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On Multi–Singular Integral Equations Involving n Weakly Singular Kernels

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Abstract. We deal with some sources of Banach spaces which are closely related to an important issue in applied mathematics i.e. the problem of existence and uniqueness of the solution for the very applicable weakly singular integral equations. In the classical mode, the uniform space $(C[a,b], \|.\|_\infty)$ is usually applied to the related discussion. Here, we apply some new types of Banach spaces, in order to extend the area of problems we could discuss. We consider a very general type of singular integral equations involving n weakly singular kernels, for an arbitrary natural number n, without any restrictive assumption of differentiability or even continuity on engaged functions. We show that in appropriate conditions the following multi–singular integral equation of weakly singular type has got exactly a solution in a defined Banach space

$$x(t) = \sum_{i=1}^{p} \frac{\gamma_i}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t}} \frac{f_i(s, x(s))}{(t_n - t_{n-1})^{1-\alpha_n^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{t} + \phi(t).$$

In particular we consider the famous fractional Langevin equation and by the method we could extend the region of variations of parameter $\alpha + \beta$ from interval $[0, 1)$ in the earlier works to interval $[0, 2)$.

1. Introduction

In this paper we focus, exclusively, on a kind of multi–singular integral equations of weakly singular type and the related spaces which are substantial for these equations. A large number of mathematical concepts are required to use such equations and consequently such spaces. The methods of integral equation may be used more effectively than techniques of differential equation in many types of problems in applied mathematics, engineering, and mathematical physics [1]. They can be applied to a wide variety of scientific fields – potential theory, mechanics, water waves, scattering of acoustic, electromagnetic and earthquake waves, statistics, and population dynamics; [2, 3]. An integral equation is said to be weakly singular if the kernel is singular within the range of integration provided that the singularity is allowed to be transformed into regularity by a suitable transformation [4]. The multi–singular integral equations that we have studied in this article are an extension of Abel's equation and some related integral equations. Abel's equation can be regarded as a basis for the definition of the operator of fractional differentiation of order $0 < \alpha < 1$, usually

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denoted by I^{α}; [5]. So this manuscript can be effectively used in the field of fractional differential equations [6]. There are several definitions for a fractional derivative of order $\alpha > 0$. Two most commonly used are the Riemann–Liouville and Caputo definitions. The difference between them is concerned with the order of evaluation. The definitions and properties of the fractional calculus theory are given in [7]. The fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non–integer orders. The fractional differential equations play an important role in various fields of science and engineering. Many mathematical models in viscoelasticity, chemistry, finance and other sciences can be described by using fractional order derivatives [8–10]. A collection of these models based on the fractional calculus are given in the book of Mainardi [11], and the papers of Čermák and Kisela [12] and Bagley and Travik [13]. Many valuable papers on the theoretical analysis have been carried on. Specially the existence and uniqueness of solutions for nonlinear singular integral equation and fractional differential equations are extensively studied [14, 15]. Among theoretical topics, the existence and uniqueness have a high altitude because the reliability of simulation methods depend extremely to these issues [16–25]. Although most of problems in the field of integral equations are considered to be scalar, many applications involve systems of weakly singular Volterra integral equations (VIEs) and weakly singular Fredholm integral equations (FIEs) with high dimensions that this paper covers them very well [26, 27]. In final section, we offer an application of these topics in the initial value problem of Langevin equation involving two fractional orders.

In [28, 29] we have sconsidered the following integral equations

$$x(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s, x(s))ds; \tag{1}$$

$$x(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, x(s))ds - \gamma \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \phi(t) \tag{2}$$

and using the tools provided in certain Banach spaces, we discussed the problem of existence and uniqueness of the solutions for them. By this way we, in fact, bring the discussion to spaces which are very broader than the uniform space $(C[a, b], \|.\|_{\infty})$. Equations (1) and (2) each, is obtained from a particular fractional differential equation so the method introduced, may be helpful in the similar issues for a such type of equations.

In this paper we first introduce some classes of Banach spaces, we have named $L^{p,\hat{\alpha}}([0, 1])$, $L^{p,n}([0, 1])$ and $L_{p,\hat{\alpha}}([0, 1])$. These classes are indeed the subclasses of real measurable functions defined on interval $[0, 1]$. We express some of their properties which are of vital importance to the topic. Then we show that the following multi–singular integral equation of weakly singular type in appropriate conditions has got exactly a fixed point in the Banach space of a kind introduced above

$$x(t_n) = \sum_{i=1}^{l} \frac{\gamma_i}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t}} \frac{f_i(s, x(s))}{(t_n - t_{n-1})^{1-\alpha_n^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{t} + \phi(t) \tag{3}$$

where t_n is in $[0, 1]$ and $\int_0^{\mathbf{t}}$ stands for a multiple integral. Note that (1) and (2) are special cases of (3). We emphasize that (3) may be stated by suitable composition of the operators I^{α}s.

