Filomat 32:6 (2018), 2207–2217
https://doi.org/10.2298/FIL1806207L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Preconditioned Normal and Skew-Hermitian Splitting Iteration Method for Continuous Sylvester Equations $AX + XB = C$[☆]

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Abstract. In this paper, we present a preconditioned normal and skew-Hermitian splitting (PNSS) iteration method for continuous Sylvester equations $AX + XB = C$ with positive definite/semi-definite matrices. Theoretical analysis shows that the PNSS methods will converge unconditionally to the exact solution of the continuous Sylvester equations. An inexact variant of the PNSS iteration method(IPNSS) and the analysis of its convergence property in detail have been established. Numerical experiments further show that this new method is more efficient and robust than the existing ones.

1. Introduction

In this paper, we consider the iteration solution of the following continuous Sylvester equations:

$$AX + XB = C, \tag{1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$. Assume that
(A_1) A, B, C are large and sparse matrices;
(A_2) at least one of A and B is non-Hermitian;
(A_3) both A and B are positive semi-definite, and at least one of them is positive definite.

The continuous Sylvester equations (1) has a unique solution, under the assumption (A_1-A_3) that there is no common eigenvalue between A and -B [22, 29]. Here and in the sequel, A^* represents the conjugate transpose of the matrix A. In [1], many important theoretical results about this kind of equations can be found. The continuous Sylvester equations (1) plays an important role in many fields, such as system theory [30, 33, 34], model reduction [2, 35, 38], power systems [26], matrix nearness problem [31], numerical solution of differential equations [3–5, 28, 48–51], finite element model updating [21], noisy image restoration [17], stability of linear systems [24] and so on.

2010 *Mathematics Subject Classification*. 65F10, 65F50
Keywords. Continuous Sylvester equations; PNSS iteration method; IPNSS iteration method; convergence.
Received: 03 January 2017; Accepted: 09 October 2017
Communicated by Dragana Cvetković Ilić
This work is supported by NNSF of China No. 11461046, NSF of Jiangxi Province with Nos.20181ACB20001,20161ACB21005, and 20171BAB211006, and the Fundamental Research Funds for the Central Universities in UIBE (CXTD9-07).
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The continuous Sylvester equation (1) is mathematically equivalent to the system of linear equations

$$\mathcal{A}x = c, \tag{2}$$

where $\mathcal{A} = I \otimes A + B^T \otimes I$, and the vectors x and c contain the concatenated columns of the matrices X and C respectively, with $\otimes$ being the Kronecker product symbol, and B^{T} representing the transpose of the matrix B. However, it is expensive and ill-conditioned to use the iteration method for solving the system of linear equations (2).

There are a large number of numerical methods for solving the continuous Sylvester equations (1). The Bartels-Stewart and the Hessenberg-Schur methods [23] are direct methods, which consist in transforming A and B into triangular or Hessenberg-Schur form by an orthogonal similarity transformation and then solving the resulting system of linear equations directly by a back-substitution process. However, they are suitable for small-scale settings, and not applicable in large-scale settings. When the matrices A and B are large and sparse, iterative methods can solve the continuous Sylvester equation (1) efficiently and accurately with its sparsity and low rank structure. The two most common iterative methods are Alternating Direction Implicit (ADI) method [6, 7, 20], gradient based algorithms[32, 37] and the Krylov subspace based algorithms [8, 18, 25, 27, 36].

The HSS iteration method was firstly proposed by Bai, Golub and Ng in [9] for non-Hermitian positive definite linear systems. Then the method was extended to other equations and conditions in [10–15, 39, 40, 42–45]. In [16], Bai presented the Hermitian and skew-Hermitian splitting (HSS) iteration method for solving large sparse continuous Sylvester equations. Wang et al. applied the idea of PSS iteration method to solve the continuous Sylvester equations in [41], Zheng and Ma applied the NSS iteration method to solve the continuous Sylvester equations in [46]. Recently, Dong and Gu presented a PMHSS iteration method [19] for Sylvester equations. Zhou and Wang presented a PPSS iteration method [47] for Sylvester equations. Thus, motivated by this, we further present and analyze a preconditioned normal and skew-Hermitian iteration method (PNSS) for solving the continuous Sylvester equations.

