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Pseudo Almost Automorphic Mild Solution of Nonautonomous Stochastic Functional Integro-differential Equations

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Abstract. In this paper, we propose a new class of stochastic process called (μ,ν)-pseudo almost automorphic in p-mean, which generalize in a natural fashion the concept of square-mean almost automorphy and its various extensions. As application, we establish the existence, uniqueness of (μ,ν)-pseudo almost automorphic in p-distribution mild solution to nonautonomous stochastic functional integro-differential equations. Finally, an example is given to illustrate the significance of the main findings.

1. Introduction

The concept of square-mean almost automorphy, introduced by Fu and Liu [20] is related to and more general than square-mean almost periodicity. Since then, this work is generalized into square-mean pseudo almost automorphy and square-mean weighted pseudo almost automorphy by Chen and Lin [10, 11], p-th mean pseudo almost automorphy ($p \geq 2$) by Bezandry and Diagana [5], square-mean μ-pseudo almost automorphy by Diop et al [15], almost automorphy in distribution sense by Liu and Sun [26], weighted pseudo almost automorphy in distribution sense by Li [24]. For more details about almost automorphy in square-mean sense or in distribution sense, its various extensions and applications in stochastic differential equations, one can see [3, 7, 9, 27, 30, 31] for more details.

So far, most of the studies on almost automorphy for stochastic differential equations are concerned in square-mean sense, except for [3, 8, 16, 17, 19, 24, 26, 33]. In this paper, we introduce the concept of (μ,ν)-pseudo almost automorphy in p-distribution by measure theory ($p \geq 2$), which generalize the concepts of almost automorphy in distribution and its various extensions. Stochastic integro-differential equations play a crucial role in qualitative theory of differential equations due to their application to physics, engineering, mechanics, population dynamics, and other subjects. By measure theory, this paper deals with existence, uniqueness of (μ,ν)-pseudo almost automorphic in p-distribution mild solution to nonautonomous stochastic functional integro-differential equations:

$$Y'(t) = A(t)Y(t) + f(t,Y(t)) + \int_{-\infty}^{t} K_1(t-s)g(s,Y(s))dW(s) + \int_{-\infty}^{t} K_2(t-s)h(s,Y(s))ds, \qquad t \in \mathbb{R}, \tag{1.1}$$

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where $A(t) : \mathcal{D}(A(t)) \subset \mathcal{L}^p(P,H) \to \mathcal{L}^p(P,H)$ is a densely defined closed (possibly unbounded) linear operator, K_1, K_2 are convolution-type kernels in $L^1(0,\infty)$. $f, h : \mathbb{R} \times \mathcal{L}^p(P,H) \to \mathcal{L}^p(P,H)$, $g : \mathbb{R} \times \mathcal{L}^p(P,H) \to \mathcal{L}^p(P, L(H,H))$ are jointly continuous functions. $W(t)$ is Q-Wiener process with values in H.

Recently, in the case of $A(t) \equiv A$ and $p = 2$, Bezanary [4] investigate the existence, uniqueness of square-mean almost periodic mild solutions to (1.1), and further generalized into (μ,ν)-pseudo almost automorphy in distribution by Xia [32]. If $p = 2$, Li study the existence, uniqueness of square-mean almost periodic mild solutions to (1.1) in [23], and further generalized to square-mean almost automorphic mild solutions in [25]. But for almost automorphy in distribution, particularly for the (μ,ν)-pseudo almost automorphy in p-distribution ($p \geq 2$) to (1.1), it is an untreated topic. This is one of the key motivations of this study.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented and the definition, properties of (μ,ν)-pseudo almost automorphy in p-distribution are given. Sections 3 is devoted to the existence, uniqueness of (μ,ν)-pseudo almost automorphic in p-distribution mild solution to (1.1). In Section 4, an application to the stochastic functional integro-differential equations is given.

2. Preliminaries and basic results

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. For A being a linear operator, $\mathcal{D}(A)$, $\rho(A)$, $R(\lambda, A)$, $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of A. We assume that $(H, \|\cdot\|)$ is real separable Hilbert spaces. $L(H,H)$ is the space of all bounded linear operators from H to H. We assume that $(\Omega, \mathcal{F}, P)$ is a probability space, and for $p \geq 2$, $\mathcal{L}^p(P,H)$ stands for the space of all H-valued random variables Y such that $\mathbb{E}\|Y\|^p = \int_\Omega \|Y\|^p dP < \infty$. For $Y \in \mathcal{L}^p(P,H)$, let $\|Y\|_p := \left(\int_\Omega \|Y\|^p dP\right)^{1/p}$, then $\mathcal{L}^p(P,H)$ is a Hilbert space equipped with the norm $\|\cdot\|_p$.