Finally we consider the famous fractional Langevin equation, considered in [23, 25, 28], and by the method we could extend the the region of variations of parameter $\alpha + \beta$ from interval $[0, 1)$ in those papers to $[0, 2)$ as well. It can be said the most important part of the article refers to this point. In fact in [28] the existence and uniqueness problem of the fractional Langevin equation is discussed. Using tools provided in certain Banach spaces introduced there, it enhances the ability to select the functions involved in the equation from $(C[a, b], \|.\|_{\infty})$ in [23], to Lebesgue measurable functions. In that method, the parameter $\alpha + \beta$ appeared in the equation has only been accepted to lie in $[0, 1)$. Here, in this paper, we generalize the method and introduce the similar spaces with multiple integral instead of simple integral, to discuss the multi–singular integral equations involving n weakly singular kernels. Such equations have fractional Langevin equations as spacial cases of their own. Due to this, we could extend the region of variations of parameter $\alpha + \beta$ from interval $[0, 1)$ in [28] to interval $[0, 2)$ in this paper. This is done getting together with extending the area of the topic in [23], as what has been done in [28]. This is useful for the practical issues, undoubtedly. We present an example to show that we speak in real situations. Note that norm and completeness of the associated space, both, are important to our needs in this part, therefore they have been placed under scrutiny as much as required.

The relationship between multi-singular integral equation and the Riemann-Liouville integral operator I^{α}, $\alpha \in \mathbb{R}^{+}$, is introduced in what follows.

2. Preliminaries

In this section, we provide some notations, definitions and basic facts in this issue which we need in the final section.

Definition 2.1. *([21, 25]). The Riemann-Liouville integral of order α for the function x is defined as*

$$I^{\alpha}x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, \quad 0 \le t \le 1, \tag{4}$$

where $m-1 < \alpha < m$, $m \in \mathbb{N}$ and $\Gamma(.)$ is the Gamma function, provided that the right-hand-side integral exists and is finite.

Definition 2.2. *([21, 25]). The α-th Caputo derivative of the function x is defined as*

$$D^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}x^{(m)}(s)ds, \quad 0 \le t \le 1, \tag{5}$$

where $m-1 < \alpha < m$ and $m \in \mathbb{N}$, provided that the right-hand-side integral exists and is finite.

We note that the Caputo derivative reduces to the conventional m-th derivative of a given function as $\alpha \to m$. Also, without loss of generality, the definitions (4) and (5) could be extended to any interval $[a, b]$.

The relationship between (4) and (5) and other properties of these notions are stated in the next theorem. For their proofs, we refer to [21, 22].

Theorem 2.3. *Let $\alpha, \beta \in \mathbb{R}^{+}$, $r, l \in \mathbb{R}$, $m, n \in \mathbb{N}^{+}$, $\alpha \in (n-1, n)$, $u \in C[a, b]$, and $f \in C^{m+n}[a, b]$. Then,*
i) $I^0 := I$, $I^{\alpha}I^{\beta}u(t) = I^{\beta}I^{\alpha}u(t) = I^{\alpha+\beta}u(t)$ where I is the identity operator;
ii) $D^0 := I$, $D^{\alpha}C = 0$, where C is a constant;
iii) $D^{\alpha}I^{\alpha}u(t) = u(t)$ and $I^{\alpha}D^{\alpha}u(t) = u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}t^k$;
iv) $I^{\alpha}(t-a)^r = \frac{\Gamma(r+1)}{r+1+\alpha}(t-a)^{r+\alpha}$, $D^{\alpha}(t-a)^l = \frac{\Gamma(l+1)}{l+1+\alpha}(t-a)^{l-\alpha}$;
v) $D^{\alpha}D^{m}f(t) = D^{m}D^{\alpha}f(t) = D^{\alpha+m}f(t)$, if $f^{s}(a) = 0$, $s = n, n+1, \cdots, m+n-1$,
vi) if $\alpha_i \in (0, 1]$, $i = 1, \ldots, n$, with $\alpha = \Sigma_{i=1}^{n}\alpha_i$, are such that, for each $k = 1, \ldots, m-1$, there exist $i_k < n$ with $\Sigma_{j=1}^{i_k}\alpha_j = k$, then the following composition formula for the Caputo fractional derivative holds:

$$D^{\alpha}u(t) = D^{\alpha_n}\ldots D^{\alpha_2}D^{\alpha_1}u(t).$$

vii) if $\alpha_i \in (0, \infty)$, $i = 1, 2, \cdots, n$, with $\alpha = \Sigma_{i=1}^{n}\alpha_i$, then the following composition formula for the Riemann-Liouville integral holds:

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha_n)\cdots\Gamma(\alpha_1)}\int_0^{t_n}\cdots\int_0^{t_1}\frac{u(s)}{(t_n-t_{n-1})^{1-\alpha_n}\cdots(t_1-s)^{1-\alpha_1}}ds\cdots dt_{n-1}.$$

viii) the general solution of the fractional differential equation $D^{\alpha}u(t) = 0$ is given by $u(t) = \sum_{i=0}^{n-1}c_i t^i$, where $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$.

3. Some classes of normed spaces

In this part we work mainly on real functions defined on interval $[0,1]$, considered as a measure space with the σ-algebra $\mathscr{L}$ consisting of all Lebesgue's measurable subsets, and Lebesgue's measure m defined on $\mathscr{L}$. Therefore when we say measurable function we refer in fact to a function that is Lebesgue's measurable on this interval. We introduce a new source of Banach spaces which are taken advantage in speaking of solution of weakly singular integral equations and fractional differential equations.

3.1. $L^{p,\hat{\alpha}}$ spaces

First we introduce some normed spaces which are generalizations of the classical L^p spaces. Most of proofs are done by appropriate adoption in associated proofs. Here we only want to express the properties we need in the final part of the paper. The others may come in another paper.