The rest of the paper is organized as follows. In Section 2, after a brief introduction of the NSS iteration method, we present the PNSS iteration method for solving the continuous Sylvester equation (1) and analyze the convergence property of the PNSS iteration method. In Section 3, we establish an inexact preconditioned normal and skew-Hermitian iteration method (IPNSS) iteration for solving the continuous Sylvester equation (1). In Section 4, some numerical examples are presented to illustrate the efficiency of the PNSS method.

In the remainder of this paper, a matrix sequence $\{Y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ is said to be convergent to a matrix $Y \in \mathbb{C}^{m\times n}$ if the corresponding vector sequence $\{y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{mn}$ is convergent to the corresponding vector $y \in \mathbb{C}^{mn}$, where the vectors $y^{(k)}$ and y contain the concatenated columns of the matrices $Y^{(k)}$ and Y, respectively. If $\{Y^{(k)}\}_{k=0}^{\infty}$ is convergent, then its convergence factor and convergence rate are defined as those of $\{y^{(k)}\}_{k=0}^{\infty}$, correspondingly. In addition, we use $sp(\cdot)$, $\|\cdot\|_2$, and $\|\cdot\|_F$ to denote the spectrum, the spectral norm, and the Frobenius norm, respectively. Note $\|\cdot\|_2$ is also used to represent the 2-norm of a vector.

2. The PNSS iteration method

In this section, we consider the scheme of PNSS iteration method and its convergence property. This iteration method is with inner and outer iterations while each step of the inner iteration is exactly computed by direct methods.

Firstly, we split A and B into normal and skew-Hermitian parts

$$A = \mathcal{N}(A) + \mathcal{S}(A), \qquad B = \mathcal{N}(B) + \mathcal{S}(B).$$

Then A and B can be rewritten as

$$A = (\alpha I + \mathcal{N}(A)) + (\mathcal{S}(A) - \alpha I) = (\alpha I + \mathcal{S}(A)) + (\mathcal{N}(A) - \alpha I),$$

$$B = (\beta I + \mathcal{N}(B)) + (\mathcal{S}(B) - \beta I) = (\beta I + \mathcal{S}(B)) + (\mathcal{N}(B) - \beta I),$$

where α , β be positive real numbers and I is the identity matrix of suitable dimension. It follows that the continuous Sylvester equations (1) can be equivalently written as follows:

$$\begin{cases} (\alpha I + \mathcal{N}(A))X + X(\beta I + \mathcal{N}(B)) = (\alpha I - \mathcal{S}(A))X + X(\beta I - \mathcal{S}(B)) + C, \\ (\alpha I + \mathcal{S}(A))X + X(\beta I + \mathcal{S}(B)) = (\alpha I - \mathcal{N}(A))X + X(\beta I - \mathcal{N}(B)) + C. \end{cases}$$

Then we can easily establish the following normal and skew-Hermitian splitting iteration method.

The NSS iteration method

Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, compute $X^{(k+1)} \in \mathbb{C}^{m\times n}$ for $k = 0, 1, 2, \cdots$, using the following iteration procedure until $\{X^{(k)}\}_{k=0}^{\infty}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I + \mathcal{N}(B)) \\ = (\alpha I - \mathcal{S}(A))X^{(k)} + X^{(k)}(\beta I - \mathcal{S}(B)) + C, \\ (\alpha I + \mathcal{S}(A))X^{(k+1)} + X^{(k+1)}(\beta I + \mathcal{S}(B)) \\ = (\alpha I - \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I - \mathcal{N}(B)) + C, \end{cases}$$

where α , β be positive real numbers and I is the identity matrix of suitable dimension.

Now, based on the above observations, we can establish the following preconditioned normal and skew-Hermitian splitting iteration for solving the continuous Sylvester equations (1).

The PNSS iteration method

Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, compute $X^{(k+1)} \in \mathbb{C}^{m\times n}$ for $k = 0, 1, 2, \cdots$, using the following iteration procedure until $\{X^{(k)}\}_{k=0}^{\infty}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha V_1 + \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta V_2 + \mathcal{N}(B)) \\ = (\alpha V_1 - \mathcal{S}(A))X^{(k)} + X^{(k)}(\beta V_2 - \mathcal{S}(B)) + C, \\ (\alpha V_1 + \mathcal{S}(A))X^{(k+1)} + X^{(k+1)}(\beta V_2 + \mathcal{S}(B)) \\ = (\alpha V_1 - \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta V_2 - \mathcal{N}(B)) + C, \end{cases}$$

where α , β be two positive real numbers and V_1, V_2 be two prescribed symmetric positive definite matrices.