Definition 2.1. A stochastic process $Y : \mathbb{R} \to \mathcal{L}^p(P,H)$ is said to be $\mathcal{L}^p$-bounded if there exists a constant $M > 0$ such that $\mathbb{E}\|Y(t)\|^p = \int_\Omega \|Y(t)\|^p dP \leq M$ for all $t \in \mathbb{R}$. Y is said to be $\mathcal{L}^p$-continuous if $\lim_{t\to s} \mathbb{E}\|Y(t) - Y(s)\|^p = 0$ for any $s \in \mathbb{R}$. Denoted by $SBC(\mathbb{R}, \mathcal{L}^p(P,H))$ the collection of all the $\mathcal{L}^p$-bounded and $\mathcal{L}^p$-continuous processes. It is a Banach space equipped with the norm $\|Y\|_\infty = \sup_{t\in\mathbb{R}} (\mathbb{E}\|Y(t)\|^p)^{1/p}$.

Definition 2.2. [5] An $\mathcal{L}^p$-continuous process $Y : \mathbb{R} \to \mathcal{L}^p(P,H)$ is said to be p-mean almost automorphic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a stochastic process $\widetilde{Y} : \mathbb{R} \to \mathcal{L}^p(P,H)$ such that

$$\lim_{n\to\infty} \mathbb{E}\|Y(t+s_n) - \widetilde{Y}(t)\|^p = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}\|\widetilde{Y}(t-s_n) - Y(t)\|^p = 0$$

hold for each $t \in \mathbb{R}$. The collection of all p-mean almost automorphic processes $Y : \mathbb{R} \to \mathcal{L}^p(P,H)$ is denoted by $SAA(\mathbb{R}, \mathcal{L}^p(P,H))$. Note that if $p = 2$, 2-mean almost automorphic stochastic is called square-mean almost automorphic stochastic which defined in [20].

Definition 2.3. An $\mathcal{L}^p$-continuous stochastic process $f(t,s) : \mathbb{R} \times \mathbb{R} \to \mathcal{L}^p(P,H)$ is said to be p-mean bi-almost automorphic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a stochastic process $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathcal{L}^p(P,H)$ such that

$$\lim_{n\to\infty} \mathbb{E}\|f(t+s_n, s+s_n) - \widetilde{f}(t,s)\|^p = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}\|\widetilde{f}(t-s_n, s-s_n) - f(t,s)\|^p = 0$$

hold for each $t, s \in \mathbb{R}$. The collection of all p-mean bi-almost automorphic processes $f : \mathbb{R} \times \mathbb{R} \to \mathcal{L}^p(P,H)$ is denoted by $bSAA(\mathbb{R} \times \mathbb{R}, \mathcal{L}^p(P,H))$. Note that if $p = 2$, it is square-mean bi-almost automorphic process which defined in [11].

Next, we introduce the concepts of (μ,ν)-pseudo almost automorphy in p-mean, and p-distribution, respectively. $\mathcal{B}$ denotes the Lebesgue σ-field of $\mathbb{R}$, $\mathcal{M}$ stands for the set of all positive measure μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 2.4. [6] Let $\mu, \nu \in \mathcal{M}$, the measure μ and ν are said to be equivalent (i.e., $\mu \sim \nu$) if there exist constants $c_0, c_1 > 0$ and a bounded interval $\Omega \subset \mathbb{R}$ (eventually $\emptyset$) such that $c_0\nu(A) \le \mu(A) \le c_1\nu(A)$ for all $A \in \mathcal{B}$ satisfying $A \cap \Omega = \emptyset$.

