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Biwarped Product Submanifolds of a Kähler Manifold

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Abstract. We study biwarped product submanifolds which are special cases of multiply warped product submanifolds in Kähler manifolds. We observe the non-existence of such submanifolds under some circumstances. We show that there exists a non-trivial biwarped product submanifold of a certain type by giving an illustrate example. We also give a necessary and sufficient condition for such submanifolds to be locally trivial. Moreover, we establish an inequality for the squared norm of the second fundamental form in terms of the warping functions for such submanifolds. The equality case is also discussed.

1. Introduction

Bishop and O' Neill [4] introduced the concept of warped product of Riemannian manifolds to construct a large class of complete manifolds of negative curvature. This concept is also a generalization of the usual product of Riemannian manifolds. Nölker [21] considered the notion of multiply warped products as a generalization of the warped products. Since that time, multiply warped products has been studied by many authors. For example, Ünal [38] studied partially the geometry of the multiply warped products when the metrics of such products are Lorentzian. Curvature properties of such products were investigated by Dobarro and Ünal [14].

The concept of warped products or multiply warped products play very important roles in physics as well as in differential geometry, especially in the theory of relativity. In fact, the standard spacetime models such as Robertson-Walker, Schwarschild, static and Kruscal are warped products. Also, the simplest models of neighborhoods of stars and black holes are warped products [23]. Moreover, many solutions to Einstein's field equation can be expressed in terms of warped products [2].

In differential geometry, especially in almost complex geometry, one of the most intensively research areas is the theory of submanifolds. In fact, the almost complex structure of an almost Hermitian manifold determines several classes of submanifolds such as holomorphic(invariant), totally real(anti-invariant) [40], CR- [3], generic [6], slant [7], semi-slant [24], hemi-slant(pseudo-slant) [5, 29], pointwise slant [10, 15], bi-slant [5], skew CR- and generic submanifolds [25]. Among them, the last one contains all other classes.

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The theory of warped product submanifolds has been becoming a popular research area since Chen [8] studied the warped product CR-submanifolds in Kähler manifolds. In fact, several classes of warped product submanifolds appeared in the last fifteen years (see [26, 28–31]). Also, warped product submanifolds have been studying in different kinds of structures such as nearly Kähler [33], para-Kähler [12], locally product Riemannian [32, 34], cosymplectic [37], Sasakian [19], generalized Sasakian [27], trans-Sasakian [17], $(\kappa, \mu)-$ [36], Kenmotsu [16, 20] and quaternion [18]. Most of the studies related to the theory of warped product submanifolds can be found in Chen's coming book [9]. Recently, Chen and Dillen [11] studied multiply warped product submanifolds in Kähler manifolds and they obtained very useful optimal inequalities. We note that such submanifolds were also studied in Kenmotsu manifolds [22].

In this paper, we consider and study biwarped product submanifolds in Kähler manifolds. Here, a biwarped product means that a multiply warped product which has only two fibers. We observe the non-existence of biwarped product submanifolds under some circumstances. After giving an illustrate example, we study such submanifolds in case of the base factor is holomorphic and one of the two fibers is totally real and the other one is pointwise slant submanifold. We also give characterization for this kind of submanifolds. Moreover, we investigate the behavior of the second fundamental form of such submanifolds and as a result, we give a necessary and sufficient condition for such manifolds to be locally trivial. Furthermore, an inequality for the squared norm of the second fundamental form in terms of the warping functions for such submanifolds is obtained. The equality case is also considered.

2. Preliminaries

In this section, we recall the fundamental definitions and notions needed further study. In fact, in subsection 2.1, we will recall the definition of the multiply warped product manifolds. In subsection 2.2, we will give the basic background for submanifolds of Riemannian manifolds. The definition of a Kähler manifold and the some classes of submanifolds of Kähler manifolds are placed in subsection 2.3.

2.1. Multiply warped product manifolds

Let (M_i, g_i) be Riemannian manifolds for any $i \in \{0, 1, ..., k\}$ and let $f_j : M_0 \to (0, \infty)$ be smooth functions for any $j \in \{1, 2, ..., k\}$. Then the *multiply warped product manifold* [21] $\bar{M} = M_0 \times_{f_1} M_1 \times ... \times_{f_k} M_k$ is the product manifold $\tilde{M} = M_0 \times M_1 \times ... \times M_k$ endowed with the metric

$$g = \pi_0^*(g_0) \oplus (f_1 \circ \pi_0)^2 \pi_1^*(g_1) \oplus ... \oplus (f_k \circ \pi_0)^2 \pi_k^*(g_k).$$

More precisely, for any vector fields X and Y of $\bar{M}$, we have

$$g(X, Y) = g_0(\pi_{0*}X, \pi_{0*}Y) + \sum_{i=1}^{k} (f_i \circ \pi_0)^2 g_i(\pi_{i*}X, \pi_{i*}Y),$$

where $\pi_i : \tilde{M} \to M_i$ is the canonical projection of $\tilde{M}$ onto M_i, $\pi_i^*(g_i)$ is the pullback of g_i by π_i and the subscript $*$ denotes the derivative map of π_i for each i. Each function f_j is called a *warping function* and each manifold (M_j, g_j), $j \in \{1, 2, ..., k\}$ is called a *fiber* of the multiply warped product $\bar{M}$. The manifold (M_0, g_0) is called a *base manifold* of $\bar{M}$. As well known, the base manifold of $\bar{M}$ is totally geodesic and the fibers of $\bar{M}$ are totally umbilic in $\bar{M}$.

Let $\bar{M} = M_0 \times_{f_1} M_1 \times ... \times_{f_k} M_k$ be multiply warped product manifold, if $k = 1$, then we get a (singly) warped product [4]. We call the multiply warped product manifolds as *biwarped product manifolds* for $k = 2$. In other words, a biwarped product manifold has the form $M_0 \times_{f_1} M_1 \times_{f_2} M_2$. We say that a biwarped product manifold is *trivial*, if the warping functions f_1 and f_2 are constants. Note that biwarped product manifolds were also studied under the name of *twice warped products* [1].

Let $\bar{M} = M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product manifold with the Levi-Civita connection $\bar{\nabla}$ and ${}^i\nabla$ denote the Levi-Civita connection of M_i for $i \in \{0, 1, 2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathcal{L}(M_i)$ and use the same notation for a vector field and for its lift. On the other hand, since the map π_0 is an isometry and π_1 and π_2 are (positive) homotheties, they preserve the Levi-Civita connections. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas for a biwarped product manifold are given by the following.

Lemma 2.1. *([1]) Let $\bar{M} = M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product manifold. Then, for any $U, V \in \mathcal{L}(M_0)$, $X \in \mathcal{L}(M_i)$ and $Z \in \mathcal{L}(M_j)$, we have*

$$\bar{\nabla}_U V = {}^0\nabla_U V, \tag{1}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln f_i)X, \tag{2}$$

$$\bar{\nabla}_X Z = \begin{cases} 0 & \text{if } i \neq j, \\ {}^i\nabla_X Z - g(X, Z)\operatorname{grad}(\ln f_i) & \text{if } i = j, \end{cases} \tag{3}$$

where, $i, j \in \{1, 2\}$.

We note that $\operatorname{grad}(\ln f_i) \in \mathcal{L}(M_0)$ for $i = 1, 2$ [21].

2.2. *Submanifolds of Riemannian manifolds*

Let M be an isometrically immersed submanifold in a Riemannian manifold $(\bar{M}, g)$. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to the metric g and let ∇ and $\nabla^{\perp}$ be the induced, and induced normal connection on M, respectively. Then, for all $U, V \in TM$ and $\xi \in T^{\perp}M$, the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{4}$$

and

$$\bar{\nabla}_U \xi = -A_{\xi} U + \nabla^{\perp}_U \xi \ , \tag{5}$$

where TM is the tangent bundle and $T^{\perp}M$ is the normal bundle of M in $\bar{M}$. Additionally, h is the *second fundamental form* of M and A_{ξ} is the Weingarten endomorphism associated with ξ. The second fundamental form h and the *shape operator* A related by

$$g(h(U, V), \xi) = g(A_{\xi} U, V) \ . \tag{6}$$

The *mean curvature vector field* H of M is given by $H = \frac{1}{m}(trace\, h)$, where $dim(M) = m$. We say that the submanifold M is *totally geodesic* in $\bar{M}$ if $h = 0$, and *minimal* if $H = 0$. The submanifold M is called *totally umbilical* if $h(U, V) = g(U, V)H$ for all $U, V \in TM$. If the manifold M is totally umbilical and its mean curvature vector field H is *parallel*, i.e. $g(\bar{\nabla}_U H, \xi) = 0$ for all $U \in TM$ and $\xi \in T^{\perp}M$, then the submanifold M is said to be *spherical* or *extrinsic sphere*.

Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be any two distributions on M. Then we say that M is $\mathcal{D}^1$*-geodesic*, if $h(U, V) = 0$ for all $U, V \in \mathcal{D}^1$ and we say that $(\mathcal{D}^1, \mathcal{D}^2)$*-mixed geodesic* if $h(V, X) = 0$ for $V \in \mathcal{D}^1$ and $X \in \mathcal{D}^2$. If for all $V \in \mathcal{D}^1$ and $X \in \mathcal{D}^2$, $\nabla_X V \in \mathcal{D}^1$, then $\mathcal{D}^1$ is called $\mathcal{D}^2$*-parallel*. We say that $\mathcal{D}^1$ is *autoparallel* if $\mathcal{D}^1$ is $\mathcal{D}^1$-parallel. If a distribution on M is autoparallel, then by the Gauss formula it is totally geodesic.

2.3. *Some classes of submanifolds of Kähler manifolds*

Let $\bar{M}$ be an almost complex manifold with almost complex structure J. If there is a Riemannian metric g on $\bar{M}$ satisfying

$$g(JX, JY) = g(X, Y) \tag{7}$$

for any $X, Y \in T\bar{M}$, then we say that $(\bar{M}, J, g)$ is an almost Hermitian manifold. Let $\bar{\nabla}$ be the Levi-Civita connection of the almost Hermitian manifold $(\bar{M}, J, g)$ with respect to g. Then $(\bar{M}, J, g)$ is called a *Kähler manifold* [40] if J is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X J)Y = 0 \tag{8}$$

for all $X, Y \in T\bar{M}$.

Let M be a Riemannian manifold isometrically immersed submanifold in a Kähler manifold $(\bar{M}, J, g)$. Then the submanifold M is called a *pointwise slant submanifold* [10, 15] if for every point p of M, the Wirtinger angle $\theta(V)$ between JV and the tangent space T_pM at p is independent of the choice of the nonzero vector $V \in T_pM$. In this case, the angle θ can be viewed as a function on M and it is called the *slant function* of M. We say that the pointwise slant submanifold M is *proper* neither $\cos\theta(p) = 0$ nor $\sin\theta(p) = 0$ at each point $p \in M$. (This condition is different from Chen's definition, see [10]).

Now, let M be a submanifold of a Kähler manifold $(\bar{M}, J, g)$. For any $V \in TM$, we put

$$JV = TV + FV \ . \tag{9}$$

Here TV is the tangential part of JV, and FV is the normal part of JV. Then M is a pointwise slant submanifold of $\bar{M}$ if and only if, for any $V \in TM$, we have

$$T^2V = -\cos^2\theta V \tag{10}$$

for some function θ defined on M [10]. For a pointwise slant submanifold of $\bar{M}$, using (9), (10) and the Kähler structure, it is not difficult to prove the following two facts.

$$g(TU, TV) = \cos^2\theta g(U, V) \ , \tag{11}$$

$$g(FU, FV) = \sin^2\theta g(U, V) \tag{12}$$

for $U, V \in TM$.

Let M be a pointwise slant submanifold with slant function θ of a Kähler manifold $(\bar{M}, J, g)$. If the function θ is a constant, i.e., it is also independent of the choice of the point $p \in M$, then we say that M is a *slant submanifold* [7].

If $\theta \equiv 0$, then M is called a *holomorphic* or *complex submanifold* of $\bar{M}$ [40]. In that case, the tangent space T_pM is invariant with respect to the almost complex structure J at each point $p \in M$, i.e., $J(T_pM) \subseteq T_pM$.

If $\theta \equiv \frac{\pi}{2}$, then M is called a *totally real submanifold* of $\bar{M}$ [40]. In which case, the tangent space T_pM is anti-invariant with respect to the almost complex structure J for every point p of M, i.e., $J(T_pM) \subseteq T_p^{\perp}M$.

3. Generalized *J*-induced submanifolds of order 1

In this section, after renaming the generic submanifolds (in the sense of Ronsse [25]), we will give some results concerning totally geodesicness and integrability of the distributions which are involved in the definition of such submanifolds.

The most general class of submanifolds determined by the almost complex structure is the class of generic submanifols which was defined by Ronsse [25]. There are two other classes of submanifolds with the same name. One of these is the class of defined by Chen [6] and the other one is the class defined by Yano and Kon [39]. Because of these facts, to avoid name confusion, we call the generic submanifolds (in the sense of Ronsse [25]) as *generalized structure induced* or *generalized J-induced submanifolds*.

Let $(\bar{M}, J, g)$ be a Kähler manifold and M be a submanifold of $\bar{M}$. For all $p \in M$ and for any tangent vectors $U, V \in T_pM$, using (7) and (9), we have

$$g(TU, V) = -g(U, TV) \ . \tag{13}$$

Hence, it follows that

$$g(T^2U, V) = g(T^2V, U) \ . \tag{14}$$

The equation (14) says that T^2 is symmetric operator on T_pM for all $p \in M$. Now, let ϱ be an eigenvalue of T^2 at $p \in M$, then using (13) and (14), we have

$$\varrho = -\frac{||TV||^2}{||V||^2}$$

for any tangent vector V at $p \in M$. Again, using (7) and (9), we obtain

$$-1 \leq -\frac{||TV||^2}{||TV||^2 + ||FV||^2} = -\frac{||TV||^2}{||JV||^2} = -\frac{||TV||^2}{||V||^2} \leq 0 \ .$$

Namely, the eigenvalues of T^2 are in the closed interval $[-1, 0]$. For arbitrary $p \in M$, we define

$$\mathcal{D}_p^{\tau} = Ker\{T_p^2 + \tau^2(p)I_p\} \ ,$$

where I is the identity operator on T_pM and τ is function defined on M with values in $[0, 1]$ such that $-\tau^2(p)$ is an eigenvalue of T_p^2. Specially, for boundary values of the function τ, we have

$$\mathcal{D}_p^1 = Ker\{F_p\} \ and \ \mathcal{D}_p^0 = Ker\{T_p\} \ .$$

Here, $\mathcal{D}_p^1$ is maximal J-invariant where as $\mathcal{D}_p^0$ is the maximal anti-J-invariant subspace of T_pM. Since, T_p^2 is symmetric and diagonalizable, there is some integer k such that $-\tau_1^2(p), ..., -\tau_k^2(p)$ are distinct eigenvalues of T_p^2. In this case, the tangent space T_pM can be decomposed as a direct sum of the mutually orthogonal T-invariant eigenspaces, i.e,

$$T_pM = \mathcal{D}_p^{\tau_1} \oplus ... \oplus \mathcal{D}_p^{\tau_k} \ .$$

Moreover, if $\tau_i \neq 0$ for $i \in \{1, ..., k\}$, then each $\mathcal{D}_p^{\tau_i}$ is even dimensional.

We now recall the definitions of generic and skew CR-submanifolds defined first by Ronsse [25].

Definition 3.1. *([25]) Let M be a submanifold of a Kähler manifold $(\bar{M}, J, g)$. Then M is called a* generic submanifold *if there exists an integer k functions τ_i, $i \in \{1, ..., k\}$ defined on M with values in $(0, 1)$ such that*

i) *Each $-\tau_i^2$, $i \in \{1, ..., k\}$ is a distinct eigenvalue of T_p^2 with*

$$T_pM = \mathcal{D}_p^1 \oplus \mathcal{D}_p^0 \oplus \mathcal{D}_p^{\tau_1} \oplus ... \oplus \mathcal{D}_p^{\tau_k}$$

for $p \in M$.

ii) *The dimensions of $\mathcal{D}_p^1, \mathcal{D}_p^0$ and $\mathcal{D}_p^{\tau_i}$, $i \in \{1, ..., k\}$ are independent of $p \in M$. In addition, if each τ_i is constant on M, then M is called a* skew CR-submanifold.