Definition 3.1. *Let f be a measurable function and $1 \leq p < \infty$ and let $\hat{\alpha} = \{\alpha_k\}_{k=1}^n$ be finite sequence in $[0,1)$. We define*

$$\|f\|_{p,\hat{\alpha}} =$$

$$\left(\int_0^1 \int_0^{t_n} \frac{1}{(t_n - t_{n-1})^{\alpha_n}} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \frac{1}{(t_2 - t_1)^{\alpha_2}} \int_0^{t_1} \frac{|f(s)|^p}{(t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_n\right)^{\frac{1}{p}} =$$

$$\left(\int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \int_0^{t_1} \frac{|f(s)|^p}{(t_n - t_{n-1})^{\alpha_n} \cdots (t_2 - t_1)^{\alpha_2} (t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_n\right)^{\frac{1}{p}}.$$

For a function f we allow that $\|f\|_{p,\hat{\alpha}} = \infty$ but those for which $\|.\|_{p,\hat{\alpha}}$ is finite form a vector space we denote by $L^{p,\hat{\alpha}}$.

Above all, we shall to verify that $\|.\|_{p,\hat{\alpha}}$ defines really a norm on $L^{p,\hat{\alpha}}$, when we identify functions that are equal almost everywhere. Among the conditions required for which, one may need to check the triangle inequality. The following theorems are related to the well-known Hölder's and Minkowski's inequalities that establish the triangle inequality as in the classical L^p spaces. Their proofs are comparatively the same and we have to drop them. Hereafter, the number $1 \leq p < \infty$ and the finite sequence $\hat{\alpha} = \{\alpha_k\}_{k=1}^n$ in $[0,1)$ are considered to be fixed. By $\hat{\alpha} = \hat{0}$, we mean $\alpha_i = 0$ for $i \in \{1, 2, \cdots, n\}$. Also $\hat{\alpha}^{(i)}$ is the finite sequence $\{\alpha_1, \cdots, \alpha_i\}$ for some $i \in \{1, \cdots, n\}$.

Theorem 3.2. *(Hölder's inequality) Let f and g are measurable functions. Then*

$$\|fg\|_{1,\hat{\alpha}} \leq \|f\|_{p,\hat{\alpha}} \|g\|_{q,\hat{\alpha}}$$

where q stands for the conjugate exponent to p.

Theorem 3.3. *(Minkowski's inequality) Let $f, g \in L^{p,\hat{\alpha}}$. Then*

$$\|f + g\|_{p,\hat{\alpha}} \leq \|f\|_{p,\hat{\alpha}} + \|g\|_{p,\hat{\alpha}}.$$

Proposition 3.4. *$L^{2,\hat{\alpha}}$ is an inner product space.*

The following theorem establishes the completeness of $L^{p,\hat{\alpha}}$. In its proof we use this fact that the necessary and sufficient condition for the completeness of a given normed space is that every absolutely convergent series in that space converges. First a lemma that is our requisiteness.

Lemma 3.5. *Let f be a positive measurable function such that $\int_0^1 \frac{f(t)}{(1-t)^\alpha} dt < \infty$ for some $\alpha \in [0,1)$. Then f is finite a.e..*

Proof. Let $E = \{t;\ f(t) = \infty\}$ and $m(E) \neq 0$. Thus

$$1 = \int_0^1 \frac{dt}{(1-t)^{\alpha}} \geq B = \int_E \frac{1}{(1-t)^{\alpha}} dt \geq \int_E dt = m(E) \neq 0.$$

Hence

$$\infty = \lim n \int_E \frac{dt}{(1-t)^{\alpha}} \leq \int_E \frac{f(t)}{(1-t)^{\alpha}} dt \leq \int_0^1 \frac{f(t)}{(1-t)^{\alpha}} dt < \infty$$

which is impossible. □

Theorem 3.6. *$L^{p,\hat{\alpha}}$ is a Banach space.*

Proof. Suppose $\{f_k\}_{k\in\mathbb{N}}$ is a sequence in $L^{p,\hat{\alpha}}$ such that $\sum_{k=1}^{\infty} \|f_k\|_{p,\hat{\alpha}} = B < \infty$. Let $G_n = \sum_1^n |f_k|$ and $G = \sum_1^{\infty} |f_k|$. Thus

$$\|G_m\|_{p,\hat{\alpha}} \leq \sum_1^m \|f_k\|_{p,\hat{\alpha}} \leq B$$

for $m \in \mathbb{N}$. Utilizing the monotone convergence theorem, several times, we reach to

$$\begin{aligned}
&\int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \int_0^{t_1} \frac{G^p}{(t_n - t_{n-1})^{\alpha_n} \cdots (t_2 - t_1)^{\alpha_2} (t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_n \quad = \\
\lim_{m\to\infty} &\int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \int_0^{t_1} \frac{G_m^p}{(t_n - t_{n-1})^{\alpha_n} \cdots (t_2 - t_1)^{\alpha_2} (t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_n \\
&\leq B^p
\end{aligned}$$

which implies that the function h_1 defined on $[0,1]$ as

$$h_1(t_n) := \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \int_0^{t_1} \frac{G^p}{(t_n - t_{n-1})^{\alpha_n} \cdots (t_2 - t_1)^{\alpha_2} (t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_{n-1}$$

is in L^1. It follows that $h_1(t) < \infty$ a.e. so $h_1(1) < \infty$. Hence the function h_2 defined on $[0,1]$ as

$$h_2(t_{n-1}) :=$$

$$\int_0^{t_{n-1}} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \int_0^{t_1} \frac{G^p}{(t_{n-1} - t_{n-2})^{\alpha_n} \cdots (t_2 - t_1)^{\alpha_2} (t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_{n-2}$$