Let $\mu, \nu \in \mathcal{M}$, define the (μ, ν)-ergodic space

$$SPAA_0(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu) = \Big\{Y \in SBC(\mathbb{R}, \mathcal{L}^p(P,H)) : \lim_{r\to\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \mathbb{E}\|Y(t)\|^p d\mu(t) = 0\Big\}.$$

Definition 2.5. Let $\mu, \nu \in \mathcal{M}$. A stochastic process $f : \mathbb{R} \to \mathcal{L}^p(P,H)$ is said to be p-mean (μ, ν)-pseudo almost automorphic if it can be decomposed as $f = g+\varphi$, where $g \in SAA(\mathbb{R}, \mathcal{L}^p(P,H))$ and $\varphi \in SPAA_0(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$. The collection of all p-mean (μ, ν)-pseudo almost automorphic processes is denoted by $SPAA(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$.

Remark 2.1. (*i*) If $\mu \sim \nu$ and μ, ν are the Lebesgue measures, then p-mean (μ, ν)-pseudo almost automorphic process $SPAA(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ is p-mean pseudo almost automorphic process $SPAA(\mathbb{R}, \mathcal{L}^p(P,H))$ [5]. Particularly, if $p = 2$ in $SPAA(\mathbb{R}, \mathcal{L}^p(P,H))$, it is square-mean pseudo almost automorphic process $SPAA(\mathbb{R}, \mathcal{L}^2(P,H))$ [10].

(*ii*) If $p = 2$, let $\rho(t) > 0$ a.e. on $\mathbb{R}$ for the Lebesgue measure. μ, ν denote the positive measure defined by $\mu(A) = \nu(A) = \int_A \rho(t)dt$ for $A \in \mathcal{B}$, where dt denotes the Lebesgue measure on $\mathbb{R}$, then p-mean (μ, ν)-pseudo almost automorphic process $PAA(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ is square-mean weighted pseudo almost automorphic process $SWPAA(\mathbb{R}, \mathcal{L}^2(P,H), \rho)$ defined in [11].

(*iii*) If $p = 2$ and $\mu \sim \nu$, p-mean (μ, ν)-pseudo almost automorphic process $SPAA(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ is square-mean μ-pseudo almost automorphic process $SPAA(\mathbb{R}, \mathcal{L}^2(P,H), \mu)$ [15].

In this paper, we formulate the following hypotheses:

(A_1) Let $\mu, \nu \in \mathcal{M}$ such that $\limsup_{r\to\infty} \frac{\mu([-r,r])}{\nu([-r,r])} < \infty$.

(A_2) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that $\mu(\{a+\tau, a \in A\}) \le \beta\mu(A)$ if $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$.

Similar as the proof of [13], one has

Lemma 2.1. *Let $\mu, \nu \in \mathcal{M}$ satisfy the condition (A_2), then $SPAA_0(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ is translation invariant, therefore $SPAA(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ is also translation invariant.*

Lemma 2.2. *Let $\mu, \nu \in \mathcal{M}$ satisfy (A_1), (A_2), then $SPAA(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ is a Banach space with the supremum norm $\|\cdot\|_\infty$.*

Now, we introduce the concept of (μ, ν)-pseudo almost automorphy in p-distribution. Let $\mathcal{P}(\mathcal{L}^p(P,H))$ be the space of all Borel probability measures on $\mathcal{L}^p(P,H)$ endowed with the metric:

$$d_{BL}(\mu, \nu) := \sup\left\{\left|\int f d\mu - \int f d\nu\right| : \|f\|_{BL} \le 1\right\}, \quad \mu, \nu \in \mathcal{P}(\mathcal{L}^p(P,H)),$$

where f are Lipschitz continuous real-valued functions on $\mathcal{L}^p(P,H)$ with the norms

$$\|f\|_{BL} = \max\{\|f\|_L, \|f\|_\infty\}, \ \|f\|_L = \sup_{x\neq y} \frac{|f(x) - f(y)|}{\|x-y\|}, \ \|f\|_\infty = \sup_{x\in\mathcal{L}^p(P,H)} |f(x)|.$$

We denote by $\mathrm{law}(Y(t))$ the p-distribution of the random variable $Y(t)$, by [3], we say that Y has almost automorphic in one-dimensional p-distribution if the mapping $t \to \mathrm{law}(Y(t))$ from $\mathbb{R}$ to $(\mathcal{P}(\mathcal{L}^p(P,H)), d_{BL})$ is almost automorphic.

Definition 2.6. [3] An $\mathcal{L}^p$-continuous stochastic process Y is almost automorphic in p-distribution:

(i) If the mapping $t \to \mathrm{law}(Y(t+\cdot))$ from $\mathbb{R}$ to $\mathcal{P}(C(\mathbb{R}, \mathcal{L}^p(P,H)))$ is almost automorphic.