As mentioned before, we call the generic submanifolds (in the sense of Ronsse [25]) as *generalized structure induced* or *generalized J-induced submanifolds*.

Based on the basic background in subsection 2.3, we see that the notion of $\mathcal{D}_p^1$ (respectively, $\mathcal{D}_p^0$) coincides with the notion of *holomorphic distribution* (respectively, *totally real distribution*). Moreover, the notion of $\mathcal{D}_p^\tau$ coincides with the notion of *pointwise slant distribution*. In that case, with the help of (10), we have $\tau^2 = \cos^2\theta$, where $-\tau^2$ is an eigenvalue of T_p^2 while θ is the slant function of the pointwise slant distribution $\mathcal{D}_p^\theta$. From now on, we denote the distributions $\mathcal{D}^1$, $\mathcal{D}^0$ and $\mathcal{D}_p^{\tau_i}$ by $\mathcal{D}^T$, $\mathcal{D}^\perp$ and $\mathcal{D}_p^{\theta_i}$, respectively. Thus, we rearrange the definition 3.1 as follows.

Definition 3.2. *Let M be a submanifold of a Kähler manifold* $(\bar{M}, J, g)$. *Then M is said to be a* generalized structure induced *or* generalized J-induced submanifold *if the tangent bundle TM of M has the form*

$$TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^{\theta_1} \oplus ... \oplus \mathcal{D}^{\theta_k} \ ,$$

where $\mathcal{D}^T$ *is a holomorphic,* $\mathcal{D}^\perp$ *is a totally real and each of* $\mathcal{D}^{\theta_i}$ *is a pointwise slant distribution on M and* θ_i*'s are distinct for* $i = 1, ..., k$. *In addition, if each of* $\mathcal{D}^{\theta_i}$ *is a slant distribution, then we say that M is a* skew CR-submanifold.

In a special case, we have the following definition.

Definition 3.3. *A submanifold M of a Kähler manifold* $(\bar{M}, J, g)$ *is called a* generalized structure induced submanifold of order 1 *or* generalized J-induced submanifold of order 1 *if it is a generalized J-induced submanifold with* $k = 1$, *i.e. the tangent bundle TM of M has the form*

$$TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \ , \tag{15}$$

where $\mathcal{D}^T$ *is a holomorphic,* $\mathcal{D}^\perp$ *is a totally real and each of* $\mathcal{D}^\theta$ *is a pointwise slant distribution on M. Additionally, if the slant function* θ *is constant, i.e.* $\mathcal{D}^\theta$ *is a slant distribution, then M is said to be a* skew CR-submanifold of order 1 *[30].*

In this case, the normal bundle $T^\perp M$ of M is decomposed as

$$T^\perp M = J(\mathcal{D}^\perp) \oplus F(\mathcal{D}^\theta) \oplus \overline{\mathcal{D}}^T \ , \tag{16}$$

where $\overline{\mathcal{D}}^T$ is the orthogonal complementary distribution of $J(\mathcal{D}^\perp) \oplus F(\mathcal{D}^\theta)$ in $T^\perp M$ and it is an invariant subbundle of $T^\perp M$ with respect to J.

We say that a generalized J-induced submanifold of order 1 is *proper*, if $\mathcal{D}^T \neq \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and the slant function θ belongs to open interval $(0, \frac{\pi}{2})$.

For the further study of generalized J-induced submanifolds of order 1 of a Kähler manifold, we need the following lemma.

Lemma 3.4. *Let M be a generalized J-induced submanifold of order 1 of a Kähler manifold* $(\bar{M}, J, g)$. *Then, we have*

$$g(\nabla_U V, Z) = \csc^2\theta g(A_{FZ}JV - A_{FTZ}V, U) \ , \tag{17}$$

$$g(\nabla_Z W, V) = \csc^2\theta g(A_{FTW}V - A_{FW}JV, Z) \ , \tag{18}$$

where $U, V \in \mathcal{D}^T$ *and* $Z, W \in \mathcal{D}^\theta$.

Proof. Using (4), (7) and (8), we have

$$\begin{aligned} g(\nabla_U V, Z) &= g(\bar{\nabla}_U JV, JZ) = g(\bar{\nabla}_U JV, TZ) + g(\bar{\nabla}_U JV, FZ) \\ &= -g(\bar{\nabla}_U V, T^2 Z) - g(\bar{\nabla}_U V, FTZ) + g(\bar{\nabla}_U JV, FZ). \end{aligned}$$

for $U, V \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. Now, using (10) and (5), we obtain

$$\sin^2\theta g(\nabla_U V, Z) = g(A_{FZ}JV - A_{FTZ}V, U) \ .$$

This gives (17). The other assertion can be obtained in a similar way. □

Lemma 3.5. *Let M be a generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Then, we have*

$$g(\nabla_U V, X) = g(A_{JX}U, JV), \tag{19}$$
$$g(\nabla_U X, Z) = -\sec^2\theta g(A_{JX}TZ - A_{FTZ}X, U), \tag{20}$$
$$g(\nabla_X Y, U) = -g(A_{JY}X, JU), \tag{21}$$
$$g(\nabla_Z W, Y) = \sec^2\theta g(A_{JY}TW - A_{FTW}Y, Z), \tag{22}$$
$$g(\nabla_Z Y, U) = -g(A_{JY}JU, Z), \tag{23}$$
$$g(\nabla_X Y, Z) = -\sec^2\theta\{g(A_{JY}X, TZ) - g(A_{FTZ}X, Y)\}, \tag{24}$$
$$g(\nabla_X U, Z) = \csc^2\theta\{g(A_{FZ}JU - A_{FTZ}U, X)\}, \tag{25}$$

where $U, V \in \mathcal{D}^T$, $X, Y \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$.

Proof. The proofs of all equations are same as the proofs of equations of Lemma 1, Lemma 2 and Lemma 3 of [30]. So, we omit them. □

As applications of Lemma (3.4) and Lemma (3.5), we have the following results.

Theorem 3.6. *Let M be a generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Then the holomorphic distribution $\mathcal{D}^T$ is totally geodesic if and only if the following equations hold*

$$g(A_{JX}U, JV) = 0 \ , \tag{26}$$

$$g(A_{FZ}U, JV) = g(A_{FTZ}U, V) \tag{27}$$

for $U, V \in \mathcal{D}^T$, $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

Proof. Let M be a generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Then, the holomorphic distribution $\mathcal{D}^T$ is totally geodesic if and only if $g(\nabla_U V, X) = 0$ and $g(\nabla_U V, Z) = 0$ for all $U, V \in \mathcal{D}^T$, $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$. Thus, both assertions follow from (19) and (17), respectively. □

Theorem 3.7. *Let M be a generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Then the pointwise slant distribution $\mathcal{D}^\theta$ is integrable if and only if the following equations hold*

$$g(A_{FTZ}V - A_{FZ}JV, W) = g(A_{FTW}V - A_{FW}JV, Z) \ , \tag{28}$$

$$g(A_{JX}TW - A_{FTW}X, Z) = g(A_{JX}TZ - A_{FTZ}X, W) \tag{29}$$

for $V \in \mathcal{D}^T$, $X \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$.

Proof. Let M be a generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Then, the pointwise slant distribution $\mathcal{D}^\theta$ is integrable if and only if $g([Z, W], V) = 0$ and $g([Z, W], X) = 0$ for all $U \in \mathcal{D}^T$, $X \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$. Thus, both assertions follow from (18) and (22), respectively. □

Remark 3.8. *From Lemma 1 of [25] or Theorem 4.2 of [35], we know that the totally real distribution $\mathcal{D}^\perp$ is always integrable.*

4. Biwarped product submanifolds of Kähler manifolds

In this section, we check that the existence of biwarped product submanifolds in the form $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ of a Kähler manifold $(\bar{M}, J, g)$, where M_0, M_1 and M_2 are one of the submanifolds given in subsection 2.3.

4.1. Existence problems

Chen proved that there do not exist (non-trivial) warped product submanifolds in the form $M_\perp \times_f M_T$ in a Kähler manifold $\bar{M}$ such that $M_\perp$ is a totally real and M_T is a holomorphic submanifold of $\bar{M}$ in Theorem 3.1 of [8], Thus, we obtain the following result.

