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On a Solvable Symmetric and a Cyclic System of Partial Difference Equations

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Abstract. It is shown that the following symmetric system of partial difference equations

$$c_{m,n} = d_{m-1,n} + c_{m-1,n-1},$$

$$d_{m,n} = c_{m-1,n} + d_{m-1,n-1},$$

is solvable on the combinatorial domain $C = \left\{(m,n) \in \mathbb{N}_0^2 : 0 \le n \le m\right\} \setminus \{(0,0)\}$, by presenting some formulas for the general solution to the system on the domain in terms of the boundary values $c_{j,j}$, $c_{j,0}$, $d_{j,j}$, $d_{j,0}$, $j \in \mathbb{N}$, and the indices m and n. The corresponding result for a related three-dimensional cyclic system of partial difference equations is also proved. These results can serve as a motivation for further studies of the solvability of symmetric, close-to-symmetric, cyclic, close-to-cyclic and other related systems of partial difference equations.

1. Introduction

By C_n^m, $0 \le n \le m$, we denote the binomial coefficients, which are obtained, for example, as the coefficients of the polynomial $P_m(x) = (1+x)^m$, $m \in \mathbb{N}_0$, that is,

$$P_m(x) = C_0^m + C_1^m x + C_2^m x^2 + \cdots + C_{m-1}^m x^{m-1} + C_m^m x^m,$$

for an $m \in \mathbb{N}_0$.

The coefficients satisfy the following equalities

$$C_0^m = C_m^m = 1, \quad m \in \mathbb{N}_0, \tag{1}$$

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and the recurrent relation

$$C_n^m = C_n^{m-1} + C_{n-1}^{m-1}, \tag{2}$$

for all $m, n \in \mathbb{N}$ such that $n < m$. By definition is taken that $C_n^m = 0$ if $m < n$, which is used in the paper.

Many books contain some results on the coefficients (see, for example, [10, 13, 15, 24, 25, 59]; books [10] and [13], among other things, contain many basic relations which include the coefficients/numbers and present basic methods for dealing with them, [15] presents many relations containing the coefficients and methods for proving them, as well as many methods for calculating finite sums of various types, [24] presents a list of numerous relations and combinatorial identities, [25] presents, among other things, some advanced methods for dealing with combinatorial identities, while [59] contains various combinatorial applications).

The notation seems a bit deceiving because it does not reveal the real character of recurrent relation (2). However, if it is written in the following way

$$c_{m,n} = c_{m-1,n} + c_{m-1,n-1}, \tag{3}$$

it becomes clear that (2) is a difference equation with two independent variables, that is, a partial difference equation, while (1) are some conditions given on the discrete half-lines $m = n$ and $n = 0$, $m \in \mathbb{N}$, that is, some boundary-value conditions on the following domain

$$\mathcal{C} = \left\{(m, n) \in \mathbb{N}_0^2 : 0 \le n \le m\right\} \setminus \{(0, 0)\},$$

which, we call *the combinatorial domain* (point $(0, 0)$ is excluded since the value $c_{0,0}$ is essentially not used in calculating the other values of the sequence $c_{m,n}$).

One of the basic problems for any kind of difference equations, as well as for other types of equations, is their solvability. Some old results on partial difference equations, devoted mostly to the problem of their solvability, can be found, for example, in [9, Chapter 12] and [11, Chapter 8]. For some results in the area up to 2003, see the monograph [3]. See, also [16]. The solvability of difference equations and systems with one independent variable is an ancient topic (see, for example, [2, 6, 9–12, 14, 15]). Some recent attention on the solvability of difference equations has been, among other things, attracted by S. Stević's note [27], which explained the solvability of the following nonlinear second-order difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_n x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{4}$$

by transforming the equation to a solvable one by using a suitable change of variables (for more details, as well as more general results see [22, 29, 30, 50] and the references therein). It has turned out that closely related methods and ideas can be applied to some other classes of difference equations (see [38, 51, 53] and the references therein), as well as to some related systems of difference equations (see, for example, [1, 28, 31–33, 47–49, 52, 54, 55]; [28] solves a two-dimensional close-to-symmetric relative of (4), while [48] studies a three-dimensional relative of (4) of the type). For some other methods for solving difference equations and systems, and related topics, see also [5, 8, 34, 38, 44]. The main feature of papers [1, 22, 33, 48–51, 53], as well as of many quoted in their lists of references, is that decisive role in their solvability plays the following difference equation

$$x_n = a_n x_{n-1} + b_n, \quad n \in \mathbb{N}, \tag{5}$$

which is solvable one (see, for example, [2, 15], where three methods for solving the equation, which essentially correspond to the three methods for solving the linear first-order differential one, are presented). We would like to mention that even behind the solvability of some product-type difference equations and systems is the solvability of equation (5) (see, for example, [35, 37, 46, 56–58] and the references therein). Moreover, in some cases, the solvability of the product-type systems is shown by using the solvability of some special cases of the corresponding product-type equation, that is, of the following equation

$$z_n = b_n z_{n-1}^{a_n}, \quad n \in \mathbb{N}.$$

For some recent results on equation (5), see [40] and [42], which are partly motivated by a well-known problem in [4] (see, also recent paper [41]). All these facts show that equation (5) occupies one of the central roles in the area of solvability of difference equations and systems.

During the investigation of solvable difference equations and systems Stević has noticed the fact that for the binomial coefficients there is a closed-form formula. Namely, it is well-known that

$$C_n^m = \frac{m!}{n!(m-n)!}, \quad 0 \le n \le m.$$

The formula presents a solution to equation (3), which suggests potential solvability of equation (3) on the combinatorial domain. That equation (3) can be "solved" is a well-known fact (see, for example, [11, p.239]), but the general solution presented therein is not suitable for solving boundary-value problems on the combinatorial domain. These facts lead us to the natural problem of trying to find closed-form formula for solutions to equation (3) on the domain in terms of its boundary values $c_{0,j}$, $c_{j,j}$, $j \in \mathbb{N}$. The problem was solved in [36], and by using the methods and ideas appearing therein, including some ideas related to solving equation (5), some related results were later obtained in [39], [43] and [45].

On the other hand, during the 90's Papaschinopoulos and Schinas have started studying concrete symmetric and related systems of difference equations ([17]-[19]), which have motivated experts to do research in the direction (see [1, 7, 20, 21, 26, 28, 31–35, 37, 46, 48, 49, 52, 54–58] and the numerous references therein). Note also that some of these papers, such as [18]-[21], study, among other things, the invariants of some systems of difference equations, which in a wider sense also belong to the area of solvability (invariants of equations and systems are not their solutions, but can help in getting some results on their long-term behavior).

These two directions in the investigation of difference equations and systems of one independent variable inspired us to study the solvability of symmetric, close to symmetric and other related classes of partial difference equations.

In this paper we start with the investigation by showing that the following system of partial difference equations

$$\begin{aligned} c_{m,n} &= d_{m-1,n} + c_{m-1,n-1}, \\ d_{m,n} &= c_{m-1,n} + d_{m-1,n-1}, \end{aligned} \tag{6}$$

which is a natural two-dimensional extension to equation (3), is solvable on $\mathcal{C}$. To do this we will use the method of half-lines described in [36], a half-constructive method which essentially uses the solvability of some special cases of equation (5) on the intersection of domain $\mathcal{C}$ and the lines $y = t + k$, $t \in \mathbb{Z}$, when $k \in \mathbb{N}$, along with an inductive argument with respect to k. More precisely, in the case of system (6) will be used the case of equation (5) when $a_n = 1$ for every n, as it was the case in [36]. The case essentially reduces application of equation (5) to telescoping summations along with iterated summations. Nevertheless the solvability of equation (5) in the special case is crucial. It should be also mentioned that the behavior of the solutions to the equation (5) in the case can be quite complex (see [14, 23]), but this problem is not considered here. The solvability of a closely related tree-dimensional cyclic system of partial difference equations is also studied. As far as we know, these are the first results of this type in the literature and this paper initiates the study of the solvability of symmetric, close-to-symmetric, cyclic, close-to-cyclic and other related systems of partial difference equations. We use the standard convention $\sum_{j=k}^{l} a_j = 0$, when $k, l \in \mathbb{Z}$ and $l < k$.

2. Main results

In this section we formulate and prove the main results in this paper.

2.1. System (6) *on the combinatorial domain*

The method of half-lines requests ”solving” system (6) on the half lines which are obtained as intersections of domain C and the lines

$$y = t + k, \quad t \in \mathbb{Z},$$

when $k \in \mathbb{N}$, for first several k-s and then based on the obtained ”solutions” should be guessed what is the general solution to the system on C. We will find solutions to the corresponding equations for $k = 1, 2, 3$, and then guess a formula for the general solution to the system (from the mathematical point of view the case $k = 3$ need not be presented, but we include it for a clearer presentation and the benefit of the reader). Before this we formulate a known lemma which will be frequently used in this paper. Beside the formulation in the terms of the binomial coefficients ([10, 15, 24]), it can be also formulated as a sum of a rational sequence and can be found in many problem books and books on finite differences ([2, 4, 15]).