Under the assumptions (A_1-A_3), we can easily know that there is no common eigenvalue between the matrices $\alpha V_1 + \mathcal{N}(A)$ and $-(\beta V_2 + \mathcal{N}(B))$, as well as between the matrices $\alpha V_1 + \mathcal{S}(A)$ and $-(\beta V_2 + \mathcal{S}(B))$, so that the two fixed-point matrix equations have unique solutions for all given right-hand side matrices. Naturally, the two half-steps involved in each step of the PNSS iteration method can be solved effectively using mostly real arithmetic. It is clear that the PNSS iteration method reduces to NSS iteration method with $V_1 = I_m, V_2 = I_n$, where I_m, I_n are the identity matrices of order m and n respectively. In particular, when $\alpha = \beta$, we have

$$\begin{cases} (\alpha V_1 + \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha V_2 + \mathcal{N}(B)) \\ = (\alpha V_1 - \mathcal{S}(A))X^{(k)} + X^{(k)}(\alpha V_2 - \mathcal{S}(B)) + C, \\ (\alpha V_1 + \mathcal{S}(A))X^{(k+1)} + X^{(k+1)}(\alpha V_2 + \mathcal{S}(B)) \\ = (\alpha V_1 - \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha V_2 - \mathcal{N}(B)) + C. \end{cases}$$

By making use of the Kronecker product, it can be rewritten the above-described PNSS iteration method in the following matrix-vector form:

$$\begin{cases} (I \otimes (\alpha V_1 + \mathcal{N}(A)) + (\alpha V_2 + \mathcal{N}(B))^T \otimes I)vec(X^{(k+\frac{1}{2})}) \\ = (I \otimes (\alpha V_1 - \mathcal{S}(A)) + (\alpha V_2 - \mathcal{S}(B))^T \otimes I)vec(X^{(k)}) + vec(C), \\ (I \otimes (\alpha V_1 + \mathcal{S}(A)) + (\alpha V_2 + \mathcal{S}(B))^T \otimes I)vec(X^{(k+1)}) \\ = (I \otimes (\alpha V_1 - \mathcal{N}(A)) + (\alpha V_2 - \mathcal{N}(B))^T \otimes I)vec(X^{(k+\frac{1}{2})}) + vec(C). \end{cases}$$

Denote by $\mathcal{A} = \mathcal{N} + \mathcal{S}$, with

$$\mathcal{N} = I \otimes \mathcal{N}(A) + \mathcal{N}(B)^T \otimes I, \qquad \mathcal{S} = I \otimes \mathcal{S}(A) + \mathcal{S}(B)^T \otimes I$$

and

$$\mathcal{K}(\alpha) = I \otimes (\alpha V_1) + (\alpha V_2)^T \otimes I = \alpha\mathcal{K}.$$

Then we obtain

$$\begin{cases} (\alpha\mathcal{K}+\mathcal{N})vec(X^{(k+\frac{1}{2})}) = (\alpha\mathcal{K}-\mathcal{S})vec(X^{(k)}) + vec(C), \\ (\alpha\mathcal{K}+\mathcal{S})vec(X^{(k+1)}) = (\alpha\mathcal{K}-\mathcal{N})vec(X^{(k+\frac{1}{2})}) + vec(C). \end{cases}$$

Evidently, the iteration scheme is the PNSS iteration method for solving the system of linear equation (2). Then after concrete operations, we can obtain:

$$vec(X^{(k+1)}) = \mathcal{M}(\alpha)vec(X^{(k)}) + \mathcal{G}(\alpha)vec(C),$$

where

$$\mathcal{M}(\alpha) = (\alpha\mathcal{K}+\mathcal{S})^{-1}(\alpha\mathcal{K}-\mathcal{N})(\alpha\mathcal{K}+\mathcal{N})^{-1}(\alpha\mathcal{K}-\mathcal{S}),$$

and

$$\mathcal{G}(\alpha) = (\alpha\mathcal{K}+\mathcal{S})^{-1}(I+(\alpha\mathcal{K}-\mathcal{N})(\alpha\mathcal{K}+\mathcal{N})^{-1}).$$

In addition, if we introduce matrices

$$\mathcal{F}_1(\alpha) = \frac{1}{2\alpha}(\alpha\mathcal{K}+\mathcal{N})\mathcal{K}^{-1}(\alpha\mathcal{K}+\mathcal{S})$$

and

$$\mathcal{G}_1(\alpha) = \frac{1}{2\alpha}(\alpha\mathcal{K}-\mathcal{N})\mathcal{K}^{-1}(\alpha\mathcal{K}-\mathcal{S}),$$

then it holds that

$$A = \mathcal{F}_1(\alpha) - \mathcal{G}_1(\alpha), \quad \mathcal{M}(\alpha) = \mathcal{F}_1(\alpha)^{-1}\mathcal{G}_1(\alpha).$$

In the following, to study the convergence property of the PNSS method. The following lemmas are required.