is in L^1 by Lemma 3.5. Going on in this process we find that $G^p \in L^1$. Therefore $G < \infty$ a.e. which implies that series $\sum_1^{\infty} f_k$ converges a.e. to a measurable function, say F. Obviously $|F| \leq G$ so $F \in L^{p,\hat{\alpha}}$. Moreover,

$$|F - \sum_1^n f_k|^p \leq (2G)^p \in L^1.$$

Therefore applying the dominated convergence theorem several times brings us to

$$\|F - \sum_1^n f_k\|_{p,\hat{\alpha}}^p =$$

$$\int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \int_0^{t_1} \frac{|F - \sum_0^n f_k|}{(t_n - t_{n-1})^{\alpha_n} \cdots (t_2 - t_1)^{\alpha_2} (t_1 - s)^{\alpha_1}} ds dt_1 \cdots dt_n \to 0$$

□

The following proposition concerned with the relationship between simple functions, i.e. the functions with finite image, and the matter of our discussion. What we see is very similar to that of classical mode.

Proposition 3.7. *The set of simple functions is dense in* $L^{p,\hat{\alpha}}$.

Proof. Obviously simple functions each one is in $L^{p,\hat{\alpha}}$. Let f be in $L^{p,\hat{\alpha}}$. There exists a sequence of simple functions $\{\phi_n\}_n$ such that $\phi_n \to f$ a.e. as n goes ∞ and $|\phi_n| \leq |f|$. This implies that $|\phi_n - f|^p \leq 2^p|f|^p$, so by invoking the dominated convergence theorem several times we have that $\|\phi_n - f\|_p \to 0$. □

Definition 3.8. *Let* $\hat{\alpha} = \hat{0}$, *the constant finite sequence* 0 *with* n *terms. Then* $L^{p,\hat{\alpha}}$ *is denoted by* $L^{p,n}$.

3.2. $L_{p,\hat{\alpha}}$ *spaces*

At this point we want to speak about one other source of normed spaces named the spaces of $L_{p,\hat{\alpha}}$.

Definition 3.9. *Let* f *be a measurable function and* $1 \leq p < \infty$ *and let* $\hat{\alpha} = \{\alpha_k\}_{k=1}^n$ *be finite sequence in* $[0,1)$. *We define*

$$_{p,\hat{\alpha}}\|f\| =$$

$$\left(\sup_{t_n\in[0,1]}\int_0^{t_n}\frac{1}{(t_n-t_{n-1})^{\alpha_n}}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{1}{(t_2-t_1)^{\alpha_2}}\int_0^{t_1}\frac{|f(s)|^p}{(t_1-s)^{\alpha_1}}ds dt_1\cdots dt_{n-1}\right)^{\frac{1}{p}} =$$

$$\left(\sup_{t_n\in[0,1]}\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\int_0^{t_1}\frac{|f(s)|^p}{(t_n-t_{n-1})^{\alpha_n}\cdots(t_2-t_1)^{\alpha_2}(t_1-s)^{\alpha_1}}ds dt_1\cdots dt_{n-1}\right)^{\frac{1}{p}}.$$

For function f *we may have* $_{p,\hat{\alpha}}\|f\| = \infty$ *and those for which* $_{p,\hat{\alpha}}\|.\|$ *is finite, form a vector space we denote by* $L_{p,\hat{\alpha}}$.

It is easy to check that $L_{p,\hat{\alpha}}$ is a normed space. In fact Minkowski's inequality holds for $_{p,\hat{\alpha}}\|.\|$ as the classical debate.

The following property is concluded immediately from the definitions.

Proposition 3.10. *Let* f *be a measurable function. Then*

$$_{p,\hat{\alpha}}\|f\| \geq \|f\|_{p,\hat{\alpha}^{(n-1)}} \geq \|f\|_{p,\hat{0}^{(n-1)}}.$$

In particular

$$L_{p,\hat{\alpha}} \subset L^{p,\hat{\alpha}^{(n-1)}} \subset L^{p,n-1}.$$

Note that if $f(s) = \frac{1}{\sqrt{1-s}}$ and $\alpha = \frac{1}{2}$, then $f \in L^1([0,1])$ but $f \notin L_{1,\alpha}([0,1])$, so the inclusions in the recent proposition is sometimes strict.

Theorem 3.11. $L_{p,\hat{\alpha}}$ *is Banach.*

Proof. Let $\{f_k\}$ be a Cauchy sequence in $L_{p,\hat{\alpha}}$. So it is Cauchy in $L^{p,\hat{\alpha}}$ by the previous Proposition, which implies that it converges to the element f of $L^{p,\hat{\alpha}^{(n-1)}}$, because of the fact that $L^{p,\hat{\alpha}^{(n-1)}}$ is Banach. Let $\varepsilon > 0$ and N be a natural number such that $_{p,\hat{\alpha}}\|f_i - f_j\| < \varepsilon$ for $i, j \geq N$. Now let $j \geq N$. A routine computation and using Fatou's lemma several times ensure

$$_{p,\hat{\alpha}}\|f - f_j\| \leq \lim_{i\to\infty} {}_{p,\hat{\alpha}}\|f_i - f_j\| \leq \varepsilon.$$

Therefore f is in $L_{p,\hat{\alpha}}$, and $f_j \to f$ as j goes ∞, which means $L_{p,\hat{\alpha}}$ is Banach. □

4. The result of existence and uniqueness

In this part we want to speak about the high ability of the spaces mentioned above in a significant discussion arises in applied mathematics. This is related to the problem of existence and uniqueness of the solution for certain weakly singular integral equations. We work in fact in continuation of the works accomplished in [28] and [29] and the method used for certain simple fractional deferential equations in those articles is extended here to singular integral equations involving n weakly singular kernels. First some notations.