Lemma 1. *Assume that* $k, r \in \mathbb{N}_0$. *Then*

$$\sum_{j=0}^{n} C_r^{k+j} = C_{r+1}^{n+k+1} - C_{r+1}^{k},$$

for every $n \in \mathbb{N}_0$.

Let $m = n + 1$, then

$$c_{n+1,n} = c_{n,n-1} + d_{n,n}, \qquad d_{n+1,n} = d_{n,n-1} + c_{n,n},$$

for $n \in \mathbb{N}$, from which it follows that

$$c_{n+1,n} = c_{1,0} + \sum_{j=1}^{n} d_{j,j}, \qquad d_{n+1,n} = d_{1,0} + \textstyle\sum_{j=1}^{n} c_{j,j}, \tag{7}$$

for $n \in \mathbb{N}_0$.

Let $m = n + 2$, then

$$c_{n+2,n} = c_{n+1,n-1} + d_{n+1,n}, \qquad d_{n+2,n} = d_{n+1,n-1} + c_{n+1,n},$$

for $n \in \mathbb{N}$, and consequently

$$\begin{aligned} c_{n+2,n} &= c_{2,0} + \sum_{j=1}^{n} d_{j+1,j}, \\ d_{n+2,n} &= d_{2,0} + \sum_{j=1}^{n} c_{j+1,j}, \end{aligned} \tag{8}$$

for $n \in \mathbb{N}_0$.

From (7), (8) and some calculation, it follows that

$$\begin{aligned} c_{n+2,n} &= c_{2,0} + \sum_{j=1}^{n} \Big(d_{1,0} + \sum_{i=1}^{j} c_{i,i} \Big) \\ &= c_{2,0} + n d_{1,0} + \sum_{i=1}^{n} (n - i + 1) c_{i,i}, \end{aligned} \tag{9}$$

$$\begin{aligned} d_{n+2,n} &= d_{2,0} + \sum_{j=1}^{n} \Big(c_{1,0} + \sum_{i=1}^{j} d_{i,i} \Big) \\ &= d_{2,0} + n c_{1,0} + \sum_{i=1}^{n} (n - i + 1) d_{i,i}, \end{aligned} \tag{10}$$

for $n \in \mathbb{N}_0$.

Let $m = n + 3$, then

$$\begin{aligned} c_{n+3,n} &= c_{n+2,n-1} + d_{n+2,n}, \\ d_{n+3,n} &= d_{n+2,n-1} + c_{n+2,n}, \end{aligned}$$

for $n \in \mathbb{N}$, from which it follows that

$$\begin{aligned} c_{n+3,n} &= c_{3,0} + \sum_{j=1}^{n} d_{j+2,j}, \\ d_{n+3,n} &= d_{3,0} + \sum_{j=1}^{n} c_{j+2,j}, \end{aligned} \tag{11}$$

for $n \in \mathbb{N}_0$.

From (9)-(11) and some calculation, it follows that

$$\begin{aligned} c_{n+3,n} &= c_{3,0} + \sum_{j=1}^{n} \left(d_{2,0} + jc_{1,0} + \sum_{i=1}^{j} (j - i + 1) d_{i,i} \right) \\ &= c_{3,0} + nd_{2,0} + \frac{n(n+1)}{2} c_{1,0} + \sum_{i=1}^{n} d_{i,i} \sum_{j=i}^{n} (j - i + 1) \\ &= c_{3,0} + nd_{2,0} + \frac{n(n+1)}{2} c_{1,0} + \sum_{i=1}^{n} d_{i,i} \frac{(n - i + 1)(n - i + 2)}{2} \end{aligned}$$

and

$$\begin{aligned} d_{n+3,n} &= d_{3,0} + \sum_{j=1}^{n} \left(c_{2,0} + jd_{1,0} + \sum_{i=1}^{j} (j - i + 1) c_{i,i} \right) \\ &= d_{3,0} + nc_{2,0} + \frac{n(n+1)}{2} d_{1,0} + \sum_{i=1}^{n} c_{i,i} \frac{(n - i + 1)(n - i + 2)}{2} \end{aligned}$$

for $n \in \mathbb{N}_0$.

One of the crucial points in this analysis is to note that the last two formulas can be written in the following, combinatorial, from (forms which include some binomial coefficients):

$$c_{n+3,n} = C_0^{n-1} c_{3,0} + C_1^{n} d_{2,0} + C_2^{n+1} c_{1,0} + \sum_{i=1}^{n} C_2^{n-i+2} d_{i,i}, \tag{12}$$

and

$$d_{n+3,n} = C_0^{n-1} d_{3,0} + C_1^{n} c_{2,0} + C_2^{n+1} d_{1,0} + \sum_{i=1}^{n} C_2^{n-i+2} c_{i,i}, \tag{13}$$

for $n \in \mathbb{N}_0$.

Equalities (7), (9), (10), (12), (13), suggest that the following formulas hold

$$c_{n+2l-1,n} = \sum_{j=1}^{l} C_{2l-2j}^{n+2l-2j-1} c_{2j-1,0} + \sum_{j=1}^{l-1} C_{2l-2j-1}^{n+2l-2j-2} d_{2j,0} + \sum_{i=1}^{n} C_{2l-2}^{n-i+2l-2} d_{i,i}, \tag{14}$$

$$c_{n+2l,n} = \sum_{j=1}^{l} C_{2l-2j}^{n+2l-2j-1} c_{2j,0} + \sum_{j=1}^{l} C_{2l-2j+1}^{n+2l-2j} d_{2j-1,0} + \sum_{i=1}^{n} C_{2l-1}^{n-i+2l-1} c_{i,i}, \tag{15}$$

$$d_{n+2l-1,n} = \sum_{j=1}^{l} C_{2l-2j}^{n+2l-2j-1} d_{2j-1,0} + \sum_{j=1}^{l-1} C_{2l-2j-1}^{n+2l-2j-2} c_{2j,0} + \sum_{i=1}^{n} C_{2l-2}^{n-i+2l-2} c_{i,i}, \tag{16}$$

$$d_{n+2l,n} = \sum_{j=1}^{l} C_{2l-2j}^{n+2l-2j-1} d_{2j,0} + \sum_{j=1}^{l} C_{2l-2j+1}^{n+2l-2j} c_{2j-1,0} + \sum_{i=1}^{n} C_{2l-1}^{n-i+2l-1} d_{i,i}, \tag{17}$$

for $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$.

Let $m = n + 2l + 1$, then

$$\begin{aligned} c_{n+2l+1,n} &= c_{n+2l,n-1} + d_{n+2l,n}, \\ d_{n+2l+1,n} &= d_{n+2l,n-1} + c_{n+2l,n}, \end{aligned} \tag{18}$$

for $n \in \mathbb{N}$.

From (18), it follows that

$$\begin{aligned} c_{n+2l+1,n} &= c_{2l+1,0} + \sum_{s=1}^{n} d_{s+2l,s}, \\ d_{n+2l+1,n} &= d_{2l+1,0} + \sum_{j=s}^{n} c_{s+2l,s}, \end{aligned} \tag{19}$$

for $n \in \mathbb{N}_0$.

Using the hypotheses (15) and (17) in (19), it follows that

$$\begin{aligned} c_{n+2l+1,n} &= c_{2l+1,0} + \sum_{s=1}^{n} \Big(\sum_{j=1}^{l} C_{2l-2j}^{s+2l-2j-1} d_{2j,0} + \sum_{j=1}^{l} C_{2l-2j+1}^{s+2l-2j} c_{2j-1,0} + \sum_{i=1}^{s} C_{2l-1}^{s-i+2l-1} d_{i,i} \Big) \\ &= c_{2l+1,0} + \sum_{j=1}^{l} d_{2j,0} \sum_{s=1}^{n} C_{2l-2j}^{s+2l-2j-1} + \sum_{j=1}^{l} c_{2j-1,0} \sum_{s=1}^{n} C_{2l-2j+1}^{s+2l-2j} + \sum_{i=1}^{n} d_{i,i} \sum_{s=i}^{n} C_{2l-1}^{s-i+2l-1}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} d_{n+2l+1,n} &= d_{2l+1,0} + \sum_{s=1}^{n} \Big(\sum_{j=1}^{l} C_{2l-2j}^{s+2l-2j-1} c_{2j,0} + \sum_{j=1}^{l} C_{2l-2j+1}^{s+2l-2j} d_{2j-1,0} + \sum_{i=1}^{s} C_{2l-1}^{s-i+2l-1} c_{i,i} \Big) \\ &= d_{2l+1,0} + \sum_{j=1}^{l} c_{2j,0} \sum_{s=1}^{n} C_{2l-2j}^{s+2l-2j-1} + \sum_{j=1}^{l} d_{2j-1,0} \sum_{s=1}^{n} C_{2l-2j+1}^{s+2l-2j} + \sum_{i=1}^{n} c_{i,i} \sum_{s=i}^{n} C_{2l-1}^{s-i+2l-1}, \end{aligned} \tag{21}$$

for every $n \in \mathbb{N}_0$.