Corollary 4.1. *There do not exist (non-trivial) biwarped product submanifolds in the form $M_\perp \times_f M_T \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$ such that $M_\perp$ is a totally real, M_T is a holomorphic and M_θ is a pointwise slant or slant submanifold of $\bar{M}$.*

In Theorem 3.1 of [28], Şahin showed that there is no (non-trivial) warped product submanifolds of the form $M_\phi \times_f M_T$ in a Kähler manifold $\bar{M}$ such that M_ϕ is a proper slant and M_T is a holomorphic submanifold of $\bar{M}$. In Theorem 4.1 of [31], he also proved that there exist no (non-trivial) warped product submanifolds of the form $M_\theta \times_f M_T$ in a Kähler manifold $\bar{M}$ such that M_θ is a proper pointwise slant and M_T is a holomorphic submanifold of $\bar{M}$. Hence, we conclude that:

Corollary 4.2. *Let $\bar{M}$ be a Kähler manifold. Then there exist no (non-trivial) biwarped product submanifolds of type $M_\theta \times_f M_T \times_\sigma M_\perp$ of $\bar{M}$ such that M_θ is a proper pointwise slant or proper slant, M_T is a holomorphic and $M_\perp$ is a totally real submanifold of $\bar{M}$.*

On the other hand, in Theorem 3.2 of [28], Şahin proved that there exists no (non-trivial) warped product submanifolds of the form $M_T \times_f M_\phi$ in a Kähler manifold $\bar{M}$ such that M_T is a holomorphic and M_ϕ is a proper slant submanifold of $\bar{M}$. Thus, we get the following result.

Corollary 4.3. *There exist no (non-trivial) biwarped product submanifolds of type $M_T \times_f M_\phi \times_\sigma M_\perp$ of a Kähler manifold $(\bar{M}, J, g)$ such that M_T is a holomorphic, M_ϕ is a proper slant and $M_\perp$ is a totally real submanifold of $\bar{M}$.*

Now, we consider (non-trivial) biwarped product submanifolds of the form $M_T \times_f M_\perp \times_\sigma M_\theta$ in a Kähler manifold $(\bar{M}, J, g)$ such that M_T is a holomorphic, $M_\perp$ is a totally real and M_ϕ is a pointwise slant submanifold of $\bar{M}$. We first present an example of such a submanifold.

Example 4.4. *Consider the Kähler manifold $\mathbb{R}^{14}$ with usual Kähler structure. For $u, v \neq 0, 1$ and $x, z, w \in (0, \frac{\pi}{2})$, we consider a submanifold M in $\mathbb{R}^{14}$ given by*

$$y_1 = u\cos z,\ y_2 = v\cos z,\ y_3 = u\cos w,\ y_4 = v\cos w,$$
$$y_5 = u\sin z,\ y_6 = v\sin z,\ y_7 = u\sin w,\ y_8 = v\sin w,$$
$$y_9 = z,\ y_{10} = w,\ y_{11} = u\cos x,\ y_{12} = v\cos x,\ y_{13} = u\sin x,\ y_{14} = v\sin x.$$

Then, we see that the local frame of the tangent bundle TM of M is given by

$$\begin{aligned}
U &= \cos z\frac{\partial}{\partial y_1} + \cos w\frac{\partial}{\partial y_3} + \sin z\frac{\partial}{\partial y_5} + \sin w\frac{\partial}{\partial y_7} + \cos x\frac{\partial}{\partial y_{11}} + \sin x\frac{\partial}{\partial y_{13}},\\
V &= \cos z\frac{\partial}{\partial y_2} + \cos w\frac{\partial}{\partial y_4} + \sin z\frac{\partial}{\partial y_6} + \sin w\frac{\partial}{\partial y_8} + \cos x\frac{\partial}{\partial y_{12}} + \sin x\frac{\partial}{\partial y_{14}},\\
X &= -u\sin x\frac{\partial}{\partial y_{11}} - v\sin x\frac{\partial}{\partial y_{12}} + u\cos x\frac{\partial}{\partial y_{13}} + v\cos x\frac{\partial}{\partial y_{14}},\\
Z &= -u\sin z\frac{\partial}{\partial y_1} - v\sin z\frac{\partial}{\partial y_2} + u\cos z\frac{\partial}{\partial y_5} + v\cos z\frac{\partial}{\partial y_6} + \frac{\partial}{\partial y_9},\\
W &= -u\sin w\frac{\partial}{\partial y_3} - v\sin w\frac{\partial}{\partial y_4} + u\cos w\frac{\partial}{\partial y_7} + v\cos w\frac{\partial}{\partial y_8} + \frac{\partial}{\partial y_{10}},
\end{aligned}$$

where $(y_1, ..., y_{14})$ are natural coordinates of $\mathbb{R}^{14}$. Then $\mathcal{D}^T = span\{U, V\}$ is a holomorphic, $\mathcal{D}^\perp = span\{X\}$ is a totally real and $\mathcal{D}^\theta = span\{Z, W\}$ is a (proper) pointwise slant distribution with the slant function $\theta = \cos^{-1}(\frac{1}{1+u^2+v^2})$. Thus, M is a generalized J-induced submanifold of order 1 of $\mathbb{R}^{14}$. Also, one can easily see the distribution $\mathcal{D}^T$ is totally *geodesic and the distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ are integrable. If we denote the integral submanifolds of $\mathcal{D}^{T}$, $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ by $M_T, M_{\perp}$ and M_{θ}, respectively, then the induced metric tensor of M is*

$$\begin{aligned} ds^2 &= 3(du^2+dv^2)+(u^2+v^2)dx^2+(1+u^2+v^2)(dz^2+dw^2) \\ &= g_{M_T}+(u^2+v^2)g_{M_{\perp}}+(1+u^2+v^2)g_{M_{\theta}}. \end{aligned}$$

Thus, $M = M_T \times_f M_{\perp} \times_{\sigma} M_{\theta}$ is a (non-trivial) biwarped product generalized J-induced submanifold of order 1 of $\mathbb{R}^{14}$ with warping functions $f = \sqrt{u^2+v^2}$ and $\sigma = \sqrt{1+u^2+v^2}$.

5. Biwarped product generalized J-induced submanifolds of order 1 in the form $M_T \times_f M_{\perp} \times_{\sigma} M_{\theta}$

In this section, we give a characterization for biwarped product generalized J-induced submanifolds of order 1 in the form $M_T \times_f M_{\perp} \times_{\sigma} M_{\theta}$, where M_T is a holomorphic, $M_{\perp}$ is a totally real and M_{θ} is a pointwise slant submanifold of a Kähler manifold $(\bar{M}, J, g)$. After that, we investigate the behavior of the second fundamental form of such submanifolds and as a result, we give a necessary and sufficient condition for such manifolds to be locally trivial. Now, we give one of the main theorems of this paper. We first recall the following fact given in [13] to prove our theorem.

Remark 5.1. *(Remark 2.1 [13]) Suppose that the tangent bundle of a Riemannian manifold M splints into an orthogonal sum $TM = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus ... \oplus \mathcal{D}_k$ of non-trivial distributions such that each $\mathcal{D}_j$ is spherical and its complement in TM is autoparallel for $j \in \{1, 2, ..., k\}$. Then the manifold M is locally isometric to a multiply warped product $M_0 \times_{f_1} M_1 \times ... \times_{f_k} M_k$.*

Theorem 5.2. *Let M be a proper generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Then M is a locally biwarped product submanifold of type $M_T \times_f M_{\perp} \times_{\sigma} M_{\theta}$ if and only if, we have*

$$A_{JX}V = -JV(\lambda)X, \tag{30}$$
$$A_{FTZ}V - A_{FZ}JV = -\sin^2\theta V(\mu)Z, \tag{31}$$

for some functions λ and μ satisfying $X(\lambda) = Z(\lambda) = 0$ and $X(\mu) = Z(\mu) = 0$, and

$$g(A_{JY}TZ, X) = g(A_{FTZ}Y, X), \tag{32}$$
$$g(A_{JY}TW, Z) = g(A_{FTZ}Y, W). \tag{33}$$

for $V \in \mathcal{D}^T$, $X, Y \in \mathcal{D}^{\perp}$ and $Z, W \in \mathcal{D}^{\theta}$.