Lemma 1[15] Let $\tilde{S} \in \mathbb{C}^{mn\times mn}$ be a skew-Hermitian matrix. Then for any $\alpha > 0$, the Cayley transform $Q(\alpha) = (\alpha I - \tilde{S})(\alpha I + \tilde{S})^{-1}$ of $\tilde{S}$ is an unitary matrix.

Lemma 2[46] If $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{m\times n}$, let $A = M_i(A) - N_i(A)(i = 1, 2)$, $B = M_i(B) - N_i(B)(i = 1, 2)$ be two splittings of the matrices A, B, respectively. Let $X^{(0)} \in \mathbb{C}^{m\times n}$ be a given initial matrix, if $\{X^{(k)}\}$ is a two-step iteration sequence defined by

$$\begin{cases} M_1(A)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}M_1(B) = N_1(A)X^{(k)} + X^{(k)}N_1(B) + C, \\ M_2(A)X^{(k+1)} + X^{(k+1)}M_2(B) = N_2(A)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}N_2(B) + C, \end{cases}$$

$k = 0, 1, 2, \ldots$, $x^{(k+1)} = vec(X^{(k+1)})$, $x^{(k+\frac{1}{2})} = vec(X^{(k+\frac{1}{2})})$, $x^{(k)} = vec(X^{(k)})$, then we can obtain

$$x^{(k+1)} = Mx^k + g,$$

where

$$\begin{aligned} M = {} & [I\otimes M_2(A) + M_2(B)^T\otimes I]^{-1}[I\otimes N_2(A) + N_2(B)^T\otimes I] \\ & \times [I\otimes M_1(A) + M_1(B)^T\otimes I]^{-1}[I\otimes N_1(A) + N_1(B)^T\otimes I] \end{aligned}$$

and

$$g = [I\otimes M_2(A) + M_2(B)^T\otimes I]^{-1}[I + (I\otimes N_2(A) + N_2(B)^T\otimes I)(I\otimes M_1(A) + M_1(B)^T\otimes I)^{-1}]vec(F).$$

Moreover, if the spectral radius $\rho(M)$ of the iteration matrix M is less than 1,i.e., $\rho(M) < 1$, then the iterative sequence $\{X^{(k)}\}_{k=0}^{\infty}$ converges to the unique solution of the linear matrix equation (1) for arbitrary initial matrix $X^{(0)} \in \mathbb{C}^{m\times n}$.

Concerning the convergence property of the PNSS method, we apply the above results to obtain the following theorem.

Theorem 1 Assume that $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ are positive semi-definite matrices, and at least one of them is positive definite. Let $\mathcal{N}(A) \in \mathbb{C}^{m\times m}$, $\mathcal{N}(B) \in \mathbb{C}^{n\times n}$ be two normal matrices and $\mathcal{S}(A) \in \mathbb{C}^{m\times m}$,

$\mathcal{S}(B) \in \mathbb{C}^{n\times n}$ be two skew-Hermitian matrices, such that $A = \mathcal{N}(A) + \mathcal{S}(A)$, $B = \mathcal{N}(B) + \mathcal{S}(B)$. Denote by $\mathcal{N} = I \otimes \mathcal{N}(A) + \mathcal{N}(B)^T \otimes I, \mathcal{S} = I \otimes \mathcal{S}(A) + \mathcal{S}(B)^T \otimes I$, $\mathcal{K}(\alpha) = I \otimes (\alpha V_1) + (\alpha V_2)^T \otimes I = \alpha\mathcal{K}$, where α is a positive constant, $V_1 \in \mathbb{C}^{m\times m}$, $V_2 \in \mathbb{C}^{n\times n}$ are prescribed symmetric positive definite matrices. Noting that $\tilde{\mathcal{N}} = \mathcal{K}^{-\frac{1}{2}}\mathcal{N}\mathcal{K}^{-\frac{1}{2}}, \tilde{\mathcal{S}} = \mathcal{K}^{-\frac{1}{2}}\mathcal{S}\mathcal{K}^{-\frac{1}{2}}$, and

$$\mathcal{M}(\alpha) = (\alpha\mathcal{K} + \mathcal{S})^{-1}(\alpha\mathcal{K} - \mathcal{N})(\alpha\mathcal{K} + \mathcal{N})^{-1}(\alpha\mathcal{K} - \mathcal{S}).$$