By Lemma 1, we have

$$\sum_{s=1}^{n} C_{2l-2j+t}^{s+2l-2j-1+t} = C_{2l-2j+1+t}^{n+2l-2j+t} - C_{2l-2j+1+t}^{2l-2j+t} = C_{2l-2j+1+t}^{n+2l-2j+t}, \tag{22}$$

for every $1 \le j \le l$ and $t \in \mathbb{N}_0$, and

$$\sum_{s=i}^{n} C_{2l-1+t}^{s-i+2l-1+t} = C_{2l+t}^{n-i+2l+t} - C_{2l+t}^{2l-1+t} = C_{2l+t}^{n-i+2l+t}, \tag{23}$$

for every $1 \le i \le n$ and $t \in \mathbb{N}_0$.

Using (22) and (23) in (20) and (21), we get

$$\begin{aligned} c_{n+2l+1,n} &= c_{2l+1,0} + \sum_{j=1}^{l} d_{2j,0} C_{2l-2j+1}^{n+2l-2j} + \sum_{j=1}^{l} c_{2j-1,0} C_{2l-2j+2}^{n+2l-2j+1} + \sum_{i=1}^{n} d_{i,i} C_{2l}^{n-i+2l}, \\ &= \sum_{j=1}^{l+1} c_{2j-1,0} C_{2l-2j+2}^{n+2l-2j+1} + \sum_{j=1}^{l} d_{2j,0} C_{2l-2j+1}^{n+2l-2j} + \sum_{i=1}^{n} d_{i,i} C_{2l}^{n-i+2l}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} d_{n+2l+1,n} &= d_{2l+1,0} + \sum_{j=1}^{l} c_{2j,0} C_{2l-2j+1}^{n+2l-2j} + \sum_{j=1}^{l} d_{2j-1,0} C_{2l-2j+2}^{n+2l-2j+1} + \sum_{i=1}^{n} c_{i,i} C_{2l}^{n-i+2l}, \\ &= \sum_{j=1}^{l+1} d_{2j-1,0} C_{2l-2j+2}^{n+2l-2j+1} + \sum_{j=1}^{l} c_{2j,0} C_{2l-2j+1}^{n+2l-2j} + \sum_{i=1}^{n} c_{i,i} C_{2l}^{n-i+2l}, \end{aligned} \tag{25}$$

for every $n \in \mathbb{N}_0$.

Let $m = n + 2l + 2$, then

$$c_{n+2l+2,n} = c_{n+2l+1,n-1} + d_{n+2l+1,n}, \quad d_{n+2l+2,n} = d_{n+2l+1,n-1} + c_{n+2l+1,n}, \tag{26}$$

for $n \in \mathbb{N}$.

From (26), it follows that

$$c_{n+2l+2,n} = c_{2l+2,0} + \sum_{s=1}^{n} d_{s+2l+1,s}, \quad d_{n+2l+2,n} = d_{2l+2,0} + \sum_{s=1}^{n} c_{s+2l+1,s}, \tag{27}$$

for $n \in \mathbb{N}_0$.

Using (24) and (25), as well as (22) and (23) in (27), it follows that

$$\begin{aligned} c_{n+2l+2,n} &= c_{2l+2,0} + \sum_{s=1}^{n} \Big(\sum_{j=1}^{l+1} d_{2j-1,0} C_{2l-2j+2}^{s+2l-2j+1} + \sum_{j=1}^{l} c_{2j,0} C_{2l-2j+1}^{s+2l-2j} + \sum_{i=1}^{s} c_{i,i} C_{2l}^{s-i+2l} \Big) \\ &= c_{2l+2,0} + \sum_{j=1}^{l+1} d_{2j-1,0} \sum_{s=1}^{n} C_{2l-2j+2}^{s+2l-2j+1} + \sum_{j=1}^{l} c_{2j,0} \sum_{s=1}^{n} C_{2l-2j+1}^{s+2l-2j} + \sum_{i=1}^{n} c_{i,i} \sum_{s=i}^{n} C_{2l}^{s-i+2l} \\ &= c_{2l+2,0} + \sum_{j=1}^{l+1} d_{2j-1,0} C_{2l-2j+3}^{n+2l-2j+2} + \sum_{j=1}^{l} c_{2j,0} C_{2l-2j+2}^{n+2l-2j+1} + \sum_{i=1}^{n} c_{i,i} C_{2l+1}^{n-i+2l+1} \\ &= \sum_{j=1}^{l+1} c_{2j,0} C_{2l-2j+2}^{n+2l-2j+1} + \sum_{j=1}^{l+1} d_{2j-1,0} C_{2l-2j+3}^{n+2l-2j+2} + \sum_{i=1}^{n} c_{i,i} C_{2l+1}^{n-i+2l+1}, \end{aligned} \tag{28}$$

and

$$
\begin{aligned}
d_{n+2l+2,n} &= d_{2l+2,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} c_{2j-1,0}C_{2l-2j+2}^{s+2l-2j+1} + \sum_{j=1}^{l} d_{2j,0}C_{2l-2j+1}^{s+2l-2j} + \sum_{i=1}^{s} d_{i,i}C_{2l}^{s-i+2l}\Big)\\
&= d_{2l+2,0} + \sum_{j=1}^{l+1} c_{2j-1,0}\sum_{s=1}^{n} C_{2l-2j+2}^{s+2l-2j+1} + \sum_{j=1}^{l} d_{2j,0}\sum_{s=1}^{n} C_{2l-2j+1}^{s+2l-2j} + \sum_{i=1}^{n} d_{i,i}\sum_{s=i}^{n} C_{2l}^{s-i+2l}\\
&= d_{2l+2,0} + \sum_{j=1}^{l+1} c_{2j-1,0}C_{2l-2j+3}^{n+2l-2j+2} + \sum_{j=1}^{l} d_{2j,0}C_{2l-2j+2}^{n+2l-2j+1} + \sum_{i=1}^{n} d_{i,i}C_{2l+1}^{n-i+2l+1}\\
&= \sum_{j=1}^{l+1} d_{2j,0}C_{2l-2j+2}^{n+2l-2j+1} + \sum_{j=1}^{l+1} c_{2j-1,0}C_{2l-2j+3}^{n+2l-2j+2} + \sum_{i=1}^{n} d_{i,i}C_{2l+1}^{n-i+2l+1}, \qquad (29)
\end{aligned}
$$

for every $n \in \mathbb{N}_0$.

From (7), (9), (10), (24), (25), (28), (29) and the induction we get that (14)-(17) hold for all $n, l \in \mathbb{N}_0$.

Now we are in a position to formulate and prove our first main result in this paper.

Theorem 1. *Assume that $(\alpha_k)_{k\in\mathbb{N}}$, $(\beta_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$ are given sequences of complex numbers. Then the solution to system* (6) *with the following boundary-value conditions*

$$
c_{k,0} = \alpha_k, \quad c_{k,k} = \beta_k, \quad d_{k,0} = \gamma_k, \quad d_{k,k} = \delta_k, \quad k \in \mathbb{N}, \qquad (30)
$$

is given by

$$
c_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}\alpha_{2j-1} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1}\gamma_{2j} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}\delta_i, \qquad (31)
$$

$$
d_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}\gamma_{2j-1} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1}\alpha_{2j} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}\beta_i, \qquad (32)
$$

when $m - n \equiv 1 \ (\bmod\ 2)$

$$
c_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}\alpha_{2j} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j}\gamma_{2j-1} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}\beta_i, \qquad (33)
$$

$$
d_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}\gamma_{2j} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j}\alpha_{2j-1} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}\delta_i, \qquad (34)
$$

when $m - n \equiv 0 \ (\bmod\ 2)$, for every $m, n \in \mathbb{N}$ such that $m > n$.

Proof. If we put $2l = m - n + 1$, when $m - n \equiv 1(\bmod\ 2)$ in (14) and (16), that is, $2l = m - n$, when $m - n \equiv 0(\bmod\ 2)$, in (15) and (17) and employ in such obtained formulas the boundary-value conditions in (30), formulas (31)-(34) are easily obtained. □

Remark 1. From Theorem 1 we see that the general solution to system (6) on C is given by

$$c_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j} c_{2j-1,0} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1} d_{2j,0} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} d_{i,i},$$
$$d_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j} d_{2j-1,0} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1} c_{2j,0} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} c_{i,i},$$

when $m - n \equiv 1 \ (\bmod\ 2)$

$$c_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1} c_{2j,0} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j} d_{2j-1,0} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} c_{i,i},$$
$$d_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1} d_{2j,0} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j} c_{2j-1,0} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} d_{i,i},$$

when $m - n \equiv 0 \ (\bmod\ 2)$, for $m, n \in \mathbb{N}$ such that $m > n$.