(ii) If $p > 0$, the family $(\|Y(t)\|^p)_{t\in\mathbb{R}}$ is uniformly integrable.

Remark 2.2. Note that if $p = 2$, then similar definition for almost periodicity are defined in [29], that is almost periodicity in one-dimensional distribution, almost periodicity in distribution, one can see [29] for more details.

Definition 2.7. Let $\mu, \nu \in \mathcal{M}$. A stochastic process $X : \mathbb{R} \to \mathcal{L}^p(P,H)$ is said to be (μ,ν)-pseudo almost automorphic in p-distribution if it can be decomposed as $X = Y + Z$, where Y is almost automorphic in p-distribution and $Z \in SPAA_0(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$.

3. Stochastic Functional Integro-differential Equations

In this section, we investigate the existence, uniqueness of (μ,ν)-pseudo almost automorphic in p-distribution mild solution to (1.1). First, we make the following assumptions:

(H_1) There exists constants $\lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi), \widetilde{L}, \widetilde{M} \geq 0$ and $\beta, \gamma \in (0,1)$ with $\beta + \gamma > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{\widetilde{M}}{1+|\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \widetilde{L}|t-s|^\beta|\lambda|^{-\gamma}$$

for $t, s \in \mathbb{R}, \Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |arg\lambda| \leq \theta\}$.

(H_2) $A(t) : \mathcal{D}(A(t)) \subset \mathcal{L}^p(P,H) \to \mathcal{L}^p(P,H)$ is a family of densely defined closed linear operators which generates a uniformly exponentially stable evolution family $(U(t,s))_{t\geq s}$, that is, there exist constants $M > 0$, $\delta > 0$ such that $\|U(t,s)\| \leq Me^{-\delta(t-s)}$, $-\infty < s \leq t < +\infty$.

(H_3) $U(t,s)x \in bSAA(\mathbb{R}\times\mathbb{R}, \mathcal{L}^p(P,H))$ uniformly for all $x \in \mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathcal{L}^p(P,H)$.

(H_4) K_1, K_2 are convolution-type kernels in $L^2(0,\infty)$ and $L^1(0,\infty)$, respectively.

(H_5) $\sup_{t\in\mathbb{R}} \mathbb{E}\|f(t,0)\|^p < \infty$, $\sup_{t\in\mathbb{R}} \mathbb{E}\|g(t,0)\|^p_{L(H,H)} < \infty$, $\sup_{t\in\mathbb{R}} \mathbb{E}\|f(h,0)\|^p < \infty$, and there exists a constant $L > 0$ such that

$$\begin{aligned}
&\mathbb{E}\|f(t,Y) - f(t,Z)\|^p \leq L \cdot \mathbb{E}\|Y - Z\|^p,\\
&\mathbb{E}\|g(t,Y) - g(t,Z)\|^p_{L(H,H)} \leq L \cdot \mathbb{E}\|Y - Z\|^p,\\
&\mathbb{E}\|h(t,Y) - h(t,Z)\|^p \leq L \cdot \mathbb{E}\|Y - Z\|^p,
\end{aligned}$$

for all $t \in \mathbb{R}$, $Y, Z \in \mathcal{L}^p(P,H)$.

(H_6) $f = f_1 + f_2, h = h_1 + h_2 \in SPAA(\mathbb{R}\times\mathcal{L}^p(P,H), \mathcal{L}^p(P,H), \mu, \nu)$, where $f_1, h_1 \in SAA(\mathbb{R}\times\mathcal{L}^p(P,H), \mathcal{L}^p(P,H))$, $f_2, h_2 \in SPAA_0(\mathbb{R}\times\mathcal{L}^p(P,H), \mathcal{L}^p(P,H), \mu, \nu)$. $g = g_1 + g_2 \in SPAA(\mathbb{R}\times\mathcal{L}^p(P,H), \mathcal{L}^p(P, L(H,H)), \mu, \nu)$, where $g_1 \in SAA(\mathbb{R}\times\mathcal{L}^p(P,H), \mathcal{L}^p(P, L(H,H)))$, $g_2 \in SPAA_0(\mathbb{R}\times\mathcal{L}^p(P,H), \mathcal{L}^p(P, L(H,H)), \mu, \nu)$.