Then the convergence factor of the PNSS iteration method is given by the spectral radius $\rho(\mathcal{M}(\alpha))$ of the matrix $\mathcal{M}(\alpha)$, which is bounded by

$$\sigma(\alpha) = \max_{\lambda_j \in \lambda(\tilde{\mathcal{N}})} \left|\frac{\alpha - \lambda_j}{\alpha + \lambda_j}\right|.$$

Therefore, it holds that

$$\rho(\mathcal{M}(\alpha)) \le \sigma(\alpha) < 1, \quad \forall \alpha > 0.$$

i.e. the PNSS iteration is convergent to the exact solution $X^* \in \mathbb{C}^{m\times n}$ of the continuous Sylvester equations (1). Morever, we denote $a_j = Re(\lambda_j), b_j = Im(\lambda_j)$.

Proof: By putting

$$M_1(A) = \alpha V_1 + \mathcal{N}(A), M_1(B) = \alpha V_2 + \mathcal{N}(B), N_1(A) = \alpha V_1 - \mathcal{S}(A), N_1(B) = \alpha V_2 - \mathcal{S}(B)$$

and

$$M_2(A) = \alpha V_1 + \mathcal{S}(A), M_2(B) = \alpha V_2 + \mathcal{S}(B), N_2(A) = \alpha V_1 - \mathcal{N}(A), N_2(B) = \alpha V_2 - \mathcal{N}(B).$$

in Lemma 2, we obtain

$$\mathcal{M}(\alpha) = (\alpha\mathcal{K} + \mathcal{S})^{-1}(\alpha\mathcal{K} - \mathcal{N})(\alpha\mathcal{K} + \mathcal{N})^{-1}(\alpha\mathcal{K} - \mathcal{S}).$$

We can easily verify that $\mathcal{K}$ is symmetric positive definite matrix, $\tilde{\mathcal{N}}$ is normal and $\tilde{\mathcal{S}}$ is skew-Hermitian. By the similarity invariance of the matrix spectrum, we have

$$\begin{aligned}
\rho(\mathcal{M}(\alpha)) &= \rho((\alpha\mathcal{K} - \mathcal{N})(\alpha\mathcal{K} + \mathcal{N})^{-1}(\alpha\mathcal{K} - \mathcal{S})(\alpha\mathcal{K} + \mathcal{S})^{-1}) \\
&\le \| (\alpha\mathcal{K} - \mathcal{N})(\alpha\mathcal{K} + \mathcal{N})^{-1}(\alpha\mathcal{K} - \mathcal{S})(\alpha\mathcal{K} + \mathcal{S})^{-1}) \|_2 \\
&= \| \mathcal{K}^{-\frac{1}{2}}(\alpha\mathcal{K} - \mathcal{N})K^{-\frac{1}{2}}\mathcal{K}^{\frac{1}{2}}(\alpha\mathcal{K} + \mathcal{N})^{-1}K^{\frac{1}{2}}\mathcal{K}^{-\frac{1}{2}}(\alpha\mathcal{K} - \mathcal{S})\mathcal{K}^{-\frac{1}{2}}\mathcal{K}^{\frac{1}{2}}(\alpha\mathcal{K} + \mathcal{S})^{-1})\mathcal{K}^{\frac{1}{2}} \|_2 \\
&= \| (\alpha I - \tilde{\mathcal{N}})(\alpha I + \tilde{\mathcal{N}})^{-1}(\alpha I - \tilde{\mathcal{S}})(\alpha I + \tilde{\mathcal{S}})^{-1} \|_2 \\
&\le \| (\alpha I - \tilde{\mathcal{N}})(\alpha I + \tilde{\mathcal{N}})^{-1} \|_2 \| (\alpha I - \tilde{\mathcal{S}})(\alpha I + \tilde{\mathcal{S}})^{-1} \|_2 \\
&\le \| (\alpha I - \tilde{\mathcal{N}})(\alpha I + \tilde{\mathcal{N}})^{-1} \|_2 .
\end{aligned}$$