Remark 2. If

$$\alpha_k = \gamma_k \quad \text{and} \quad \beta_k = \delta_k, \quad k \in \mathbb{N},$$

then from Theorem 1 we see that

$$c_{m,n} = d_{m,n},$$

for every $(m, n) \in C$. Moreover, after some calculation from (31) and (33), we obtain

$$c_{m,n} = \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} \alpha_j + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \beta_i,$$

which is the formula for general solution to equation (3) on domain C obtained in [36]. □

From formulas (31)-(34) we see that to get closed-form formulas for solutions to equation (6) we should know some closed-form formulas for the sums of the following forms:

$$\sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j} c_j, \quad \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1} c_j, \quad \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} c_i,$$

when $m - n \equiv 1 \ (\bmod\ 2)$, that is, of the following ones:

$$\sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1} c_j, \quad \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j} c_j, \quad \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} c_i,$$

when $m - n \equiv 0 \ (\bmod\ 2)$, for every $m, n \in \mathbb{N}$ such that $m > n$.

One of the cases where we can give some more compact formulas for the solution to system (6) is presented in the following corollary.

Corollary 1. *Assume that $\beta, \delta \in \mathbb{C}$, and $(\alpha_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$ are given sequences of complex numbers. Then the solution to system* (6) *with the following boundary value conditions*

$$c_{k,0} = \alpha_k, \quad c_{k,k} = \beta, \quad d_{k,0} = \gamma_k, \quad d_{k,k} = \delta, \quad k \in \mathbb{N}, \tag{35}$$

is given by

$$c_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}\alpha_{2j-1} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1}\gamma_{2j} + \delta C_{m-n}^{m-1}, \tag{36}$$

$$d_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}\gamma_{2j-1} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1}\alpha_{2j} + \beta C_{m-n}^{m-1}, \tag{37}$$

when $m - n \equiv 1 \pmod 2$

$$c_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}\alpha_{2j} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j}\gamma_{2j-1} + \beta C_{m-n}^{m-1}, \tag{38}$$

$$d_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}\gamma_{2j} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j}\alpha_{2j-1} + \delta C_{m-n}^{m-1}, \tag{39}$$

when $m - n \equiv 0 \pmod 2$, *for every* $m, n \in \mathbb{N}$ *such that* $m > n$.

Proof. If we put boundary-value conditions (35) in (31)-(34), we get

$$c_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}\alpha_{2j-1} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1}\gamma_{2j} + \delta \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}, \tag{40}$$

$$d_{m,n} = \sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}\gamma_{2j-1} + \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1}\alpha_{2j} + \beta \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}, \tag{41}$$

when $m - n \equiv 1 \pmod 2$

$$c_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}\alpha_{2j} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j}\gamma_{2j-1} + \beta \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}, \tag{42}$$

$$d_{m,n} = \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}\gamma_{2j} + \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j}\alpha_{2j-1} + \delta \sum_{i=1}^{n} C_{m-n-1}^{m-i-1}, \tag{43}$$

On the other hand, by Lemma 1, we have

$$\sum_{i=1}^{n} C_{m-n-1}^{m-i-1} = C_{m-n}^{m-1} - C_{m-n}^{m-n-1} = C_{m-n}^{m-1}. \tag{44}$$

By using equality (44) into (40)-(43), the formulas in (36)-(39) follow, finishing the proof. $\square$

Remark 3. Note that in the special case, when

$$\alpha_k = \gamma_k = 0, \quad k \in \mathbb{N},$$

from (36)-(39) it follows that

$$c_{m,n} = \delta C_{m-n}^{m-1}, \qquad d_{m,n} = \beta C_{m-n}^{m-1},$$

when $m - n \equiv 1 \pmod 2$, while

$$c_{m,n} = \beta C_{m-n}^{m-1}, \qquad d_{m,n} = \delta C_{m-n}^{m-1},$$

when $m - n \equiv 0 \pmod 2$, for every $m, n \in \mathbb{N}$ such that $m > n$.

The case when the sequences $(\alpha_{2j-1})_{j\in\mathbb{N}}$, $(\alpha_{2j})_{j\in\mathbb{N}}$, $(\gamma_{2j-1})_{j\in\mathbb{N}}$ and $(\gamma_{2j})_{j\in\mathbb{N}}$, are constant is one of the interesting ones. From (36)-(39) we see that to find closed-form formulas for the general solutions to system (6) in the case, it is necessary to calculate the following sums:

$$\sum_{j=1}^{\frac{m-n+1}{2}} C_{m-n+1-2j}^{m-2j}, \qquad \sum_{j=1}^{\frac{m-n-1}{2}} C_{m-n-2j}^{m-2j-1},$$

when $m - n \equiv 1 \pmod 2$, and

$$\sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j}^{m-2j-1}, \qquad \sum_{j=1}^{\frac{m-n}{2}} C_{m-n-2j+1}^{m-2j},$$

when $m - n \equiv 0 \pmod 2$, for every $m, n \in \mathbb{N}$ such that $m > n$, which can be written in the following somewhat neater forms:

$$a_{m,n} := \sum_{j=0}^{\frac{m-n-1}{2}} C_{2j}^{n+2j-1}, \qquad b_{m,n} := \sum_{j=0}^{\frac{m-n-3}{2}} C_{2j+1}^{n+2j},$$

when $m - n \equiv 1 \pmod 2$, and

$$\widehat{a}_{m,n} := \sum_{j=0}^{\frac{m-n-2}{2}} C_{2j}^{n+2j-1}, \qquad \widehat{b}_{m,n} := \sum_{j=0}^{\frac{m-n-2}{2}} C_{2j+1}^{n+2j},$$

when $m - n \equiv 0 \pmod 2$, for every $m, n \in \mathbb{N}$ such that $m > n$, or to avoid using the modulo function as follows:

$$a_{n+2k+1,n} := \sum_{j=0}^{k} C_{2j}^{n+2j-1}, \qquad b_{n+2k+1,n} := \sum_{j=0}^{k-1} C_{2j+1}^{n+2j},$$

$$\widehat{a}_{n+2k+2,n} := \sum_{j=0}^{k} C_{2j}^{n+2j-1}, \qquad \widehat{b}_{n+2k+2,n} := \sum_{j=0}^{k} C_{2j+1}^{n+2j},$$

when $k \in \mathbb{N}_0$.

Note that

$$a_{n+2k+1,n} + b_{n+2k+1,n} = \sum_{j=0}^{2k} C_j^{n+j-1} = \sum_{j=0}^{2k} \left(C_j^{n+j} - C_{j-1}^{n+j-1} \right) = C_{2k}^{n+2k}, \tag{45}$$

and

$$\widehat{a}_{n+2k+2,n} + \widehat{b}_{n+2k+2,n} = \sum_{j=0}^{2k+1} C_j^{n+j-1} = \sum_{j=0}^{2k+1} \left(C_j^{n+j} - C_{j-1}^{n+j-1} \right) = C_{2k+1}^{n+2k+1},$$

for $k \in \mathbb{N}_0$.

However, we are not able to find closed-form formulas for $a_{n+2k+1,n}$, $b_{n+2k+1,n}$, $\widehat{a}_{n+2k+2,n}$ and $\widehat{b}_{n+2k+2,n}$, at the moment. Namely, all the methods that we have used so far in trying to find the closed-form formulas only transformed the sums which define the sequences to some other ones, for which we are not able to find closed-form formulas, either. Nevertheless, we have obtained several interesting relations and facts which could serve as a motivation for further investigation in the area. Now we will present some of them.

First, note that

$$\begin{aligned} a_{n+2k+1,n} &= \sum_{j=0}^{k} C_{2j}^{n+2j-1} = 1 + \sum_{j=1}^{k}\left(C_{2j-1}^{n+2j-2} + C_{2j}^{n+2j-2}\right) \\ &= \sum_{j=0}^{k-1} C_{2j+1}^{n+2j} + \sum_{j=0}^{k} C_{2j}^{n-1+2j-1} \\ &= b_{n+2k+1,n} + a_{n+2k,n-1}, \end{aligned} \tag{46}$$

for $k \in \mathbb{N}_0$.

From (45) and (46), it follows that

$$a_{n+2k+1,n} = \frac{a_{n+2k,n-1}}{2} + \frac{1}{2}C_{2k}^{n+2k}, \tag{47}$$

for $k \in \mathbb{N}_0$.

Using the change of variables

$$x_n := a_{n+2k+1,n}, \quad n \in \mathbb{N},$$

for a fixed k, equation (47) can be written in the form of the equation in (5) and by solving it, we obtain

$$a_{n+2k+1,n} = \frac{a_{2k+1,0}}{2^n} + \sum_{j=1}^{n} \frac{1}{2^{n-j+1}} C_{2k}^{2k+j}, \tag{48}$$

for $k \in \mathbb{N}_0$.

From (46) and (48), we see that the problem of calculating the sums $a_{n+2k+1,n}$ and $b_{n+2k+1,n}$ is equivalent to the calculation of the sum

$$s_{n,k} := \sum_{j=1}^{n} 2^j C_{2k}^{2k+j}, \tag{49}$$

for $k \in \mathbb{N}_0$.