Remark 3.1. (H_1) is usually called "Acquistapace-Terreni" conditions, which was introduced in [1] and widely used to study nonautonomous differential equations [1, 2, 14, 18, 21]. If (H_1) holds, there exists a unique evolution family $(U(t,s))_{t\geq s}$, which governs the homogeneous version of (1.1) [2].

Before starting our main results, we recall the definition of the mild solution to (1.1).

Definition 3.1. [23] An $\mathcal{F}_t$-progressively measurable stochastic process $\{Y(t)\}_{t\in\mathbb{R}}$ is called a mild solution to (1.1) if it satisfies the corresponding stochastic integral equation:

$$\begin{aligned} Y(t) &= U(t,a)Y(a) + \int_a^t U(t,s)f(s,Y(s))ds + \int_a^t U(t,\sigma)\int_a^\sigma K_1(\sigma-s)g(s,Y(s))dW(s)d\sigma \\ &\quad + \int_a^t U(t,\sigma)\int_a^\sigma K_2(\sigma-s)h(s,Y(s))dsd\sigma, \end{aligned} \tag{3.1}$$

for all $t \geq a$ and each $a \in \mathbb{R}$.

3.1. Almost automorphy in p-distribution

In this subsection, assume that $f \in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,H))$, $h \in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,H))$, $g \in AA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,L(H,H)))$, and investigate the existence, uniqueness of almost automorphic in p-distribution mild solution to (1.1).

Theorem 3.1. *Let* (A_1)-(A_2) *and* (H_1)-(H_5) *be satisfied, and* $f,h \in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,H))$, $g \in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,L(H,H)))$, *then* (1.1) *has a unique mild solution in* $SBC(\mathbb{R},\mathcal{L}^p(P,H))$ *if*

$$\Theta := 3^{p-1}LM^p\delta^{-p}\left(1 + C_p(trQ)^{p/2}\|K_1\|_{L^2}^{p/2} + \|K_2\|_{L^1}^p\right) < 1.$$

Furthermore, this unique mild solution is almost automorphic in p-distribution if

$$\vartheta := 3^{2p-2}LM^p\delta^{-p}\left(1 + C_p(trQ)^{p/2}\|K_1\|_{L^2}^{p/2} + \|K_2\|_{L^1}^p\right) < 1.$$

Proof. Note that

$$\begin{aligned} Y(t) &= \int_{-\infty}^t U(t,s)f(s,Y(s))ds + \int_{-\infty}^t U(t,\sigma)\int_{-\infty}^\sigma K_1(\sigma-s)g(s,Y(s))dW(s)d\sigma \\ &\quad + \int_{-\infty}^t U(t,\sigma)\int_{-\infty}^\sigma K_2(\sigma-s)h(s,Y(s))dsd\sigma, \end{aligned} \tag{3.2}$$

is well defined and satisfies (3.1). Hence Y is a mild solution to (1.1).

Define the operator $\mathcal{F}$ by

$$\begin{aligned} \mathcal{F}Y(t) &= \int_{-\infty}^t U(t,s)f(s,Y(s))ds + \int_{-\infty}^t U(t,\sigma)\int_{-\infty}^\sigma K_1(\sigma-s)g(s,Y(s))dW(s)d\sigma \\ &\quad + \int_{-\infty}^t U(t,\sigma)\int_{-\infty}^\sigma K_2(\sigma-s)h(s,Y(s))dsd\sigma \\ &:= (\mathcal{F}_1Y)(t) + (\mathcal{F}_2Y)(t) + (\mathcal{F}_3Y)(t), \end{aligned}$$

where

$$\begin{aligned} (\mathcal{F}_1Y)(t) &= \int_{-\infty}^t U(t,s)f(s,Y(s))ds, \\ (\mathcal{F}_2Y)(t) &= \int_{-\infty}^t U(t,\sigma)\int_{-\infty}^\sigma K_1(\sigma-s)g(s,Y(s))dW(s)d\sigma, \\ (\mathcal{F}_3Y)(t) &= \int_{-\infty}^t U(t,\sigma)\int_{-\infty}^\sigma K_2(\sigma-s)h(s,Y(s))dsd\sigma. \end{aligned}$$

It is not difficult to see that $\mathcal{F}$ maps $SBC(\mathbb{R},\mathcal{L}^p(P,H))$ into itself.