Here we use the special property of the matrix $Q = (\alpha I - \tilde{\mathcal{S}})(\alpha I + \tilde{\mathcal{S}})^{-1}$ which can be obtained from Lemma 1, i.e., Q is an unitary matrix with its spectral norm equal to 1 (Q is also called the Cayley transform of $\tilde{\mathcal{S}}$). Hence, we can further get

$$\rho(\mathcal{M}(\alpha)) \le \sigma(\alpha) = \max_{\lambda_j \in \lambda(\tilde{\mathcal{N}})} \left|\frac{\alpha - \lambda_j}{\alpha + \lambda_j}\right| \le \max_j \sqrt{\frac{(\alpha - a_j)^2 + b_j^2}{(\alpha + a_j)^2 + b_j^2}}.$$

since the $a_j = Re(\lambda_j) > 0, j = 1, 2, ..., n$, and α is a positive constant, it is easy to see that

$$\rho(\mathcal{M}(\alpha)) \le \sigma(\alpha) < 1, \quad \forall \alpha > 0.$$

Therefore the PNSS iteration method converges unconditionally to the exact solution $X^* \in \mathbb{C}^{m\times n}$ of (1), with the convergence factor being $\rho(\mathcal{M}(\alpha))$.

3. The inexact PNSS iteration method

The two-half steps at each step of the PNSS iteration method for solving the continuous Sylvester equation (1) require finding solutions of two continuous Sylvester equations

$$(\alpha V_1 + N(A))X + X(\alpha V_2 + N(B)) = C_S$$

and

$$(\alpha V_1 + S(A))X + X(\alpha V_2 + S(B)) = C_H$$

where C_S and C_H are prescribed $m - by - n$ complex matrices. However, this may be very costly and impractical in actual implementations, particularly when the sizes of the matrices involved are very large. To further improve the computational efficiency of the PNSS iteration, we can solve the two subproblems inexactly by utilizing certain effective iteration methods such as Gauss-Seidel, SOR, ADI or Krylov subspace based methods, which results in the following inexact preconditioned normal and skew-Hermitian splitting iteration for solving the continuous Sylvester equation(1).

The IPNSS iteration method

Give an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, for $k = 0, 1, 2, \cdots$ until $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ satisfies the stopping criterion, solve $X^{(k+\frac{1}{2})} \in \mathbb{C}^{m\times n}$ approximately from

$$(\alpha V_1 + \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha V_2 + \mathcal{N}(B)) \approx (\alpha V_1 - \mathcal{S}(A))X^{(k)} + X^{(k)}(\alpha V_2 - \mathcal{S}(B)) + C$$

by employing an inner iteration with $X^{(k)}$ as the initial guess, then solve $\{X^{(k+1)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ approximately from

$$(\alpha V_1 + \mathcal{S}(A))X^{(k+1)} + X^{(k+1)}(\alpha V_2 + \mathcal{S}(B)) \approx (\alpha V_1 - \mathcal{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha V_2 - \mathcal{N}(B)) + C$$

by employing an inner iteration with $X^{(k+\frac{1}{2})}$ as the initial guess, where α is given positive constant.

To simplify numerical implementation and convergence analysis, we may rewrite the above iteration method as the following scheme.

The IPNSS iteration method

Give an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, for $k = 0, 1, 2, \cdots$ until $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ converges:

Step 1. Approximate the solution of

$$(\alpha V_1 + \mathcal{N}(A))Z^{(k)} + Z^{(k)}(\alpha V_2 + \mathcal{N}(B)) = R^{(k)},$$

with $R^{(k)} = C - AX^{(k)} - X^{(k)}B$, by iterating until $Z^{(k)}$ is such that the residual

$$P^k = R^{(k)} - ((\alpha V_1 + \mathcal{N}(A))Z^{(k)} + Z^{(k)}(\alpha V_2 + \mathcal{N}(B))),$$

satisfies

$$\|P^{(k)}\|_F \leq \varepsilon_k \|R^{(k)}\|_F.$$

Then compute

$$X^{(k+\frac{1}{2})} = X^{(k)} + Z^{(k)};$$

Step 2. Approximate the solution of

$$(\alpha V_1 + \mathcal{S}(A))Z^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}(\alpha V_2 + \mathcal{S}(B)) = R^{(k+\frac{1}{2})},$$

with $R^{(k+\frac{1}{2})} = C - AX^{(k+\frac{1}{2})} - X^{(k+\frac{1}{2})}B$, by iterating until $Z^{(k+\frac{1}{2})}$ is such that the residual

$$Q^{(k+\frac{1}{2})} = R^{(k+\frac{1}{2})} - ((\alpha V_1 + \mathcal{S}(A))Z^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}(\alpha V_2 + \mathcal{S}(B))),$$

satisfies

$$\|Q^{(k+\frac{1}{2})}\|_F \leq \eta_k \|R^{(k+\frac{1}{2})}\|_F.$$

Then compute

$$X^{(k+1)} = X^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}.$$

Here, ε_k and η_k are prescribed tolerances which are used to control the accuracies of the inner iterations.