Another method for calculating the sum $a_{n+2k+1,n}$ is by using suitable polynomials such that the sum in (49) is one of their coefficients (the method is used in [10, 15, 25]). One of the most natural polynomials is the following one:

$$P_{n+2k-1}(x) = \sum_{j=0}^{k} (1+x)^{n-1+2j}(x^2)^{k-j},$$

whose coefficients at x^{2k} is equal to $a_{n+2k+1,n}$.

By some calculation we have

$$\begin{aligned} P_{n+2k-1}(x) &= (1+x)^{n-1}\frac{(1+x)^{2k+2} - x^{2k+2}}{(1+x)^2 - x^2} \\ &= ((1+x)^{n+2k+1} - (1+x)^{n-1}x^{2k+2})(1+2x)^{-1} \\ &= ((1+x)^{n+2k+1} - (1+x)^{n-1}x^{2k+2})\sum_{j=0}^{\infty}(-2)^j x^j, \end{aligned}$$

where, of course, the last equality holds for $|x| < 1/2$.

Comparing the coefficients at x^{2k} in these two expansions of the polynomial is obtained

$$a_{n+2k+1,n} = \sum_{j=0}^{2k} (-2)^j C_{2k-j}^{n+2k+1}. \tag{50}$$

However, the sum is also not so easy to calculate.

2.2. *On a three-dimensional relative of system* (6)

Having solved system (6) on C it is natural to ask if some of its three-dimensional cousins are also solvable. One of the most natural ones is the following cyclic system:

$$\begin{aligned} b_{m,n} &= c_{m-1,n} + b_{m-1,n-1}, \\ c_{m,n} &= d_{m-1,n} + c_{m-1,n-1}, \\ d_{m,n} &= b_{m-1,n} + d_{m-1,n-1}, \end{aligned} \tag{51}$$

where $(m, n) \in C$.

Let $m = n + 1$, then

$$b_{n+1,n} = b_{n,n-1} + c_{n,n}, \quad c_{n+1,n} = c_{n,n-1} + d_{n,n}, \quad d_{n+1,n} = d_{n,n-1} + b_{n,n},$$

for $n \in \mathbb{N}$, and consequently

$$\begin{aligned} b_{n+1,n} &= b_{1,0} + \sum_{j=1}^{n} c_{j,j}, \\ c_{n+1,n} &= c_{1,0} + \sum_{j=1}^{n} d_{j,j}, \\ d_{n+1,n} &= d_{1,0} + \sum_{j=1}^{n} b_{j,j}, \end{aligned} \tag{52}$$

for $n \in \mathbb{N}_0$.

Let $m = n + 2$, then

$$b_{n+2,n} = b_{n+1,n-1} + c_{n+1,n}, \quad c_{n+2,n} = c_{n+1,n-1} + d_{n+1,n}, \quad d_{n+2,n} = d_{n+1,n-1} + b_{n+1,n},$$

for $n \in \mathbb{N}$, and consequently

$$\begin{aligned} b_{n+2,n} &= b_{2,0} + \sum_{j=1}^{n} c_{j+1,j}, \\ c_{n+2,n} &= c_{2,0} + \sum_{j=1}^{n} d_{j+1,j}, \\ d_{n+2,n} &= d_{2,0} + \sum_{j=1}^{n} b_{j+1,j}, \end{aligned} \tag{53}$$

for $n \in \mathbb{N}_0$.

From (52), (53) and some calculation, it follows that

$$b_{n+2,n}=b_{2,0}+\sum_{j=1}^{n}\Big(c_{1,0}+\sum_{i=1}^{j}d_{i,i}\Big)=b_{2,0}+nc_{1,0}+\sum_{i=1}^{n}(n-i+1)d_{i,i},\tag{54}$$

$$c_{n+2,n}=c_{2,0}+\sum_{j=1}^{n}\Big(d_{1,0}+\sum_{i=1}^{j}b_{i,i}\Big)=c_{2,0}+nd_{1,0}+\sum_{i=1}^{n}(n-i+1)b_{i,i},\tag{55}$$

$$d_{n+2,n}=d_{2,0}+\sum_{j=1}^{n}\Big(b_{1,0}+\sum_{i=1}^{j}c_{i,i}\Big)=d_{2,0}+nb_{1,0}+\sum_{i=1}^{n}(n-i+1)c_{i,i},\tag{56}$$

for $n\in\mathbb{N}_0$.

Let $m=n+3$, then

$$b_{n+3,n}=b_{n+2,n-1}+c_{n+2,n},\quad c_{n+3,n}=c_{n+2,n-1}+d_{n+2,n},\quad d_{n+3,n}=d_{n+2,n-1}+b_{n+2,n},$$

for $n\in\mathbb{N}$, from which it follows that

$$\begin{aligned}b_{n+3,n}&=b_{3,0}+\sum_{j=1}^{n}c_{j+2,j},\\ c_{n+3,n}&=c_{3,0}+\sum_{j=1}^{n}d_{j+2,j},\\ d_{n+3,n}&=d_{3,0}+\sum_{j=1}^{n}b_{j+2,j},\end{aligned}\tag{57}$$

for $n\in\mathbb{N}_0$.

From (54)-(57) and some calculation, it follows that

$$\begin{aligned}b_{n+3,n}&=b_{3,0}+\sum_{j=1}^{n}\Big(c_{2,0}+jd_{1,0}+\sum_{i=1}^{j}(j-i+1)b_{i,i}\Big)\\ &=b_{3,0}+nc_{2,0}+\frac{n(n+1)}{2}d_{1,0}+\sum_{i=1}^{n}b_{i,i}\sum_{j=i}^{n}(j-i+1)\\ &=C_0^{n-1}b_{3,0}+C_1^{n}c_{2,0}+C_2^{n+1}d_{1,0}+\sum_{i=1}^{n}C_2^{n-i+2}b_{i,i},\end{aligned}\tag{58}$$

$$\begin{aligned}c_{n+3,n}&=c_{3,0}+\sum_{j=1}^{n}\Big(d_{2,0}+jb_{1,0}+\sum_{i=1}^{j}(j-i+1)c_{i,i}\Big)\\ &=C_0^{n-1}c_{3,0}+C_1^{n}d_{2,0}+C_2^{n+1}b_{1,0}+\sum_{i=1}^{n}C_2^{n-i+2}c_{i,i},\end{aligned}\tag{59}$$

and

$$\begin{aligned}d_{n+3,n}&=d_{3,0}+\sum_{j=1}^{n}\Big(b_{2,0}+jc_{1,0}+\sum_{i=1}^{j}(j-i+1)d_{i,i}\Big)\\ &=C_0^{n-1}d_{3,0}+C_1^{n}b_{2,0}+C_2^{n+1}c_{1,0}+\sum_{i=1}^{n}C_2^{n-i+2}d_{i,i},\end{aligned}\tag{60}$$

for $n \in \mathbb{N}_0$.

Let $m = n + 4$, then

$$b_{n+4,n} = b_{n+3,n-1} + c_{n+3,n}, \quad c_{n+4,n} = c_{n+3,n-1} + d_{n+3,n}, \quad d_{n+4,n} = d_{n+3,n-1} + b_{n+3,n},$$

for $n \in \mathbb{N}$, from which it follows that

$$\begin{aligned} b_{n+4,n} &= b_{4,0} + \sum_{j=1}^{n} c_{j+3,j}, \\ c_{n+4,n} &= c_{4,0} + \sum_{j=1}^{n} d_{j+3,j}, \\ d_{n+4,n} &= d_{4,0} + \sum_{j=1}^{n} b_{j+3,j}, \end{aligned} \tag{61}$$

for $n \in \mathbb{N}_0$.

From (58)-(61), some calculation and Lemma 1, it follows that

$$\begin{aligned} b_{n+4,n} &= b_{4,0} + \sum_{j=1}^{n} \left(C_0^{j-1} c_{3,0} + C_1^{j} d_{2,0} + C_2^{j+1} b_{1,0} + \sum_{i=1}^{j} C_2^{j-i+2} c_{i,i} \right) \\ &= b_{4,0} + C_1^{n} c_{3,0} + C_2^{n+1} d_{2,0} + C_3^{n+2} b_{1,0} + \sum_{i=1}^{n} C_3^{n-i+3} c_{i,i}, \end{aligned} \tag{62}$$

$$\begin{aligned} c_{n+4,n} &= c_{4,0} + \sum_{j=1}^{n} \left(C_0^{j-1} d_{3,0} + C_1^{j} b_{2,0} + C_2^{j+1} c_{1,0} + \sum_{i=1}^{j} C_2^{j-i+2} d_{i,i} \right) \\ &= c_{4,0} + C_1^{n} d_{3,0} + C_2^{n+1} b_{2,0} + C_3^{n+2} c_{1,0} + \sum_{i=1}^{n} C_3^{n-i+3} d_{i,i}, \end{aligned} \tag{63}$$

and

$$\begin{aligned} d_{n+4,n} &= d_{4,0} + \sum_{j=1}^{n} \left(C_0^{j-1} b_{3,0} + C_1^{j} c_{2,0} + C_2^{j+1} d_{1,0} + \sum_{i=1}^{j} C_2^{j-i+2} b_{i,i} \right) \\ &= d_{4,0} + C_1^{n} b_{3,0} + C_2^{n+1} c_{2,0} + C_3^{n+2} d_{1,0} + \sum_{i=1}^{n} C_3^{n-i+3} b_{i,i}, \end{aligned} \tag{64}$$

for $n \in \mathbb{N}_0$.