Let $\hat{\alpha} = \{\alpha_1, \cdots \alpha_{n_1}\}$ and $\hat{\beta} = \{\beta_1, \cdots \beta_{n_2}\}$ be two sequences in $[0,1)$. By $\hat{\alpha} \leq \hat{\beta}$ we mean $\alpha_i \leq \beta_i$ for $i \in \{1, \cdots, \min\{n_1, n_2\}\}$. In this case it is easy to see that if $n_1 \leq n_2$, then ${}_{p,\hat{\alpha}}\|f\| \leq {}_{p,\hat{\beta}}\|f\|$ for a measurable function f and number $p \geq 1$. We use $\int_0^{\mathbf{t_i}}$ instead of $\int_0^t \int_0^{t_{n_i-1}} \cdots \int_0^{t_1}$ and $d\mathbf{s_i}$ instead of $dsdt_1 \cdots dt_{n_i-1}$. $\Gamma(\hat{\alpha}_i)$ is in fact $\Gamma(\alpha_1) \cdots \Gamma(\alpha_{n_i})$.

Theorem 4.1. *Let* $\widehat{\alpha_1} = \{\alpha_1^1, \cdots, \alpha_{n_1}^1\}$, $\widehat{\alpha_2} = \{\alpha_1^2, \cdots, \alpha_{n_2}^2\}, \cdots, \widehat{\alpha_l} = \{\alpha_1^l, \cdots, \alpha_{n_l}^l\}$ *and*

$$n_1 \leq n_2 \leq \cdots \leq n_l, \quad \widehat{\alpha_1} \leq \widehat{\alpha_2} \leq \cdots \leq \widehat{\alpha_l}$$

and $\frac{1}{p} + \frac{1}{q} = 1$ *for* $\frac{1}{\alpha} < p \leq \infty$ *where* $\alpha = \min_{i,j}\{\alpha_i^j\}$ *and let* $f_1, f_2 \cdots, f_l$ *be functions satisfying in the following conditions*

(i) $f_i(s, 0) \in L_{1,1-\widehat{\alpha_i}}$ *for* $i \in \{1, \cdots, l\}$;

(ii) there exists $\lambda_i \in L^{p,n_i}$ *such that* $|f_i(t, s_1) - f_i(t, s_2)| \leq \lambda_i(t)|s_1 - s_2|$ *for* $i \in \{1, \cdots, l\}$, $t \in [0, 1]$ *and* $s_1, s_2 \in \mathbb{R}^k$;

(iii) γ_i*'s are real numbers such that* $R := M \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \|\lambda_i(s)\|_{p,\hat{0}} < 1$ *where* $M := {}_{q,q-q\widehat{\alpha_1}}\|1\|$.

Then the following integral equation has a unique solution in $L_{q,q-q\widehat{\alpha_1}}$

$$x(t) = \sum_{i=1}^{l} \frac{\gamma_i}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{f_i(s, x(s))}{(t_- t_{n_i-1})^{1-\alpha_{n_i}^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{s_i} + \phi(t). \tag{6}$$

Proof. Let $x \in L_{q,q-q\widehat{\alpha_1}}$ and define operator T as follows:

$$Tx(t) = \sum_{i=1}^{l} \frac{\gamma_i}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{f_i(s, x(s))}{(t - t_{n_i-1})^{1-\alpha_{n_i}^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{s_i} + \phi(t). \tag{7}$$

We want to show that T has got exactly a fixed point in L^{q,n_1}. Note that

$$\begin{aligned} |Tx(t)| &\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{|f_i(s, x(s))|}{(t - t_{n_i-1})^{1-\alpha_{n_i}^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{s_i} + |\phi(t)| \\ &\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{|f_i(s, 0)|}{(t - t_{n_i-1})^{1-\alpha_{n_i}^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{s_i} + \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{|f_i(s, x(s)) - f_i(s, 0)|}{(t - t_{n_i-1})^{1-\alpha_{n_i}^i} \cdots (t_1 - s)^{1-\alpha_1^i}} d\mathbf{s_i} + |\phi(t)| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})}\, {}_{1,1-\widehat{\alpha_i}}\|f_i(s,0)\| + \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{\lambda_i(s)|x(s)|}{(t-t_{n_i-1})^{1-\alpha^i_{n_i}}\cdots(t_1-s)^{1-\alpha^i_1}} d\mathbf{s_i} + |\phi(t)| \\
&\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})}\, {}_{1,1-\widehat{\alpha_i}}\|f_i(s,0)\| + \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \left(\int_0^{\mathbf{t_i}} \lambda_i^p(s) d\mathbf{s_i}\right)^{\frac{1}{p}} \left(\int_0^{\mathbf{t_i}} \frac{|x(s)|^q}{(t-t_{n_i-1})^{q-q\alpha^i_{n_i}}\cdots(t_1-s)^{q-q\alpha^i_1}} d\mathbf{s_i}\right)^{\frac{1}{q}} + \|\phi\|_\infty \\
&\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})}\, {}_{1,1-\widehat{\alpha_i}}\|f_i(s,0)\| + \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \|\lambda_i\|_{p,n_i}\, {}_{q,q-q\widehat{\alpha_i}}\|x\| + \|\phi\|_\infty \\
&\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})}\, {}_{1,1-\widehat{\alpha_i}}\|f_i(s,0)\| + \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \|\lambda_i\|_{p,n_i}\, {}_{q,q-q\widehat{\alpha_1}}\|x\| + \|\phi\|_\infty \\
&:= B < \infty.
\end{aligned}
$$