Theorem 2 Assume that the assumptions of Theorem 1 hold. If $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ is the iteration sequence generated by the IPNSS iteration method and if $X^* \in \mathbb{C}^{m\times n}$ is the exact solution of the continuous Sylvester equation (1), then we have

$$\| X^{(k+1)} - X^* \|_{\mathcal{S}} \leq (\sigma(\alpha) + \theta\varrho\eta_k)(1 + \theta\varepsilon_k) \| X^{(k)} - X^* \|_{\mathcal{S}},$$

where the norm $\| \cdot \|_{\mathcal{S}}$ is defined as $\| Y \|_{\mathcal{S}} = \| (\alpha V_1 + \mathcal{S}(A))Y + Y(\alpha V_1 + \mathcal{S}(B)) \|_F$, for any matrix $Y \in \mathbb{C}^{m\times n}$, and the constants ϱ and θ are given by

$$\varrho = \| (\alpha\mathcal{K} + \mathcal{S})(\alpha\mathcal{K} + \mathcal{N})^{-1} \|_2,$$

$$\theta = \| A(\alpha\mathcal{K} + \mathcal{S})^{-1} \|_2,$$

$$\sigma(\alpha) = \| (\alpha I - \tilde{\mathcal{N}})(\alpha I + \tilde{\mathcal{N}})^{-1} \|_2 .$$

In particular, if

$$(\sigma(\alpha) + \theta\varrho\eta_{max})(1 + \theta\varepsilon_{max}) < 1,$$

then the iteration sequence $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ converges to $X^* \in \mathbb{C}^{m\times n}$, where $\varepsilon_{max} = \max_k \varepsilon_k$, and $\eta_{max} = \max_k \eta_k$.

Proof: The conclusion is straightforward according to Theorem 3.1 in [16].

Theorem 3 Assume that the conditions of Theorem 1 hold. Suppose that both $r_1(k)$ and $r_2(k)$ are nondecreasing and positive sequence satisfying $r_1(k) \geq 1$ and $r_2(k) \geq 1$, and $\lim_{k\to\infty}\sup r_1(k) = \lim_{k\to\infty}\sup r_2(k) = +\infty$, and that both δ_1 and δ_2 are real constants in the interval $(0, 1)$ satisfying

$$\varepsilon_k \leq c_1\delta_1^{r_1(k)} \quad and \quad \eta_k \leq c_2\delta_2^{r_2(k)}, \quad k = 0, 1, 2, \ldots$$

where c_1 and c_2 are nonnegative constants. Then it holds that

$$\|X^{(k+1)} - X^*\|_{\mathcal{S}} \leq (\sqrt{\sigma(\alpha) + \varphi\theta\delta^{r(k)}})^2 \|X^{(k)} - X^*\|_{\mathcal{S}},$$

where ϱ and θ are defined in the above theorem and $r(k)$ and δ are defined as

$$r(k) = \min\{r_1(k), r_2(k)\}, \quad \delta = \max\{\delta_1, \delta_2\},$$

and

$$\varphi = \max\{\sqrt{c_1c_2\varrho}, \frac{1}{2\sqrt{\sigma(\alpha)}}(c_1\sigma(\alpha) + c_2\varrho)\}.$$

Proof: The conclusion is straightforward according to Theorem 3.2 in [16].