Let $m = n + 5$, then

$$b_{n+5,n} = b_{n+4,n-1} + c_{n+4,n}, \quad c_{n+5,n} = c_{n+4,n-1} + d_{n+4,n}, \quad d_{n+5,n} = d_{n+4,n-1} + b_{n+4,n},$$

for $n \in \mathbb{N}$, from which it follows that

$$\begin{aligned} b_{n+5,n} &= b_{5,0} + \sum_{j=1}^{n} c_{j+4,j}, \\ c_{n+5,n} &= c_{5,0} + \sum_{j=1}^{n} d_{j+4,j}, \\ d_{n+5,n} &= d_{5,0} + \sum_{j=1}^{n} b_{j+4,j}, \end{aligned} \tag{65}$$

for $n \in \mathbb{N}_0$.

From (62)-(65), some calculation and Lemma 1, it follows that

$$\begin{aligned} b_{n+5,n} &= b_{5,0} + \sum_{j=1}^{n} \left(c_{4,0} + C_1^j d_{3,0} + C_2^{j+1} b_{2,0} + C_3^{j+2} c_{1,0} + \sum_{i=1}^{j} C_3^{j-i+3} d_{i,i} \right) \\ &= b_{5,0} + C_1^n c_{4,0} + C_2^{n+1} d_{3,0} + C_3^{n+2} b_{2,0} + C_4^{n+3} c_{1,0} + \sum_{i=1}^{n} C_4^{n-i+4} d_{i,i}, \end{aligned} \tag{66}$$

$$\begin{aligned} c_{n+5,n} &= c_{5,0} + \sum_{j=1}^{n} \left(d_{4,0} + C_1^j b_{3,0} + C_2^{j+1} c_{2,0} + C_3^{j+2} d_{1,0} + \sum_{i=1}^{j} C_3^{j-i+3} b_{i,i} \right) \\ &= c_{5,0} + C_1^n d_{4,0} + C_2^{n+1} b_{3,0} + C_3^{n+2} c_{2,0} + C_4^{n+3} d_{1,0} + \sum_{i=1}^{n} C_4^{n-i+4} b_{i,i}, \end{aligned} \tag{67}$$

and

$$\begin{aligned} d_{n+5,n} &= d_{5,0} + \sum_{j=1}^{n} \left(b_{4,0} + C_1^j c_{3,0} + C_2^{j+1} d_{2,0} + C_3^{j+2} b_{1,0} + \sum_{i=1}^{j} C_3^{j-i+3} c_{i,i} \right) \\ &= d_{5,0} + C_1^n b_{4,0} + C_2^{n+1} c_{3,0} + C_3^{n+2} d_{2,0} + C_4^{n+3} b_{1,0} + \sum_{i=1}^{n} C_4^{n-i+4} c_{i,i}, \end{aligned} \tag{68}$$

for $n \in \mathbb{N}_0$.

Let $m = n + 6$, then

$$b_{n+6,n} = b_{n+5,n-1} + c_{n+5,n}, \quad c_{n+6,n} = c_{n+5,n-1} + d_{n+5,n}, \quad d_{n+6,n} = d_{n+5,n-1} + b_{n+5,n},$$

for $n \in \mathbb{N}_0$, from which it follows that

$$\begin{aligned} b_{n+6,n} &= b_{6,0} + \sum_{j=1}^{n} c_{j+5,j}, \\ c_{n+6,n} &= c_{6,0} + \sum_{j=1}^{n} d_{j+5,j}, \\ d_{n+6,n} &= d_{6,0} + \sum_{j=1}^{n} b_{j+5,j}, \end{aligned} \tag{69}$$

for $n \in \mathbb{N}_0$.

From (66)-(69), some calculation and Lemma 1, it follows that

$$\begin{aligned} b_{n+6,n} &= b_{6,0} + \sum_{j=1}^{n} \left(c_{5,0} + C_1^j d_{4,0} + C_2^{j+1} b_{3,0} + C_3^{j+2} c_{2,0} + C_4^{j+3} d_{1,0} + \sum_{i=1}^{j} C_4^{j-i+4} b_{i,i} \right) \\ &= b_{6,0} + C_1^n c_{5,0} + C_2^{n+1} d_{4,0} + C_3^{n+2} b_{3,0} + C_4^{n+3} c_{2,0} + C_5^{n+4} d_{1,0} + \sum_{i=1}^{n} C_5^{n-i+5} b_{i,i}, \end{aligned} \tag{70}$$

$$\begin{aligned} c_{n+6,n} &= c_{6,0} + \sum_{j=1}^{n} \left(d_{5,0} + C_1^j b_{4,0} + C_2^{j+1} c_{3,0} + C_3^{j+2} d_{2,0} + C_4^{j+3} b_{1,0} + \sum_{i=1}^{j} C_4^{j-i+4} c_{i,i} \right) \\ &= c_{6,0} + C_1^n d_{5,0} + C_2^{n+1} b_{4,0} + C_3^{n+2} c_{3,0} + C_4^{n+3} d_{2,0} + C_5^{n+4} b_{1,0} + \sum_{i=1}^{n} C_5^{n-i+5} c_{i,i}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} d_{n+6,n} &= d_{6,0} + \sum_{j=1}^{n}\left(b_{5,0} + C_1^j c_{4,0} + C_2^{j+1} d_{3,0} + C_3^{j+2} b_{2,0} + C_4^{j+3} c_{1,0} + \sum_{i=1}^{j} C_4^{j-i+4} d_{i,i}\right) \\ &= d_{6,0} + C_1^n b_{5,0} + C_2^{n+1} c_{4,0} + C_3^{n+2} d_{3,0} + C_4^{n+3} b_{2,0} + C_5^{n+4} c_{1,0} + \sum_{i=1}^{n} C_5^{n-i+5} d_{i,i}, \end{aligned} \tag{72}$$

for $n \in \mathbb{N}_0$.

Equalities (52), (54)-(56), (58)-(60), (62)-(64), (66)-(68), (70)-(72), suggest that the following formulas hold:

$$b_{n+3l-2,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} b_{3j-2,0} + \sum_{j=1}^{l-1} C_{3l-3j-2}^{n+3l-3j-3} c_{3j,0} + \sum_{j=1}^{l-1} C_{3l-3j-1}^{n+3l-3j-2} d_{3j-1,0} + \sum_{i=1}^{n} C_{3l-3}^{n-i+3l-3} c_{i,i}, \tag{73}$$

$$b_{n+3l-1,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} b_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} c_{3j-2,0} + \sum_{j=1}^{l-1} C_{3l-3j-1}^{n+3l-3j-2} d_{3j,0} + \sum_{i=1}^{n} C_{3l-2}^{n-i+3l-2} d_{i,i}, \tag{74}$$

$$b_{n+3l,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} b_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} c_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} d_{3j-2,0} + \sum_{i=1}^{n} C_{3l-1}^{n-i+3l-1} b_{i,i}, \tag{75}$$

$$c_{n+3l-2,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} c_{3j-2,0} + \sum_{j=1}^{l-1} C_{3l-3j-2}^{n+3l-3j-3} d_{3j,0} + \sum_{j=1}^{l-1} C_{3l-3j-1}^{n+3l-3j-2} b_{3j-1,0} + \sum_{i=1}^{n} C_{3l-3}^{n-i+3l-3} d_{i,i}, \tag{76}$$

$$c_{n+3l-1,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} c_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} d_{3j-2,0} + \sum_{j=1}^{l-1} C_{3l-3j-1}^{n+3l-3j-2} b_{3j,0} + \sum_{i=1}^{n} C_{3l-2}^{n-i+3l-2} b_{i,i}, \tag{77}$$

$$c_{n+3l,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} c_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} d_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} b_{3j-2,0} + \sum_{i=1}^{n} C_{3l-1}^{n-i+3l-1} c_{i,i}, \tag{78}$$

$$d_{n+3l-2,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} d_{3j-2,0} + \sum_{j=1}^{l-1} C_{3l-3j-2}^{n+3l-3j-3} b_{3j,0} + \sum_{j=1}^{l-1} C_{3l-3j-1}^{n+3l-3j-2} c_{3j-1,0} + \sum_{i=1}^{n} C_{3l-3}^{n-i+3l-3} b_{i,i}, \tag{79}$$

$$d_{n+3l-1,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} d_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} b_{3j-2,0} + \sum_{j=1}^{l-1} C_{3l-3j-1}^{n+3l-3j-2} c_{3j,0} + \sum_{i=1}^{n} C_{3l-2}^{n-i+3l-2} c_{i,i}, \tag{80}$$

$$d_{n+3l,n} = \sum_{j=1}^{l} C_{3l-3j}^{n+3l-3j-1} d_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} b_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} c_{3j-2,0} + \sum_{i=1}^{n} C_{3l-1}^{n-i+3l-1} d_{i,i}, \tag{81}$$

for every $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$.