Thus

$$
\left(\int_0^{\mathbf{t_i}} \frac{|Tx(s)|^q}{(t-t_{n_i-1})^{q-q\alpha^1_{n_i}}\cdots(t_1-s)^{q-q\alpha^1_1}} d\mathbf{s_i}\right)^{\frac{1}{q}} \leq \left(\int_0^{\mathbf{t_i}} \frac{B^q}{(t-t_{n_i-1})^{q-q\alpha^1_{n_i}}\cdots(t_1-s)^{q-q\alpha^1_1}} d\mathbf{s_i}\right)^{\frac{1}{q}} \leq BM < \infty
$$

which implies that T carries the space of $L_{q,q-q\widehat{\alpha_1}}$ into itself so we can consider T as

$$
T : L_{q,q-q\widehat{\alpha_1}} \to L_{q,q-q\widehat{\alpha_1}}.
$$

Now, suppose x and y are in $L_{q,q-q\widehat{\alpha_1}}$. Therefore

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{|f_i(s,x(s)) - f_i(s,y(s))|}{(t-t_{n_i-1})^{1-\alpha^i_{n_i}}\cdots(t_1-s)^{1-\alpha^i_1}} d\mathbf{s_i} \\
&\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \int_0^{\mathbf{t_i}} \frac{\lambda_i(s)|x(s)-y(s)|}{(t-t_{n_i-1})^{1-\alpha^i_{n_i}}\cdots(t_1-s)^{1-\alpha^i_1}} d\mathbf{s_i} \\
&\leq \sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \left(\int_0^{\mathbf{t_i}} |\lambda_i(s)|^p d\mathbf{s_i}\right)^{\frac{1}{p}} \left(\int_0^{\mathbf{t_i}} \frac{|x(s)-y(s)|^q}{(t-t_{n_i-1})^{q-q\alpha^i_{n_i}}\cdots(t_1-s)^{q-q\alpha^i_1}} d\mathbf{s_i}\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \|\lambda_i(s)\|_{p,\hat{0}_i}\right) {}_{q,q-q\widehat{\alpha_1}}\|x-y\|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
{}_{q,q-q\widehat{\alpha_1}}\|Tx - Ty\| &\leq M\left(\sum_{i=1}^{l} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \|\lambda_i(s)\|_{p,\hat{0}_i}\right) {}_{q-q\widehat{\alpha_1}}\|x-y\| \\
&= R \times {}_{q-q\widehat{\alpha_1}}\|x-y\|.
\end{aligned}
$$

The operator T is a contractive mapping due to the assumption that $R < 1$. So we can utilize the contraction mapping principle. Accordingly there exists a unique element $x^* \in L_{q,q-q\widehat{\alpha_1}}$ so that $Tx^* = x^*$. □

Remark 4.2. *Since $L_{q,q-q\widehat{\alpha_1}} \subset L^{q,n_1}$, under the assumptions of Theorem 4.1, the integral equation (6) has a unique solution in L^{q,n_1}.*

5. Application

In this section, we apply Theorem 4.1 to verify the existence and uniqueness of solution for the initial value problem of a type of the well-known fractional Langevin equation which is in the following form:

$$\begin{cases} D^{\beta}(D^{\alpha}+\gamma)x(t) = f(t,x(t)), & 0 < t \le 1, \\ x^{(k)}(0) = \mu_k, & 0 \le k < m, \\ x^{(\alpha+k)}(0) = \nu_k, & 0 \le k < n, \end{cases} \tag{8}$$

where $\gamma \in \mathbb{R}$, $m-1 < \alpha \le m$, $n-1 < \beta \le n$, $m, n \in \mathbb{N}$, D^{α} and D^{β} are the Caputo fractional derivatives and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function.

The fractional Langevin equation has some significant applications in physics and engineering. For instance it is used for modeling of single-file diffusion [30] and for a free particle driven by power law type of noises [31]. In [32], the transformation of the Fokker-Planck equation, which corresponds to the Langevin equation with multiplicative white noise, into the Wiener process is made available for any prescription. This equation raised out of the work of French physicist Paul Langevin in 1908 who proposed an elaborate description of Brownian motion, that is the random movement of a particle submerged in a fluid, due to its collisions with the much smaller fluid molecules. Since then, Langevin was considered as one of the institutors of the branch of stochastic differential equations [33]. Many of stochastic problems in fluctuating environments are described by Langevin equation [34]. But for some complex systems, the classical Langevin equation cannot offer a correct concept of the problem. As a result, various generalizations have been offered which make up the lacks of the classic cases and they describe more physical phenomena in disordered regions [35]. In [28] the author considered this equation and proved some result related to the problem of existence and uniqueness of that. Here we generalize the method of that article and we extend the region of variations of the parameter $\alpha + \beta$.

Theorem 5.1. *Under the conditions of Theorem 4.1, the initial value problem of fractional Langevin equation involving two fractional orders (8) has a unique solution in* $L^{q,1}[0,1]$ *for some* $1 \le q \le \infty$.