4. Numerical examples

In this section, we use several examples to illustrate the effectiveness of the PNSS iteration method for solving the continuous Sylvester equation $AX + XB = C$. In addition, the numerical experiments are performed in Matlab (R2010b) on an Inter dual core processor (1.90GHz, 4GB RAM). All iterations of this section are started from zero matrix and terminated when the current residual norm satisfies

$$\|R^{(k)}\|_F / \|R^{(0)}\|_F \leq 10^{-6},$$

where $R^{(k)} = C - AX^k - X^kB$. The number of iteration steps (denoted as "IT") and the computing time in seconds (denoted as "CPU") are listed in tables. In the PNSS iteration method for the continuous

Table 1: IT and CPU for PNSS and NSS

	PNSS						NSS					
	r=0.01		r=0.1		r=1		r=0.01		r=0.1		r=1	
n	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU
8	9	0.075	9	0.085	9	0.085	13	0.128	13	0.131	14	0.166
16	12	0.541	12	0.465	13	0.463	20	0.758	19	0.646	18	0.668
32	20	2.686	18	2.378	16	2.670	36	4.497	31	4.773	28	3.685
64	26	11.361	23	9.440	22	10.391	45	22.975	37	17.229	35	18.965
128	37	129.059	35	122.163	33	110.980	62	214.199	58	192.295	55	167.192
256	45	1068.930	42	932.747	43	856.326	78	1665.597	73	1403.131	70	1399.644

Sylvester equation (1), $\mathcal{N}(A)$ and $\mathcal{N}(B)$ are two circulant matrices. We set the preconditioning matrices $V_1 = diag(\mathcal{N}(A)), V_2 = diag(\mathcal{N}(B))$.

Example 1. In this experiment, We consider the continuous Sylvester equation (1) with $m = n$ and the matrices

$$A = B = M + 2rN + \frac{100}{(n+1)^2} I,$$

where $M, N \in \mathbb{C}^{n\times n}$ are the tridiagonal matrices given by

$$M = \begin{pmatrix} 2.6 & -1 & & & \\ 1 & 2.6 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2.6 & -1 \\ & & & 1 & 2.6 \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & -0.5 & & & \\ 0.5 & 0 & -0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.5 & 0 & -0.5 \\ & & & 0.5 & 0 \end{pmatrix}.$$

This class of problems arise frequently in the preconditioned Krylov subspace iteration methods. We will solve this continuous Sylvester equation by the PNSS and the NSS iteration methods. The computing results of PNSS iteration method and the NSS iteration method are listed in Table 1, respectively. We compare the iteration steps and the computing time in seconds of both methods. We also present the optimal parameters $\alpha_{\exp}$ for the PNSS iteration method and the NSS iteration method in Table 2. From the results in Table 1, we observe that the PNSS is much better than the NSS both in terms of the number of iteration steps and computing time.

Table 2: The optimal values α_{exp} for PNSS and NSS

	PNSS			NSS		
n	r=0.01	r=0.1	r=1	r=0.01	r=0.1	r=1
8	0.68	0.68	0.82	1.99	1.97	1.95
16	0.65	0.62	0.66	3.00	2.91	2.82
32	0.79	0.69	0.56	3.73	3.20	3.02
64	0.92	0.87	0.42	4.18	3.37	3.23
128	1.26	1.20	0.30	5.52	5.10	4.91
256	1.52	1.41	0.23	6.85	6.39	6.10

Table 3: IT and CPU for PNSS and NSS

	PNSS		NSS	
n	IT	CPU	IT	CPU
8	6	0.054	10	0.396
16	6	0.307	11	0.416
32	6	1.008	13	2.545
64	6	4.566	17	8.417
128	6	17.120	21	62.988
256	6	108.034	22	521.290

Example 2. To generate large sparse matrices A and B , we build them in the following structures:

$$A = \begin{pmatrix} 10 & 1 & & & 1 \\ 2 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & 10 & 1 \\ 2 & & & 2 & 10 \end{pmatrix},$$

$$B = \begin{pmatrix} 8 & 1 & & & 1 \\ 3 & 8 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 3 & 8 & 1 \\ 3 & & & 3 & 8 \end{pmatrix}.$$

The computing results of the PNSS and the NSS are listed in the Table 3, the optimal parameters $\alpha_{\rm exp}$ for PNSS and NSS are presented in Table 4.

Table 4: The optimal parameters α_{exp} for PNSS and NSS

	PNSS	NSS
n	α_{exp}	α_{exp}
8	1.14	7.8
16	1.14	7.3
32	1.14	6.2
64	1.14	5.0
128	1.14	4.1
256	1.14	3.8

From the results we can observe that the number of iteration steps (IT) of PNSS is smaller than that of NSS, and the PNSS has much less computational workload than NSS at each of the iteration steps, and the actual computing time (CPU) of PNSS may be less than that of NSS. So when the matrices A and B are large enough, the PNSS iteration methods considerably outperform the NSS iteration methods in both iteration step and computing time.

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