Proof. Let M be a biwarped product proper generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$ in the form $M_T \times_f M_{\perp} \times_{\sigma} M_{\theta}$. Then for any $V \in \mathcal{D}^T$, $X \in \mathcal{D}^{\perp}$ and $Z \in \mathcal{D}^{\theta}$ using (4)~(5) and (7)~(8), we have

$$g(A_{JX}V, U) = -g(\bar{\nabla}_U JX, V) = g(\bar{\nabla}_U X, JV) = g(\nabla_U X, JV).$$

Here, we know $\nabla_U X = U(\ln f)X$ from (2). Thus, we obtain

$$g(A_{JX}V, U) = U(\ln f)g(X, JV) = 0 \ , \tag{34}$$

since $g(X, JV) = 0$. Similarly, we have

$$g(A_{JX}V, Z) = -g(\bar{\nabla}_Z JX, V) = g(\bar{\nabla}_Z X, JV) = g(\nabla_Z X, JV).$$

Thus, we obtain

$$g(A_{JX}V, Z) = 0 \ , \tag{35}$$

since $\nabla_Z X = 0$ from (3). Next, by a similar argument, for $Y \in \mathcal{D}^\perp$, we have

$$g(A_{JX}V, Y) = -g(\bar{\nabla}_Y JX, V) = g(\bar{\nabla}_Y X, JV) = -g(\nabla_Y JV, X).$$

Using (2), we obtain

$$g(A_{JX}V, Y) = g(-JV(\ln f)X, Y) \ . \tag{36}$$

Moreover, we have $X(\ln f) = Z(\ln f) = 0$, since f depends on only points of M_T. So, we conclude that $\lambda = \ln f$. Thus, from (34)~(36), it follows that (30). Now, we prove (31). Using (4)~(5) and (7)~(10), we have

$$\begin{aligned} g(A_{FTZ}V - A_{FZ}JV, U) &= -g(\bar{\nabla}_V FTZ, U) + g(\bar{\nabla}_{JV} FZ, U) \\ &= -g(\bar{\nabla}_V JTZ, U) + g(\bar{\nabla}_V T^2 Z, U) \\ &\quad + g(\bar{\nabla}_{JV} JZ, U) - g(\bar{\nabla}_{JV} TZ, U) \\ &= +g(\bar{\nabla}_V TZ, JU) - V[\cos^2\theta]g(Z, U) - \cos^2\theta g(\bar{\nabla}_V Z, U) \\ &\quad - g(\bar{\nabla}_{JV} Z, JU) - g(\bar{\nabla}_{JV} TZ, U) \\ &= +g(\nabla_V TZ, JU) - \cos^2\theta g(\nabla_V Z, U) \\ &\quad - g(\nabla_{JV} Z, JU) - g(\nabla_{JV} TZ, U). \end{aligned}$$

Here, if we use (2), we obtain

$$\begin{aligned} g(A_{FTZ}V - A_{FZ}JV, U) =& V(\ln\sigma)g(TZ, JU) - \cos^2\theta V(\ln\sigma)g(Z, U) \\ & - JV(\ln\sigma)g(Z, JU) - JV(\ln\sigma)g(TZ, U). \end{aligned}$$

Hence, we get

$$g(A_{FTZ}V - A_{FZ}JV, U) = 0 \ , \tag{37}$$

since $g(TZ, JU) = g(Z, U) = g(Z, JU) = g(TZ, U) = 0$. Similarly, we have

$$\begin{aligned} g(A_{FTZ}V - A_{FZ}JV, X) &= -g(\bar{\nabla}_X FTZ, V) + g(\bar{\nabla}_X FZ, JV) \\ &= -g(\bar{\nabla}_X JTZ, V) + g(\bar{\nabla}_X T^2 Z, V) \\ &\quad + g(\bar{\nabla}_X JZ, JV) - g(\bar{\nabla}_X TZ, JV) \\ &= +g(\bar{\nabla}_X TZ, JV) - X[\cos^2\theta]g(Z, V) - \cos^2\theta g(\bar{\nabla}_X Z, V) \\ &\quad - g(\bar{\nabla}_X Z, V) - g(\bar{\nabla}_X TZ, JV) \\ &= +g(\nabla_X TZ, JV) - \cos^2\theta g(\nabla_X Z, V) \\ &\quad - g(\nabla_X Z, V) - g(\nabla_X TZ, JV). \end{aligned}$$

Here, we know $\nabla_X TZ = \nabla_X Z = 0$ from (3). So, we get

$$g(A_{FTZ}V - A_{FZ}JV, X) = 0 \ . \tag{38}$$

On the other hand, using (18) , we have

$$g(A_{FTZ}V - A_{FZ}JV, W) = -\sin^2\theta g(\nabla_W V, Z) \ .$$

Using (2), we get

$$g(A_{FTZ}V - A_{FZ}JV, W) = g(-\sin^2\theta V(\ln\sigma)Z, W) \ . \tag{39}$$

Moreover, we have $X(\ln\sigma) = Z(\ln\sigma) = 0$, since σ depends on only points of M_T. So, we conclude that $\mu = \ln\sigma$. Thus, (31) follows form (37)~(39). Next, we prove (32) and (33). Using (24) and (3), we have

$$g(A_{JY}X, TZ) - g(A_{FTZ}X, Y) = -\cos^2\theta g(\nabla_X Z, Y) = 0 \ .$$

Thus, (32) follows. Similarly, using (34) and (3), we have

$$g(A_{JY}TW, Z) = g(A_{FTZ}Y, W) = -\cos^2\theta g(\nabla_Z Y, W) = 0 \ .$$

So, we get (33).

Conversely, assume that M is a proper generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$ such that (30)~(33) hold. Then we satisfy (26) and (27) by using (30) and (31), respectively. Thus, by Theorem 3.6, the holomorphic distribution $\mathcal{D}^T$ is totally geodesic and as a result it is integrable. By (32) and (33), we easily satisfy (28) and (29). Thus, by Theorem 3.7, the pointwise slant distribution $\mathcal{D}^\theta$ is integrable. Also, by Remark 3.8, the totally real distribution $\mathcal{D}^\perp$ is always integrable. Let M_T, $M_\perp$ and M_θ be the integral manifolds of $\mathcal{D}^T$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. If we denote the second fundamental form of $M_\perp$ in M by $h^\perp$, for $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$, using (4), (24) and (32), we have

$$g(h^\perp(X, Y), Z) = g(\nabla_X Y, Z) = 0 \ . \tag{40}$$

For any $X, Y \in \mathcal{D}^\perp$ and $V \in \mathcal{D}^T$, using (4), (20) and (30), we have

$$g(h^\perp(X, Y), V) = g(\nabla_X Y, V) = -g(A_{JY}X, JV) = -V(\lambda)g(X, Y) \ .$$

After some calculation, we obtain

$$g(h^\perp(X, Y), V) = g(-g(X, Y)\nabla\lambda, V) \ , \tag{41}$$

where $\nabla\lambda$ is the gradient of λ. Thus, from (40) and (41), we conclude that

$$h^\perp(X, Y) = -g(X, Y)\nabla\lambda \ .$$

This equation says that $M_\perp$ is totally umbilic in M with the mean curvature vector field $-\nabla\lambda$. Now, we show that $-\nabla\lambda$ is parallel. We have to satisfy $g(\nabla_X\nabla\lambda, E) = 0$ for $X \in \mathcal{D}^\perp$ and $E \in (\mathcal{D}^\perp)^\perp = \mathcal{D}^T \oplus \mathcal{D}^\theta$. Here, we can put $E = V + Z$, where $V \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. By direct computations, we obtain

$$\begin{aligned} g(\nabla_X\nabla\lambda, E) =& \{Xg(\nabla\lambda, E) - g(\nabla\lambda, \nabla_X E)\} \\ =& \{X(E(\lambda)) - [X, E]\lambda - g(\nabla\lambda, \nabla_E X)\} \\ =& \{[X, E]\lambda + E(X(\lambda)) - [X, E]\lambda - g(\nabla\lambda, \nabla_E X)\} \\ =& -g(\nabla\lambda, \nabla_V X) - g(\nabla\lambda, \nabla_Z X), \end{aligned}$$