Proof. Let $x(t)$ be a solution of the initial problem (8). By taking the operators I^{β} and I^{α} of both sides of the equation (8), respectively, and by using Theorem 2.3, we have

$$\begin{aligned} x(t) &= I^{\alpha}I^{\beta}f(t,x(t)) - \gamma I^{\alpha}x(t) + I^{\alpha}\Big(\sum_{i=0}^{n-1} a_i t^i\Big) + \sum_{i=0}^{m-1} b_i t^i \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^{t_1} \frac{f(s,x(s))}{(t-t_1)^{1-\alpha}(t_1-s)^{1-\beta}} ds dt_1 - \frac{\gamma}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds \\ &+ \sum_{i=0}^{n-1} \frac{\Gamma(i+1)}{\Gamma(\alpha+i+1)} a_i \, t^{\alpha+i} + \sum_{i=0}^{m-1} b_i t^i. \end{aligned} \tag{9}$$

Using the initial conditions for the initial value problem of Langevin equation (8), we deduce that

$$a_i = \frac{\nu_i + \gamma\mu_i}{\Gamma(\alpha+i+1)}, \quad 0 \le i < n,$$

and

$$b_i = \frac{\mu_i}{\Gamma(i+1)}, \quad 0 \le i < m.$$

Substituting the values of a_i and b_i in (9), the solution $x(t)$ can be obtained as the following integral equation

$$\begin{aligned} x(t) = &-\frac{\gamma}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^{t_1} \frac{f(s,x(s))}{(t-t_1)^{1-\alpha}(t_1-s)^{1-\beta}} ds dt_1 \\ &+ \sum_{i=0}^{n-1} \frac{\nu_i + \gamma\mu_i}{\Gamma(\alpha+i+1)} t^{\alpha+i} + \sum_{j=0}^{m-1} \frac{\mu_j}{\Gamma(j+1)} t^j. \end{aligned} \tag{10}$$

Obviously, the equation (10) is a special case of (6) with $l := 2$, $n_1 := 1$, $\gamma_1 := -\gamma$, $\widehat{\alpha_1} := \{\alpha\}$, $f_1(t,s) := s$, $n_2 := 2$, $\gamma_2 := 1$, $\widehat{\alpha_2} := \{\alpha, \beta\}$, $f_2(t,s) := f(t,s)$, and $\phi(t) := \sum_{i=0}^{n-1} \frac{\nu_i+\gamma\mu_i}{\Gamma(\alpha+i+1)} t^{\alpha+i} + \sum_{j=0}^{m-1} \frac{\mu_j}{\Gamma(j+1)} t^j$. Then the initial problem (8) has a unique solution in $L^{q,1}[0,1]$ for some $1 \leq q \leq \infty$.

It is necessary to mention that our article is a real extension of the results of the articles [23] and [28]. The article [23] has discussed the existence and uniqueness of solutions for the initial value problem of Langevin equation involving two fractional order (8) with the continuity and differentiability assumption for f and the article [28] has discussed the existence and uniqueness of solutions with the constraint $0 < \alpha + \beta < 1$. However, in our manuscript for any measurable function f, we investigate the existence and uniqueness of solutions in initial value problem (8) with the less restrictive assumption $0 < \alpha + \beta < 2$.

Example 5.2. *Consider the following initial value problem*

$$\begin{cases} D^{5/6}(D^{3/4} + \frac{1}{3})x(t) = \frac{t-\sin(x(t))}{\Gamma(1/4)\ e^t\ t^{1/4}}, \quad 0 < t \leq 1, \\ x(0) = 0, \\ x^{(3/4)}(0) = 1. \end{cases} \tag{11}$$

Obviously, this equation is a special case of (8) with $\alpha = \frac{5}{6}, \beta = \frac{3}{4}, \gamma = \frac{1}{3}, \mu_0 = 0, \nu_0 = 1$ and $f(t,s) = \frac{t-\sin(s)}{\Gamma(1/4)\ e^t\ t^{1/4}}$, for $0 \leq s < t \leq 1$. First note that f is discontinuous of infinite type at $t = 0$. Therefore, the obtained results in [23] do not cover this example. Also, the parameter $\alpha + \beta$ does not belong to the interval $[0,1)$, so by using the results of [28], we could not speak about the existence and uniqueness of solution of this problem, too. Then, we show that the assumptions (i)–(iii) in Theorem 4.1 are satisfied.

Taking $p = 2$, $\lambda_1(t) = 1$, and $\lambda_2(t) = \frac{1}{\Gamma(1/4)} e^{-t}\, t^{-1/4}$, the assumptions (*i*) and (*ii*) of Theorem 4.1 are satisfied. Now, we are checking that $R < 1$. It is easily seen that

$$\begin{aligned} M = {}_{2,2-2\widehat{\alpha_1}}\|1\| &= \Big(\sup_{t\in[0,1]} \int_0^t \frac{1}{(t-s)^{1/3}} ds\Big)^{1/2} \cong 1.22474 \\ \|\lambda_1\|_{2,\hat{0}_1} &= \Big(\int_0^1 \int_0^{t_1} ds\, dt_1\Big)^{1/2} \cong 0.70710 \\ \|\lambda_2\|_{2,\hat{0}_2} &= \Big(\int_0^1 \int_0^{t_2} \int_0^{t_1} \frac{e^{-2s} s^{-1/2}}{\Gamma^2(1/4)} ds\, dt_1\, dt_2\Big)^{1/2} \cong 0.17868. \end{aligned}$$

Therefore,

$$R := M \sum_{i=1}^{2} \frac{|\gamma_i|}{\Gamma(\widehat{\alpha_i})} \|\lambda_i(s)\|_{p,\hat{0}_i} = 0.41394 < 1.$$

So, utilizing Theorem 4.1, we conclude that (11) has a unique solution belonging to $L^{2,1}[0,1]$.