Let $m = n + 3l + 1$, then

$$\begin{aligned} b_{n+3l+1,n} &= b_{n+3l,n-1} + c_{n+3l,n}, \\ c_{n+3l+1,n} &= c_{n+3l,n-1} + d_{n+3l,n}, \\ d_{n+3l+1,n} &= d_{n+3l,n-1} + b_{n+3l,n}, \end{aligned} \tag{82}$$

for $n \in \mathbb{N}$.

From (82), it follows that

$$\begin{aligned} b_{n+3l+1,n} &= b_{3l+1,0} + \sum_{j=1}^{n} c_{j+3l,j}, \\ c_{n+3l+1,n} &= c_{3l+1,0} + \sum_{j=1}^{n} d_{j+3l,j}, \\ d_{n+3l+1,n} &= d_{3l+1,0} + \sum_{j=1}^{n} b_{j+3l,j}, \end{aligned} \tag{83}$$

for $n \in \mathbb{N}_0$.

Using the hypotheses (75), (78) and (81) in (83), and employing Lemma 1, it follows that

$$\begin{aligned} b_{n+3l+1,n} &= b_{3l+1,0} + \sum_{s=1}^{n} \Big(\sum_{j=1}^{l} C_{3l-3j}^{s+3l-3j-1} c_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{s+3l-3j} d_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} b_{3j-2,0} + \sum_{i=1}^{s} C_{3l-1}^{s-i+3l-1} c_{i,i} \Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} b_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} c_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} d_{3j-1,0} + \sum_{i=1}^{n} C_{3l}^{n-i+3l} c_{i,i}, \end{aligned} \tag{84}$$

$$\begin{aligned} c_{n+3l+1,n} &= c_{3l+1,0} + \sum_{s=1}^{n} \Big(\sum_{j=1}^{l} C_{3l-3j}^{s+3l-3j-1} d_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{s+3l-3j} b_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} c_{3j-2,0} + \sum_{i=1}^{s} C_{3l-1}^{s-i+3l-1} d_{i,i} \Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} c_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} d_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} b_{3j-1,0} + \sum_{i=1}^{n} C_{3l}^{n-i+3l} d_{i,i}, \end{aligned} \tag{85}$$

and

$$\begin{aligned} d_{n+3l+1,n} &= d_{3l+1,0} + \sum_{s=1}^{n} \Big(\sum_{j=1}^{l} C_{3l-3j}^{s+3l-3j-1} b_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{s+3l-3j} c_{3j-1,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} d_{3j-2,0} + \sum_{i=1}^{s} C_{3l-1}^{s-i+3l-1} b_{i,i} \Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} d_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{n+3l-3j} b_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} c_{3j-1,0} + \sum_{i=1}^{n} C_{3l}^{n-i+3l} b_{i,i}, \end{aligned} \tag{86}$$

for every $n \in \mathbb{N}_0$.

Let $m = n + 3l + 2$, then

$$\begin{aligned} b_{n+3l+2,n} &= b_{n+3l+1,n-1} + c_{n+3l+1,n}, \\ c_{n+3l+2,n} &= c_{n+3l+1,n-1} + d_{n+3l+1,n}, \\ d_{n+3l+2,n} &= d_{n+3l+1,n-1} + b_{n+3l+1,n}, \end{aligned} \tag{87}$$

for $n \in \mathbb{N}$.

From (87), it follows that

$$\begin{aligned} b_{n+3l+2,n} &= b_{3l+2,0} + \sum_{j=1}^{n} c_{j+3l+1,j}, \\ c_{n+3l+2,n} &= c_{3l+2,0} + \sum_{j=1}^{n} d_{j+3l+1,j}, \\ d_{n+3l+2,n} &= d_{3l+2,0} + \sum_{j=1}^{n} b_{j+3l+1,j}, \end{aligned} \tag{88}$$

for $n \in \mathbb{N}_0$.

Using (84)-(86) in (88), and employing Lemma 1, it follows that

$$\begin{aligned} b_{n+3l+2,n} &= b_{3l+2,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} C_{3l-3j+3}^{s+3l-3j+2} c_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{s+3l-3j} d_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} b_{3j-1,0} + \sum_{i=1}^{s} C_{3l}^{s-i+3l} d_{i,i}\Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} b_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{n+3l-3j+3} c_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} d_{3j,0} + \sum_{i=1}^{n} C_{3l+1}^{n-i+3l+1} d_{i,i} \end{aligned} \tag{89}$$

$$\begin{aligned} c_{n+3l+2,n} &= c_{3l+2,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} C_{3l-3j+3}^{s+3l-3j+2} d_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{s+3l-3j} b_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} c_{3j-1,0} + \sum_{i=1}^{s} C_{3l}^{s-i+3l} b_{i,i}\Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} c_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{n+3l-3j+3} d_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} b_{3j,0} + \sum_{i=1}^{n} C_{3l+1}^{n-i+3l+1} b_{i,i} \end{aligned} \tag{90}$$

and

$$\begin{aligned} d_{n+3l+2,n} &= d_{3l+2,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} C_{3l-3j+3}^{s+3l-3j+2} b_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+1}^{s+3l-3j} c_{3j,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} d_{3j-1,0} + \sum_{i=1}^{s} C_{3l}^{s-i+3l} c_{i,i}\Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} d_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{n+3l-3j+3} b_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{n+3l-3j+1} c_{3j,0} + \sum_{i=1}^{n} C_{3l+1}^{n-i+3l+1} c_{i,i}, \end{aligned} \tag{91}$$

for every $n \in \mathbb{N}_0$.

Let $m = n + 3l + 3$, then

$$\begin{aligned} b_{n+3l+3,n} &= b_{n+3l+2,n-1} + c_{n+3l+2,n}, \\ c_{n+3l+3,n} &= c_{n+3l+2,n-1} + d_{n+3l+2,n}, \\ d_{n+3l+3,n} &= d_{n+3l+2,n-1} + b_{n+3l+2,n}, \end{aligned} \tag{92}$$

for $n \in \mathbb{N}$.

From (92), it follows that for $n \in \mathbb{N}_0$

$$\begin{aligned} b_{n+3l+3,n} &= b_{3l+3,0} + \sum_{j=1}^{n} c_{j+3l+2,j}, \\ c_{n+3l+3,n} &= c_{3l+3,0} + \sum_{j=1}^{n} d_{j+3l+2,j}, \\ d_{n+3l+3,n} &= d_{3l+3,0} + \sum_{j=1}^{n} b_{j+3l+2,j}. \end{aligned} \tag{93}$$

Using (89)-(91) in (93), and employing Lemma 1, it follows that

$$\begin{aligned} b_{n+3l+3,n} &= b_{3l+3,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} C_{3l-3j+3}^{s+3l-3j+2} c_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{s+3l-3j+3} d_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} b_{3j,0} + \sum_{i=1}^{s} C_{3l+1}^{s-i+3l+1} b_{i,i}\Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} b_{3j,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{n+3l-3j+3} c_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+5}^{n+3l-3j+4} d_{3j-2,0} + \sum_{i=1}^{n} C_{3l+2}^{n-i+3l+2} b_{i,i}, \end{aligned} \tag{94}$$

$$\begin{aligned} c_{n+3l+3,n} &= c_{3l+3,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} C_{3l-3j+3}^{s+3l-3j+2} d_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{s+3l-3j+3} b_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} c_{3j,0} + \sum_{i=1}^{s} C_{3l+1}^{s-i+3l+1} c_{i,i}\Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} c_{3j,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{n+3l-3j+3} d_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+5}^{n+3l-3j+4} b_{3j-2,0} + \sum_{i=1}^{n} C_{3l+2}^{n-i+3l+2} c_{i,i}, \end{aligned} \tag{95}$$

$$\begin{aligned} d_{n+3l+3,n} &= d_{3l+3,0} + \sum_{s=1}^{n}\Big(\sum_{j=1}^{l+1} C_{3l-3j+3}^{s+3l-3j+2} b_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{s+3l-3j+3} c_{3j-2,0} + \sum_{j=1}^{l} C_{3l-3j+2}^{s+3l-3j+1} d_{3j,0} + \sum_{i=1}^{s} C_{3l+1}^{s-i+3l+1} d_{i,i}\Big) \\ &= \sum_{j=1}^{l+1} C_{3l-3j+3}^{n+3l-3j+2} d_{3j,0} + \sum_{j=1}^{l+1} C_{3l-3j+4}^{n+3l-3j+3} b_{3j-1,0} + \sum_{j=1}^{l+1} C_{3l-3j+5}^{n+3l-3j+4} c_{3j-2,0} + \sum_{i=1}^{n} C_{3l+2}^{n-i+3l+2} d_{i,i}, \end{aligned} \tag{96}$$

for every $n \in \mathbb{N}_0$.

From this and by induction we see that formulas (73)-(81) hold for every $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$.

Now we formulate and prove the main result regarding the three-dimensional system (51).