since $X(\lambda) = 0$. Here, for any $U \in \mathcal{D}^T$, we have $g(\nabla_V X, U) = -g(\nabla_V U, X) = 0$, since M_T is totally geodesic in M. Thus, $\nabla_V X \in \mathcal{D}^\perp$ or $\nabla_V X \in \mathcal{D}^\theta$. In either case, we have

$$g(\nabla\lambda, \nabla_V X) = 0 \ . \tag{42}$$

On the other hand, from (23), we have $g(\nabla_Z X, U) = -g(A_{JX}JU, Z)$. Here, using (32), we obtain $g(\nabla_Z X, U) = 0$. That is; $\nabla_Z X \in \mathcal{D}^\perp$ or $\nabla_Z X \in \mathcal{D}^\theta$. In either case, we get

$$g(\nabla\lambda, \nabla_Z X) = 0 \ . \tag{43}$$

From (42) and (43), we find

$$g(\nabla_X\nabla\lambda, E) = 0 \ .$$

Thus, $M_\perp$ is spherical, since it is also totally umbilic. Consequently, $\mathcal{D}^\perp$ is spherical.
Next, we show that $\mathcal{D}^\theta$ is spherical. Let h^θ denote the second fundamental form of M_θ in M. Then for $Z, W \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^\perp$, using (4), (22) and (32), we have

$$g(h^\theta(Z, W), X) = g(\nabla_Z W, X) = 0 \ . \tag{44}$$

On the other hand, for any $V \in \mathcal{D}^T$, using (4) and (18), we have

$$g(h^\theta(Z,W),V) = g(\nabla_Z W, V) = \csc^2\theta g(A_{FTW}V - A_{FW}JV, Z) \ .$$

Using (31), we obtain

$$g(h^\theta(Z,W),V) = -V(\mu)g(Z,W) \ .$$

After some calculation, we get

$$g(h^\theta(Z,W),V) = -g(g(Z,W)\nabla\mu, V) \ , \tag{45}$$

where $\nabla\mu$ is the gradient of μ. Thus, from (44) and (45), we deduce that

$$h^\theta(Z,W) = -g(Z,W)\nabla\mu \ .$$

It means that M_θ is totally umbilic in M with the mean curvature vector field $-\nabla\mu$. What's left is to show that $-\nabla\mu$ is parallel. We have to satisfy $g(\nabla_Z\nabla\mu, E) = 0$ for $Z \in \mathcal{D}^\theta$ and $E \in (\mathcal{D}^\theta)^\perp = \mathcal{D}^T \oplus \mathcal{D}^\perp$. Here, $E = V + X$, for $V \in \mathcal{D}^T$ and $X \in \mathcal{D}^\perp$. Upon direct calculation, we obtain

$$\begin{aligned} g(\nabla_Z\nabla\mu, E) =& \{Zg(\nabla\mu, E) - g(\nabla\mu, \nabla_Z E)\} \\ =& \{Z(E(\mu)) - [Z,E]\mu - g(\nabla\mu, \nabla_E Z)\} \\ =& \{[Z,E]\mu + E(Z(\mu)) - [Z,E]\mu - g(\nabla\mu, \nabla_E Z)\} \\ =& -g(\nabla\mu, \nabla_V Z) - g(\nabla\mu, \nabla_X Z), \end{aligned}$$

since $Z(\mu) = 0$. Here, for any $U \in \mathcal{D}^T$, using (25) and (30), we have

$$g(\nabla_X Z, U) = -\csc^2\theta\{g(A_{FZ}JU - A_{FTZ}U, X)\} = 0 \ .$$

So, $\nabla_X Z \in \mathcal{D}^\perp$ or $\nabla_X Z \in \mathcal{D}^\theta$. Hence

$$g(\nabla\mu, \nabla_X Z) = 0 \ , \tag{46}$$

since $\nabla\mu \in \mathcal{D}^T$. On the other hand, since M_T is totally geodesic in M, we have $g(\nabla_V Z, U) = -g(\nabla_V U, Z) = 0$. Hence, $\nabla_V Z \in \mathcal{D}^\perp$ or $\nabla_V Z \in \mathcal{D}^\theta$. So, we get

$$g(\nabla\mu, \nabla_V Z) = 0 \ . \tag{47}$$

By (46) and (47), we find

$$g(\nabla_Z\nabla\mu, E) = 0 \ .$$

Lastly, we prove that $(\mathcal{D}^\perp)^\perp = \mathcal{D}^T \oplus \mathcal{D}^\theta$ and $(\mathcal{D}^\theta)^\perp = \mathcal{D}^T \oplus \mathcal{D}^\perp$ are autoparallel. In fact, $\mathcal{D}^T \oplus \mathcal{D}^\theta$ is autoparallel if and only if all four types of covariant derivatives $\nabla_U V, \nabla_U Z, \nabla_Z U, \nabla_Z W$ are again in $\mathcal{D}^T \oplus \mathcal{D}^\theta$ for $U, V \in \mathcal{D}^T$ and $Z, W \in \mathcal{D}^\theta$. This is equivalent to say that all four inner products $g(\nabla_U V, X), g(\nabla_U Z, X), g(\nabla_Z U, X)$, $g(\nabla_Z W, X)$ vanish, where $X \in \mathcal{D}^\perp$. Using (19) and (30), we get

$$g(\nabla_U V, X) = g(\nabla_Z U, X) = 0 \ .$$

By (20), (22) and (32), we get

$$g(\nabla_U Z, X) = g(\nabla_Z W, X) = 0 \ .$$

Thus, $\mathcal{D}^T \oplus \mathcal{D}^\theta$ is autoparallel. On the other hand, $\mathcal{D}^T \oplus \mathcal{D}^\perp$ is autoparallel if and only if all four inner products $g(\nabla_U V, Z), g(\nabla_U X, Z), g(\nabla_X U, Z), g(\nabla_X Y, Z)$ vanish, where $U, V \in \mathcal{D}^T$, $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$. Firstly, we have already $g(\nabla_U X, Z) = 0$ from above. Using (17), (25) and (31), we get

$$g(\nabla_U V, Z) = g(\nabla_X U, Z) = 0 \ .$$

Using (24) and (32), we find

$$g(\nabla_X Y, Z) = 0 \ .$$

So, $\mathcal{D}^T \oplus \mathcal{D}^\perp$ is autoparallel. Thus, by Remark 5.1, M is locally biwarped product submanifold of the form $M_T \times_f M_\perp \times_\sigma M_\theta$. □

Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a non-trivial biwarped product generalized J-induced submanifold of order 1 of a Kähler manifold $(\bar{M}, J, g)$. Next, we investigate the behavior of the second fundamental form h of such submanifolds.

Lemma 5.3. *Let M be a biwarped product generalized J-induced submanifold of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$. Then for the second fundamental form h of M in $\bar{M}$, we have*

$$g(h(U,V), JX) = 0 \,, \tag{48}$$
$$g(h(V,X), JY) = -JV(\ln f)g(X,Y) \,, \tag{49}$$
$$g(h(V,Z), JX) = 0 \,, \tag{50}$$

where $U, V \in \mathcal{D}^T$, $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

Proof. Using (4),(7) and (8), we have

$$g(h(U,V), JX) = g(\bar{\nabla}_U V, JX) = -g(\bar{\nabla}_U JX, V) = g(\bar{\nabla}_U X, JV)$$

for $U, V \in \mathcal{D}^T$ and $X \in \mathcal{D}^\perp$. Again, using (4), we obtain $g(h(U,V), JX) = g(\nabla_U X, JV)$. Here, we know $\nabla_U X = U(\ln f)X$ from (2). Thus, we obtain $g(h(U,V), JX) = U(\ln f)g(X,V) = 0$, since $g(X,V) = 0$. So, (48) follows. Now, let $V \in \mathcal{D}^T$ and $X, Y \in \mathcal{D}^\perp$. Then using (4), (7) and (8), we have

$$g(h(V,X), JY) = g(\bar{\nabla}_X V, JY) = -g(\bar{\nabla}_X JV, Y) = -g(\nabla_X JV, Y) \ .$$

Here, we know $\nabla_X JV = JV(\ln f)X$ from (2). Thus, we get (49). The last assertion can be obtained in a similar method. □

The previous lemma shows partially us the behavior of the second fundamental form h of the biwarped product generalized J-induced submanifolds of order 1 has the form $M_T \times_f M_\perp \times_\sigma M_\theta$ in the normal subbundle $J(\mathcal{D}^T)$.