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References

[1] R. Estrada, R. P. Kanwal, Singular Integral Equations, Birkhäuser, Boston, 2000.
[2] G.B. Arfken, H. J. Weber, Mathematical Methods for Physicists, Academic Press, New York, 2001.
[3] S. R. Manam, Multiple integral equations arising in the theory of water waves, Appl. Math. Lett., 24 (2011) 1369–1373.
[4] B. N. Mandal, A. Chakrabarti, Applied Singular Integral Equations, Science Publishers, New York, 2011.
[5] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[7] K. B. Oldham, J. Spanier, The fractional calculus: Theory and Application of Differential Equations, Academic Press, New York, 1974.
[8] A. W. Wharmby, R. L. Bagley, Modifying Maxwell's equations for dielectric materials based on techniques from viscoelasticity and concepts from fractional calculus, Int. J. Eng. Sci., 79 (2014) 59–80.
[9] Z. H. Wang, X. Wang, General solution of the Bagley–Torvik equation with fractional-order derivative, Commun. Nonlinear Sci. Numer. Simulat., 15 (2010) 1279–1285.
[10] M. Bonforte, J. L. Vázquez, Quantitative local and global a priori estimates for fractional nonlinear diffusion equations, Adv. Math., 250 (2014) 242–284.
[11] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, World Scientific, New York, 2010.
[12] J. Čermák, T. Kisela, Exact and discretized stability of the Bagley–Torvik equation, J. Comput. Appl. Math., 269 (2014) 53–67.
[13] R. Bagley, P. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol., 27 (1983) 201–210.
[14] M. Bonforte, J. L. Vázquez, Fractional nonlinear degenerate diffusion equations on bounded domains part I: existence, uniqueness and upper bounds , Nonlinear Anal., 131 (2016) 363–398.
[15] F. F. Wang, D. Y. Chen, X. G. Zhang, Y. Wu, The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay, Appl. Math. Lett., 53 (2016) 45–51.
[16] B. Wu, S. Wu, Existence and uniqueness of an inverse source problem for a fractional integro-differential equation, Comput. Math. Appl., 68 (2014) 1123–1136.
[17] Y. Chen, T. Tang, Spectral methods for weakly singular Volterra integral equations with smooth solutions, J. Comput. Appl. Math., 233 (2009) 938–950.
[18] X. Q. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, Appl. Math. Lett., 39 (2015) 22–27.
[19] G. Arthi, J. H. Park, H. Y. Jung, Existence and exponential stability for neutral stochastic integrodifferential equations with impulses driven by a fractional Brownian motion, Commun. Nonlinear Sci. Numer. Simulat., 32 (2016) 145–157.
[20] W. Lin, Global existence theory and chaos control of fractional differential equations, J. Math. Anal. Appl., 332 (2007) 709–726.
[21] N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Anal., 70 (2009) 2521–2529.
[22] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[23] T. Yu, K. Deng, M. Luo, Existence and uniqueness of solutions of initial value problems for nonlinear Langevin equation involving two fractional orders, Commun. Nonlinear Sci. Numer. Simulat., 19 (2014) 1661–1668.
[24] J. Deng, L. Ma, Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Appl. Math. Lett., 23 (2010) 676–680.
[25] J. Deng, Z. Deng, Existence of solutions of initial value problems for nonlinear fractional differential equations, Appl. Math. Lett., 32 (2014) 6–12.
[26] D. O'Regan, R. P. Agarwal, K. Perera, Nonlinear integral equations singular in the dependent variable, Appl. Math. Lett., 20 (2007) 1137–1141.
[27] E. Babolian, S. Bazm, P. Lima, Numerical solution of nonlinear two–dimensional integral equations using rationalized Haar functions, Commun. Nonlinear Sci. Numer. Simulat., 16 (2011) 1164–1175.
[28] O. Baghani, On fractional Langevin equation involving two fractional orders, Commun. Nonlinear Sci. Numer. Simulat., 42 (2017) 675–681.
[29] O. Baghani, S. M. S. Nabavi Sales, Existence and relaxation results in initial value type problems for nonlinear fractional differential equations(submitted).
[30] C. H. Eab, S. C. Lim, Fractional generalized Langevin equation approach to single–file diffusion, Phys. A, 389 (2010) 2510–2521.
[31] T. Sandev, Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, Phys. Lett. A, 378 (2014) 1–9.
[32] S. F. Kwok, Langevin equation with multiplicative white noise: Transformation of diffusion processes into the Wiener process in different prescriptions, Ann. Phys., 327 (2012) 1989–1997.
[33] C. Beck, G. Roepstorff, From dynamical systems to the Langevin Equation, Phys. A, 145 (1987) 1–14.
[34] W. T. Coffey, Y. P. Kalmykov, J. T. Waldron, The Langevin Equation, second ed., World Scientific, Singapore, 2004.
[35] R. Klages, G. Radons, I. M. Sokolov, Anomalous Transport: Foundations and Applications, Wiley-VCH, Weinheim, 2008.