{d^{m-1}}{dt^{m-1}}\left(\frac{\partial L}{\partial x^{(m-1)}}\right) = 0. \tag{17}$$

At least we remark that for the problem of calculus of variations (P_L) the endpoint condition (b) of Theorem 3.6 is superfluous. Indeed, for problem (P_L)

$$M_k = \left\{x_k^0\right\},\ Q_k = \left\{x_k^1\right\},\ k = 0,1,...,m-1$$

and so

$$K_{M_k}\big(\tilde{x}^{(k)}(0)\big) = \{0\} \text{ and } K_{Q_k}\big(\tilde{x}^{(k)}(1)\big) = \{0\},\ \ k = 0,1,...,m-1.$$

Consequently

$$K^*_{M_k}\big(\tilde{x}^{(k)}(0)\big) = \mathbb{R}^n\ K^*_{Q_k}\big(\tilde{x}^{(k)}(1)\big) = \mathbb{R}^n,\ \ k = 0,1,...,m-1. \tag{18}$$

Therefore, the endpoint conditions at the endpoints $t = 0$ and $t = 1$ for problem (P_L) is unnecessary.

It is well known that in calculus of variations for optimality of the trajectory $\tilde{x}(t)$ the Euler-Poisson equation (17) is necessary condition. Thus, we can formulate the following corollary.

Corollary 4.1. *Let the Lagrangian L be a real-valued function with continuous first partial derivatives such that the inequality (14) holds. Then for $\tilde{x}(t)$, $t \in [0,1]$ to be optimal solution in calculus of variations (P_L) it is necessary and sufficient that the Euler- Poisson equation is satisfied:*

$$\sum_{k=1}^{m}(-1)^{k-1}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial x^{(k-1)}}\right)=0.$$

This equation is an ordinary differential equation of order $2(m-1)$ and requires $2(m-1)$ boundary conditions.

Remark 4.2. *In fact, in the proof of Corollary 4.1 it was shown that for the problem (P_H) with higher order differential inclusions, where $M_k=\{x_k^0\}$, $Q_k=\{x_k^1\}$, $k=0,1,...,m-1$ by virtue of relations (18) the endpoint conditions at the endpoints $t=0$ and $t=1$ are unnecessary. Nevertheless, in this particular case, since the equality relations*

$$\frac{d^{m-k-1}\big(x(0)-\tilde{x}(0)\big)}{dt^{m-k-1}}=\frac{d^{m-k-1}\big(x(1)-\tilde{x}(1)\big)}{dt^{m-k-1}}=0$$

hold it follows that the right hand side of the basic inequality (11) is equal to zero and so

$$\int_0^1\Big[L\big(x(t),x'(t),...,x^{(m-1)}(t),t\big)-L\big(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(m-1)}(t),t\big)\Big]dt\geq 0$$

for all feasible solutions $x(t)$, $t\in[0,1]$, that is, $\tilde{x}(t)$ is optimal. □

Now, let us consider optimization of the following higher order "linear" differential inclusion with initial value problem, labelled by (P_{HIV}):

$$(P_{HIV})\qquad \begin{aligned}&\text{minimize } J[x(\cdot)]=\int_0^1 L_0(x(t))dt\\ &\frac{d^m x(t)}{dt^m}\in F\big(x(t),x'(t),x''(t),...,x^{(m-1)}(t)\big),\ \text{a.e. } t\in[0,1],\\ &x(0)=x_0^0,\ x'(0)=x_0^1,\ x''(0)=x_0^2,\ ...,\ x^{(m-1)}(0)=x_0^{m-1},\\ &F\big(x,v_1,...,v_{m-1}\big)\equiv A_0x+\sum_{k=1}^{m-1}A_kv_k+BU.\end{aligned}$$

Here the integrand $L_0(x)$ is equal to the quadratic form $L_0(x)=\frac{1}{2}\langle x,Qx\rangle+\langle b,x\rangle$, where Q is a symmetric nonnegative semidefinite $n\times n$ matrix and b, x_0^k, $k=0,...,m-1$, are fixed points; $x_0^k\in\mathbb{R}^n$. Obviously, this function is convex and by Theorem 3.1 $\partial_{(x,v)}L(x,v_1,...,v_{m-1},t)\equiv\Big\{\partial_x L_0(x)\times\underbrace{(0,...,0)}_{m}\Big\}$, where $\partial_x L_0(x)=\{Qx+b\}$.
Moreover, A_i, $i=0,...,m-1$ and B are $n\times n$ and $n\times r$ matrices, respectively, U is a convex closed subset of $\mathbb{R}^r$. In fact, the problem is to find a controlling parameter $\tilde{u}(t)\in U$ for initial value problem with higher order "linear" differential inclusions and free endpoint constraints such that the arc $\tilde{x}(t)$ corresponding to it minimizes $J[x(\cdot)]$.