Lemma 5.4. *Let M be a biwarped product generalized J-induced submanifold of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$. Then for the second fundamental form h of M in $\bar{M}$, we have*

$$g(h(U,V), FZ) = 0 \,, \tag{51}$$
$$g(h(V,X), FZ) = 0 \,, \tag{52}$$
$$g(h(V,Z), FW) = -JV(\ln\sigma)g(Z,W) - V(\ln\sigma)g(Z,TW) \,, \tag{53}$$

where $U, V \in \mathcal{D}^T$, $X \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$.

Proof. Using (4),(7) and (8), we have

$$g(h(U,V), FZ) = g(\bar{\nabla}_U V, FZ) = g(\bar{\nabla}_U V, JZ) - g(\bar{\nabla}_U V, TZ)$$

for $U, V \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. After some calculation, we obtain

$$g(h(U,V), FZ) = g(\nabla_U Z, JV) + g(\nabla_U TZ, V) \ .$$

Here, we know $\nabla_U Z = U(\ln\sigma)Z$ and $\nabla_U Z = U(\ln\sigma)TZ$ from (2). Thus, we get

$$g(h(U,V), FZ) = U(\ln\sigma)g(Z, JV) + U(\ln\sigma)g(TZ, V) = 0 \ ,$$

since $g(Z, JV) = g(TZ, V) = 0$. So, (51) follows. Now, we prove (53). Let $V \in \mathcal{D}^T$ and $Z, W \in \mathcal{D}^\theta$. Then using (4),(7) and (8), we have

$$g(h(V, Z), FW) = -g(\nabla_Z JV, W) - g(\nabla_Z V, TW) \ .$$

Again, by (2), we easily get (53). Similarly, we can obtain (52). □

The last lemma shows partially us the behavior of the second fundamental form h of the biwarped product generalized J-induced submanifolds of order 1 of type $M_T \times_f M_\perp \times_\sigma M_\theta$ in the normal subbundle $F(\mathcal{D}^\theta)$.

Remark 5.5. *Equation (48) of Lemma 5.3 and equation (51) of Lemma 5.4 also hold for skew CR-warped product submanifolds of Kähler manifolds, see Lemma 4 of [30]. Moreover, equations (51) and (53) of Lemma 5.4 are also valid for warped product pointwise semi-slant submanifolds in Kähler manifolds, see Lemma 5.3 of [31].*

By (48) and (51), we immediately get the following result.

Corollary 5.6. *Let M be a biwarped product generalized J-induced submanifold of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\overline{\mathcal{D}}^T = \{0\}$. Then M is $\mathcal{D}^T$-geodesic.*

Lastly, we give a necessary and sufficient condition for such submanifolds to be locally trivial.

Theorem 5.7. *Let M be a biwarped product proper generalized J-induced submanifold in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\overline{\mathcal{D}}^T = \{0\}$. Then M is locally trivial if and only if M is both $(\mathcal{D}^T, \mathcal{D}^\perp)$ and $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic.*

Proof. Let M be a biwarped product proper generalized J-induced submanifold of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\overline{\mathcal{D}}^T = \{0\}$. If M is locally trivial, then the warping functions f and σ are constants. By (49), we have $g(h(V, X), JY) = 0$ for $V \in \mathcal{D}^T$ and $X, Y \in \mathcal{D}^\perp$, since $JV(\ln f) = 0$. Taking into account the equation (16) and (52), we get $h(V, X) = 0$. It means that M is $(\mathcal{D}^T, \mathcal{D}^\perp)$-mixed geodesic.
On the other hand, for any $V \in \mathcal{D}^T$ and $Z, W \in \mathcal{D}^\theta$, we have $g(h(V, Z), FW) = 0$ from (53), since $JV(\ln \sigma) = 0$ and $V(\ln \sigma) = 0$. Taking into account the equation (16) and (50), we obtain $h(V, Z) = 0$. Which says us M is $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic.

Conversely, let M be both $(\mathcal{D}^T, \mathcal{D}^\perp)$ and $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic. Then, for any $V \in \mathcal{D}^T$, from (49) we conclude that $JV(\ln f) = 0$, since M is $(\mathcal{D}^T, \mathcal{D}^\perp)$-mixed geodesic. Hence, it follows that f is a constant. Since M is also $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic, for $V \in \mathcal{D}^T$ and $Z, W \in \mathcal{D}^\theta$, we have

$$JV(\ln \sigma) g(Z, W) + V(\ln \sigma) g(Z, TW) = 0 \tag{54}$$

from (53). If we put $V = JV$ in (54), we obtain

$$-V(\ln \sigma) g(Z, W) + JV(\ln \sigma) g(Z, TW) = 0 \ .$$

If we take $W = TW$ in the last equation and use (10), the last equation becomes

$$-V(\ln \sigma) g(Z, TW) - \cos^2\theta JV(\ln \sigma) g(Z, W) = 0 \ . \tag{55}$$

From (54) and (55), we get

$$\sin^2\theta JV(\ln \sigma) g(Z, W) = 0 \ . \tag{56}$$

Since M is proper, $\sin\theta \neq 0$. So, we deduce that $JV(\ln \sigma) = 0$ from (56). Hence, it follows that σ is a constant. Thus, M must be locally trivial, since we found the warping functions f and σ as constants. □

6. An inequality for non-trivial biwarped product generalized J-induced submanifolds of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$

In this section, by using the results given the preceding section, we shall establish an inequality for the squared norm of the second fundamental form in terms of the warping functions for biwarped product generalized J-induced submanifolds of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$, where M_T is a holomorphic, $M_\perp$ is a totally real and M_θ is a pointwise slant submanifold of a Kähler manifold$(\bar{M}, J, g)$.

Let M be a $(k+n+m)$-dimensional biwarped product generalized J-induced submanifold of order 1 of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $\bar{M}$. We choose a canonical orthonormal basis $\{e_1, ..., e_k, \tilde{e}_1, ..., \tilde{e}_n, \bar{e}_1, ..., \bar{e}_m, e^*_1, ..., e^*_m, J\tilde{e}_1, ..., J\tilde{e}_n, \hat{e}_1, ..., \hat{e}_l\}$ of $\bar{M}$ such that $\{e_1, ..., e_k\}$ is an orthonormal basis of $\mathcal{D}^T$, $\{\tilde{e}_1, ..., \tilde{e}_n\}$ is an orthonormal basis of $\mathcal{D}^\perp$, $\{\bar{e}_1, ..., \bar{e}_m\}$ is an orthonormal basis of $\mathcal{D}^\theta$, $\{J\tilde{e}_1, ..., J\tilde{e}_n\}$ is an orthonormal basis of $J\mathcal{D}^\perp$, $\{e^*_1, ..., e^*_m\}$ is an orthonormal basis of $F\mathcal{D}^\theta$ and $\{\hat{e}_1, ..., \hat{e}_l\}$ is an orthonormal basis of $\overline{\mathcal{D}}^T$. Here, $k = dim(\mathcal{D}^T)$, $n = dim(\mathcal{D}^\perp)$, $m = dim(\mathcal{D}^\theta)$ and $l = dim(\overline{\mathcal{D}}^T)$.