Theorem 2. *Assume that $(\beta_k)_{k\in\mathbb{N}}$, $(\widehat{\beta_k})_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$, $(\widehat{\gamma_k})_{k\in\mathbb{N}}$, $(\delta_k)_{k\in\mathbb{N}}$ and $(\widehat{\delta_k})_{k\in\mathbb{N}}$ are given sequences of complex numbers. Then the solution to system* (51) *with the following boundary value conditions*

$$b_{k,0} = \beta_k, \quad b_{k,k} = \widehat{\beta_k}, \quad c_{k,0} = \gamma_k, \quad c_{k,k} = \widehat{\gamma_k}, \quad d_{k,0} = \delta_k, \quad d_{k,k} = \widehat{\delta_k}, \tag{97}$$

$k \in \mathbb{N}$, is given by

$$b_{m,n} = \sum_{j=1}^{\frac{m-n+1}{3}} C_{m-n+1-3j}^{m-3j} \beta_{3j-1} + \sum_{j=1}^{\frac{m-n+1}{3}} C_{m-n-3j+2}^{m-3j+1} \gamma_{3j-2} + \sum_{j=1}^{\frac{m-n-2}{3}} C_{m-n-3j}^{m-3j-1} \delta_{3j} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \widehat{\delta}_i, \tag{98}$$

$$c_{m,n} = \sum_{j=1}^{\frac{m-n+1}{3}} C_{m-n-3j+1}^{m-3j} \gamma_{3j-1} + \sum_{j=1}^{\frac{m-n+1}{3}} C_{m-n-3j+2}^{m-3j+1} \delta_{3j-2} + \sum_{j=1}^{\frac{m-n-2}{3}} C_{m-n-3j}^{m-3j-1} \beta_{3j} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \widehat{\beta}_i, \tag{99}$$

$$d_{m,n} = \sum_{j=1}^{\frac{m-n+1}{3}} C_{m-n-3j+1}^{m-3j} \delta_{3j-1} + \sum_{j=1}^{\frac{m-n+1}{3}} C_{m-n-3j+2}^{m-3j+1} \beta_{3j-2} + \sum_{j=1}^{\frac{m-n-2}{3}} C_{m-n-3j}^{m-3j-1} \gamma_{3j} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \widehat{\gamma}_i, \tag{100}$$

when $m - n \equiv 2 \pmod 3$, by

$$b_{m,n} = \sum_{j=1}^{\frac{m-n+2}{3}} C_{m-n-3j+2}^{m-3j+1} \beta_{3j-2} + \sum_{j=1}^{\frac{m-n-1}{3}} C_{m-n-3j}^{m-3j-1} \gamma_{3j} + \sum_{j=1}^{\frac{m-n-1}{3}} C_{m-n-3j+1}^{m-3j} \delta_{3j-1} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \widehat{\gamma}_i, \tag{101}$$

$$c_{m,n} = \sum_{j=1}^{\frac{m-n+2}{3}} C_{m-n-3j+2}^{m-3j+1} \gamma_{3j-2} + \sum_{j=1}^{\frac{m-n-1}{3}} C_{m-n-3j}^{m-3j-1} \delta_{3j} + \sum_{j=1}^{\frac{m-n-1}{3}} C_{m-n-3j+1}^{m-3j} \beta_{3j-1} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \widehat{\delta}_i, \tag{102}$$

$$d_{m,n} = \sum_{j=1}^{\frac{m-n+2}{3}} C_{m-n-3j+2}^{m-3j+1} \delta_{3j-2} + \sum_{j=1}^{\frac{m-n-1}{3}} C_{m-n-3j}^{m-3j-1} \beta_{3j} + \sum_{j=1}^{\frac{m-n-1}{3}} C_{m-n-3j+1}^{m-3j} \gamma_{3j-1} + \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} \widehat{\beta}_i, \tag{103}$$

when $m-n\equiv 1\ (\bmod\ 3)$, *by*

$$b_{m,n}=\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j}^{m-3j-1}\beta_{3j}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+1}^{m-3j}\gamma_{3j-1}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+2}^{m-3j+1}\delta_{3j-2}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}\widehat{\beta_i},\tag{104}$$

$$c_{m,n}=\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j}^{m-3j-1}\gamma_{3j}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+1}^{m-3j}\delta_{3j-1}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+2}^{m-3j+1}\beta_{3j-2}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}\widehat{\gamma_i},\tag{105}$$

$$d_{m,n}=\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j}^{m-3j-1}\delta_{3j}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+1}^{m-3j}\beta_{3j-1}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+2}^{m-3j+1}\gamma_{3j-2}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}\widehat{\delta_i},\tag{106}$$

when $m-n\equiv 0\ (\bmod\ 3)$, *for every* $m,n\in\mathbb{N}_0$ *such that* $m\ge n$.

Proof. If we put $m=n+3l-1$, when $m-n\equiv 2(\bmod\ 3)$ in (74), (77) and (80), put $m=n+3l-2$, when $m-n\equiv 1(\bmod\ 3)$ in (73), (76) and (79), and put $3l=m-n$, when $m-n\equiv 0(\bmod\ 3)$ in (75), (78) and (81), we get

$$b_{m,n}=\sum_{j=1}^{\frac{m-n+1}{3}}C_{m-n+1-3j}^{m-3j}b_{3j-1,0}+\sum_{j=1}^{\frac{m-n+1}{3}}C_{m-n-3j+2}^{m+1-3j}c_{3j-2,0}+\sum_{j=1}^{\frac{m-n-2}{3}}C_{m-n-3j}^{m-3j-1}d_{3j,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}d_{i,i},\tag{107}$$

$$c_{m,n}=\sum_{j=1}^{\frac{m-n+1}{3}}C_{m-n-3j+1}^{m-3j}c_{3j-1,0}+\sum_{j=1}^{\frac{m-n+1}{3}}C_{m-n-3j+2}^{m-3j+1}d_{3j-2,0}+\sum_{j=1}^{\frac{m-n-2}{3}}C_{m-n-3j}^{m-3j-1}b_{3j,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}b_{i,i},\tag{108}$$

$$d_{m,n}=\sum_{j=1}^{\frac{m-n+1}{3}}C_{m-n-3j+1}^{m-3j}d_{3j-1,0}+\sum_{j=1}^{\frac{m-n+1}{3}}C_{m-n-3j+2}^{m-3j+1}b_{3j-2,0}+\sum_{j=1}^{\frac{m-n-2}{3}}C_{m-n-3j}^{m-3j-1}c_{3j,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}c_{i,i},\tag{109}$$

when $m-n\equiv 2\ (\bmod\ 3)$, by

$$b_{m,n}=\sum_{j=1}^{\frac{m-n+2}{3}}C_{m-n-3j+2}^{m-3j+1}b_{3j-2,0}+\sum_{j=1}^{\frac{m-n-1}{3}}C_{m-n-3j}^{m-3j-1}c_{3j,0}+\sum_{j=1}^{\frac{m-n-1}{3}}C_{m-n-3j+1}^{m-3j}d_{3j-1,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}c_{i,i},\tag{110}$$

$$c_{m,n}=\sum_{j=1}^{\frac{m-n+2}{3}}C_{m-n-3j+2}^{m-3j+1}c_{3j-2,0}+\sum_{j=1}^{\frac{m-n-1}{3}}C_{m-n-3j}^{m-3j-1}d_{3j,0}+\sum_{j=1}^{\frac{m-n-1}{3}}C_{m-n-3j+1}^{m-3j}b_{3j-1,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}d_{i,i},\tag{111}$$

$$d_{m,n}=\sum_{j=1}^{\frac{m-n+2}{3}}C_{m-n-3j+2}^{m-3j+1}d_{3j-2,0}+\sum_{j=1}^{\frac{m-n-1}{3}}C_{m-n-3j}^{m-3j-1}b_{3j,0}+\sum_{j=1}^{\frac{m-n-1}{3}}C_{m-n-3j+1}^{m-3j}c_{3j-1,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}b_{i,i},\tag{112}$$

when $m-n\equiv 1\ (\bmod\ 3)$, by

$$b_{m,n}=\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j}^{m-3j-1}b_{3j,0}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+1}^{m-3j}c_{3j-1,0}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+2}^{m-3j+1}d_{3j-2,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}b_{i,i},\tag{113}$$

$$c_{m,n}=\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j}^{m-3j-1}c_{3j,0}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+1}^{m-3j}d_{3j-1,0}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+2}^{m-3j+1}b_{3j-2,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}c_{i,i},\tag{114}$$

$$d_{m,n}=\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j}^{m-3j-1}d_{3j,0}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+1}^{m-3j}b_{3j-1,0}+\sum_{j=1}^{\frac{m-n}{3}}C_{m-n-3j+2}^{m-3j+1}c_{3j-2,0}+\sum_{i=1}^{n}C_{m-n-1}^{m-i-1}d_{i,i},\tag{115}$$

Employing (97) in (107)-(115), are easily obtained formulas (98)-(106). □

Remark 4. Theorems 1 and 2 can be extended for the case of their natural k-dimensional extension. We leave the formulation and the proof of the result to the interested reader as an exercise.

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