We will apply the Theorem 3.1. Since $M_k=\{x_0^k\}$, $k=0,...,m-1$, by the first formula of (18) it follows that the endpoint condition (ii) of Theorem 3.1 at the point $t=0$ is superfluous. On the other hand in the problem (P_{HIV}) $Q_k\equiv\mathbb{R}^n$, $k=0,...,m-1$ and $K_{Q_k}\big(\tilde{x}^{(k)}(1)\big)\equiv\mathbb{R}^n$. Then $K^*_{Q_k}\big(\tilde{x}^{(k)}(1)\big)=\{0\}$. Consequently, the endpoint condition (iii) of Theorem 3.1 at the point $t=1$ is transformed into equalities

$$\eta_k^*(1) + (-1)^{k+1}\frac{d^k x^*(1)}{dt^k} = 0\ \left(\eta_0^*(1) = 0\right),\ k = 0, 1, ..., m-1. \tag{19}$$

To formulate of m th-order adjoint Euler-Lagrange differential inclusion for the convex optimization problem (P_{HIV}) we should compute $F^*\big(v_m^*; (x, v, v_m)\big)$. Taking into account that $F\big(x, v_1, ..., v_{m-1}\big) \equiv A_0 x + \sum_{k=1}^{m-1} A_k v_k + BU$ it can be easily computed that

$$H_F(x, v, v_m^*) = \sup_{v_m}\left\{\langle v_m, v_m^*\rangle : v_m \in F(x, v)\right\} = \sup_u \left\{\left\langle A_0 x + \sum_{k=1}^{m-1} A_k v_k + BU, v_m^*\right\rangle : u \in U\right\}$$
$$= \langle x, A_0^* v_m^*\rangle + \sum_{k=1}^{m-1}\left\langle v_k, A_k^* v_m^*\right\rangle + \sup_u\left\{\langle Bu, v_m^*\rangle : u \in U\right\},$$

where A^* is the adjoint (transposed) matrix of A.
Then if $\tilde{v}_m = A_0\tilde{x} + \sum_{k=1}^{m-1} A_k\tilde{v}_k + B\tilde{u},\ \tilde{u} \in U$ one has

$$F^*\left(v_m^*; (\tilde{x}, \tilde{v}, \tilde{v}_m)\right) = \begin{cases} \left(A_0^* v_m^*, A_1^* v_m^*, ..., A_{m-1}^* v_m^*\right), & -B^* v_m^* \in K_U^*(\tilde{u}), \\ \varnothing\ , & -B^* v_m^* \notin K_U^*(\tilde{u}). \end{cases} \tag{20}$$

Thus, using (20) and the relation $\partial_x L_0(x) = \{Qx + b\}$ by the condition (i) of Theorem 3.1 we have the following system of Euler-Lagrange's type linear adjoint equations:

$$\begin{aligned} (-1)^m \frac{d^m x^*(t)}{dt^m} + \frac{d\eta_{m-1}^*(t)}{dt} &= A_0^* x^*(t) - Qx - b, \\ \eta_{m-1}^*(t) + \frac{d\eta_{m-2}^*(t)}{dt} &= A_1^* x^*(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} &= A_{m-2}^* x^*(t), \\ \eta_1^*(t) &= A_{m-1}^* x^*(t). \end{aligned} \tag{21}$$

By sequential substitution, we find that

$$\begin{aligned} \eta_1^*(t) = A_{m-1}^* x^*(t), \eta_2^*(t) &= A_{m-2}^* x^*(t) - A_{m-1}^*\frac{dx^*(t)}{dt}, \\ \eta_3^*(t) &= A_{m-3}^* x^*(t) - A_{m-2}^*\frac{dx^*(t)}{dt} + A_{m-1}^*\frac{d^2x^*(t)}{dt^2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \eta_{m-1}^*(t) = A_1^* x^*(t) - A_2^*\frac{dx^*(t)}{dt} + A_3^*\frac{d^2x^*(t)}{dt^2} &- ... - (-1)^{m-1} A_{m-1}^*\frac{d^{m-2}x^*(t)}{dt^{m-2}}. \end{aligned} \tag{22}$$

Then taking into account the relations (22) in (19) we can easily see that

$$\sum_{k=1}^{s}\left[(-1)^{s-k} A_{m-k}^*\frac{d^{s-k}x^*(1)}{dt^{s-k}} + (-1)^{s+1}\frac{d^s x^*(1)}{dt^s}\right] = 0\ (x^*(1) = 0),\ s = 1, 2, ..., m-1. \tag{23}$$

In turn by substituting the expression for $\eta_k^*(t),\ k = 1, ..., m-1$ into the first equation in (21) we can define the following Euler-Lagrange type adjoint differential inclusion (equation);

$$(-1)^m \frac{d^m x^*(t)}{dt^m} = \sum_{k=0}^{m-1} (-1)^k A_k^* \frac{d^k x^*(t)}{dt^k} - Qx - b. \tag{24}$$

On the other hand, the Weierstrass-Pontryagin maximum principle [17]-[19] of theorem is an immediate consequence of the conditions (iv) of Theorem 3.1 and formula (20):

$$\left\langle B\tilde{u}(t), x^*(t) \right\rangle = \sup_{u \in U} \left\langle Bu, x^*(t) \right\rangle. \tag{25}$$

Finally, we can formulate the obtained result as follows.

Theorem 4.3. *The arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{u}(t)$ minimizes the quadratic cost functional in the higher order linear optimal control problem (P_{HIV}) with initial value problem and free endpoint constraints, if there exists an absolutely continuous function $x^*(t)$ satisfying the higher order adjoint differential equation (24), the endpoint condition (23) and Weierstrass-Pontryagin maximum principle (25).*

Note that for $m = 2$ in problem (P_{HIV}), we have the following second order linear problem:

$$\text{minimize } J_0[x(\cdot)] \text{ subject to } \frac{d^2x(t)}{dt^2} \in F\big(x(t), x'(t)\big), \text{ a.e. } t \in [0, 1],\ x(0) = x_0^0,\ x'(0) = x_0^1,$$

where $F(x, v_1) \equiv A_0x + A_1v_1 + BU$. Then we can verify easily that the conditions (23), (24) can be simplified as follows:

$$\frac{d^2x^*(t)}{dt^2} = A_0^*x^*(t) - A_1^*\frac{dx^*(t)}{dt} - Qx - b;\ \frac{dx^*(1)}{dt} = 0,\ x^*(1) = 0. \ \square$$

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