Remark 6.1. *In view of (7), we can observe that* $\{Je_1, ..., Je_k\}$ *is also an orthonormal basis of* $\mathcal{D}^T$*. On the other hand, with the help of (11) and (12), we can see that* $\{\sec\theta T\bar{e}_1, ..., \sec\theta T\bar{e}_m\}$ *is also an orthonormal basis of* $\mathcal{D}^\theta$ *and* $\{\csc\theta F\bar{e}_1, ..., \csc\theta F\bar{e}_m\}$ *is also an orthonormal basis of* $F\mathcal{D}^\theta$*, where* θ *is the slant function of* $\mathcal{D}^\theta$*.*

Theorem 6.2. *Let M be a biwarped product proper generalized J-induced submanifold of order 1 in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of a Kähler manifold $(\bar{M}, J, g)$. Then the squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq 2\{n\|\nabla(\ln f)\|^2 + m(\csc^2\theta + \cot^2\theta)\|\nabla(\ln\sigma)\|^2\} \tag{57}$$

where $n = dim(M_\perp)$ and $m = dim(M_\theta)$. The equality case of (57) holds identically if and only if the following assertions are true.
a) *M_T is a totally geodesic submanifold in $\bar{M}$.*
b) *$M_\perp$ and M_θ are totally umbilic submanifolds in $\bar{M}$ with their mean curvature vector fields $-\nabla(\ln f)$ and $-\nabla(\ln\sigma)$, respectively.*
c) *M is minimal in $\bar{M}$.*
d) *M is $(\mathcal{D}^\perp, \mathcal{D}^\theta)$-mixed geodesic.*

Proof. By the decomposition (15), the squared norm of the second fundamental form h can be written as

$$\begin{aligned}\|h\|^2 &= \|h(\mathcal{D}^T, \mathcal{D}^T)\|^2 + \|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 + \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2 \\ &\quad + 2\Big\{\|h(\mathcal{D}^T, \mathcal{D}^\perp)\|^2 + \|h(\mathcal{D}^T, \mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\perp, \mathcal{D}^\theta)\|^2\Big\}.\end{aligned}$$

In view of decomposition (16) and by (48)~(53), which can be explicitly written as follows:

$$\begin{aligned}\|h\|^2 &= \sum_{a,b,c=1}^{n} g(h(\tilde{e}_a, \tilde{e}_b), J\tilde{e}_c)^2 + \sum_{a,b=1}^{n}\sum_{r=1}^{m} g(h(\tilde{e}_a, \tilde{e}_b), e^*_r)^2 \\ &\quad + \sum_{r,s=1}^{m}\sum_{a=1}^{n} g(h(\bar{e}_r, \bar{e}_s), J\tilde{e}_a)^2 + \sum_{r,s,q=1}^{m} g(h(\bar{e}_r, \bar{e}_s), e^*_q)^2 \\ &\quad + 2\sum_{i=1}^{k}\sum_{a,b=1}^{n} g(h(e_i, \tilde{e}_a), J\tilde{e}_b)^2 + 2\sum_{i=1}^{k}\sum_{r,s=1}^{m} g(h(e_i, \bar{e}_r), e^*_s)^2 \\ &\quad + 2\sum_{a,b=1}^{n}\sum_{r=1}^{m} g(h(\tilde{e}_a, \bar{e}_r), J\tilde{e}_b)^2 + 2\sum_{a=1}^{n}\sum_{r,s=1}^{m} g(h(\tilde{e}_a, \bar{e}_r), e^*_s)^2 \\ &\quad + \sum_{A,B=1}^{k+n+m}\sum_{t=1}^{l} g(h(e_A, e_B), \hat{e}_t)^2.\end{aligned} \tag{58}$$

Where the set $\{e_A\}_{1\leq A\leq(k+n+m)}$ is an orthonormal basis of M. Hence, we get

$$||h||^2 \geq 2\Big\{\sum_{i=1}^{k}\sum_{a,b=1}^{n} g(h(e_i,\tilde{e}_a),J\tilde{e}_b)^2 + \sum_{i=1}^{k}\sum_{r,s=1}^{m} g(h(e_i,\bar{e}_r),e_s^*)^2\Big\}\ . \tag{59}$$

Using (49) and Remark 6.1, we arrive

$$||h||^2 \geq 2\Big\{\sum_{i=1}^{k}\sum_{a,b=1}^{n} (Je_i(\ln f)g(\tilde{e}_a,\tilde{e}_b))^2 + \sum_{i=1}^{k}\sum_{r,s=1}^{m} g(h(e_i,\bar{e}_r),\csc\theta F\bar{e}_s))^2\Big\}\ .$$

from the inequality (59). Using (53) and after some calculation we find

$$\begin{aligned} ||h||^2 \geq 2n||\nabla(\ln f)||^2 + 2\sum_{i=1}^{k}\sum_{r,s=1}^{m}(\csc^2\theta)\Big\{&(Je_i(\ln\sigma)g(\bar{e}_r,\bar{e}_s))^2 \\ &+2Je_i(\ln\sigma)g(\bar{e}_r,\bar{e}_s)e_i(\ln\sigma)g(\bar{e}_r,T\bar{e}_s) + (e_i(\ln\sigma)g(\bar{e}_r,T\bar{e}_s))^2\Big\}. \end{aligned} \tag{60}$$

from the last inequality. Here,

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{r,s=1}^{m}\Big\{Je_i(\ln\sigma)g(\bar{e}_r,\bar{e}_s)e_i(\ln\sigma)g(\bar{e}_r,T\bar{e}_s)\Big\} \\ &= \sum_{i=1}^{k}\sum_{r,s=1}^{m}\Big\{g(\nabla\ln\sigma,Je_i)g(\nabla\ln\sigma,e_i)g(\bar{e}_r,\bar{e}_s)g(\bar{e}_r,T\bar{e}_s)\Big\} \\ &= -\sum_{r,s=1}^{m}\Big\{\sum_{i=1}^{k} g(g(\nabla(\ln\sigma),e_i)e_i,J\nabla(\ln\sigma))\Big\}g(\bar{e}_r,\bar{e}_s)g(\bar{e}_r,T\bar{e}_s) \\ &= -g(\nabla(\ln\sigma),J\nabla(\ln\sigma))\sum_{r,s=1}^{m} g(\bar{e}_r,\bar{e}_s)g(\bar{e}_r,T\bar{e}_s) = 0. \end{aligned}$$

Thus, by Remark 6.1, the equation (11) and the last yield, we deduce the inequality (57) from the inequality (60).

Next, from (58) we see that the equality case of (57) holds identically if and only if the following conditions hold.

$$h(\mathcal{D}^T,\mathcal{D}^T) = \{0\},\qquad h(\mathcal{D}^\perp,\mathcal{D}^\perp) = \{0\},\qquad h(\mathcal{D}^\theta,\mathcal{D}^\theta) = \{0\} \tag{61}$$

and

$$h(\mathcal{D}^\perp,\mathcal{D}^\theta) = \{0\}. \tag{62}$$

Since M_T is totally geodesic in M, from the first condition in (61) it follows that M_T is also totally geodesic in $\bar{M}$. So, assertion **a)** follows. Now, let $h^\perp$ denote the second fundamental of $M_\perp$ in M. We know that $h^\perp(\mathcal{D}^\perp,\mathcal{D}^\perp)\subseteq\mathcal{D}^T$ from [21]. Then for $V\in\mathcal{D}^T$ and $X,Y\in\mathcal{D}^\perp$, we have $g(h^\perp(X,Y),V) = g(\nabla_X Y,V)$. Here, we know $\nabla_X Y = {}^\perp\nabla_X Y - g(X,Y)\nabla(\ln f)$ from (3), where ${}^\perp\nabla$ is an induced connection on $M_\perp$. Hence, we obtain

$$g(h^\perp(X,Y),V) = -V(\ln f)g(X,Y) = -g(g(X,Y)\nabla(\ln f),V)\ .$$

It follows that

$$h^\perp(X,Y) = -g(X,Y)\nabla(\ln f) \tag{63}$$

from the last equation. Thus, combining the second condition in (61) and (63), we can deduce that $M_\perp$ is a totally umbilic submanifold in $\bar{M}$ with its mean curvature vector field $-\nabla(\ln f)$. By a similar argument, we can find M_θ as a totally umbilic submanifold in $\bar{M}$ with its mean curvature vector field $-\nabla(\ln\sigma)$. So, assertion **b)** is obtained. Assertions **c)** and **d)** immediately follow from (61) and (62), respectively. □

Remark 6.3. *In case $\mathcal{D}^\theta = \{0\}$, Theorem 6.2 coincides with Theorem 5.1 of [8]. In other words, Theorem 6.2 is a generalization of Theorem 5.1 of [8]. Moreover, Theorem 6.2 coincides with Theorem 5.2 of [31] if $\mathcal{D}^\perp = \{0\}$. Thus, Theorem 6.2 is also a generalization of Theorem 5.2 of [31].*

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