(*i*) $\mathcal{F}$ has a unique fixed point in $SBC(\mathbb{R},\mathcal{L}^p(P,H))$.

For $Y_1, Y_2 \in SBC(\mathbb{R}, \mathcal{L}^p(P,H))$, $t \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}\|(\mathcal{F}Y_1)(t)-(\mathcal{F}Y_2)(t)\|^p &\le 3^{p-1}\mathbb{E}\|(\mathcal{F}_1Y_1)(t)-(\mathcal{F}_1Y_2)(t)\|^p + 3^{p-1}\mathbb{E}\|(\mathcal{F}_2Y_1)(t)-(\mathcal{F}_2Y_2)(t)\|^p\\ &\quad + 3^{p-1}\mathbb{E}\|(\mathcal{F}_3Y_1)(t)-(\mathcal{F}_3Y_2)(t)\|^p := \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned}$$

- $\mathcal{I}_1$.

By Hölder inequality, one has

$$\begin{aligned}\mathbb{E}\|(\mathcal{F}_1Y_1)(t)-(\mathcal{F}_1Y_2)(t)\|^p &\le M^p\mathbb{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\|f(s,Y_1(s))-f(s,Y_2(s))\|ds\right)^p\\ &\le M^p\left(\int_{-\infty}^t e^{-\delta(t-s)}ds\right)^{p-1}\cdot\left(\int_{-\infty}^t e^{-\delta(t-s)}\mathbb{E}\|f(s,Y_1(s))-f(s,Y_2(s))\|^p ds\right)\\ &\le LM^p\delta^{-p}\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_1(s)-Y_2(s)\|^p.\end{aligned}$$

- $\mathcal{I}_2$.

By Hölder inequality and [28, Lemma 2.2], $\mathcal{I}_2$ can be estimated as follows:

$$\begin{aligned}&\mathbb{E}\|(\mathcal{F}_2Y_1)(t)-(\mathcal{F}_2Y_2)(t)\|^p\\ &\le \mathbb{E}\left[\int_{-\infty}^t Me^{-\delta(t-\sigma)}\cdot\left\|\int_{-\infty}^\sigma K_1(\sigma-s)[g(s,Y_1(s))-g(s,Y_2(s))]dW(s)\right\|d\sigma\right]^p\\ &\le M^p\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}d\sigma\right)^{p-1}\cdot\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^\sigma K_1(\sigma-s)[g(s,Y_1(s))-g(s,Y_2(s))]dW(s)\right\|^p d\sigma\right)\\ &\le M^p\delta^{1-p}\cdot\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^\sigma K_1(\sigma-s)[g(s,Y_1(s))-g(s,Y_2(s))]dW(s)\right\|^p d\sigma\right)\\ &\le M^p\delta^{1-p}C_p\cdot\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\mathbb{E}\left[trQ\int_{-\infty}^\sigma \|K_1(\sigma-s)\|^{2-4/p}\|K_1(\sigma-s)\|^{4/p}\right.\right.\\ &\qquad\left.\left.\times\ \|(g(s,Y_1(s))-g(s,Y_2(s)))\|^2_{L(H,H)}ds\right]^{p/2}d\sigma\right)\\ &\le M^p\delta^{1-p}C_p(trQ)^{p/2}\cdot\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\left[\int_{-\infty}^\sigma\|K_1(\sigma-s)\|^2ds\right]^{p/2-1}\right.\\ &\qquad\left.\times\left[\int_{-\infty}^\sigma\|K_1(\sigma-s)\|^2\cdot\mathbb{E}\|g(s,Y_1(s))-g(s,Y_2(s))\|^p_{L(H,H)}ds\right]d\sigma\right)\\ &\le LM^p\delta^{-p}C_p(trQ)^{p/2}\|K_1\|^{p/2}_{L^2}\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_1(s)-Y_2(s)\|^p,\end{aligned}$$

where $C_p>0$ is a constant and $C_2=1$ if $p=2$.

- $\mathcal{I}_3$.

By Hölder inequality, we have

$$\begin{aligned}&\mathbb{E}\|(\mathcal{F}_3Y_1)(t)-(\mathcal{F}_3Y_2)(t)\|^p\\ &\le \mathbb{E}\left[\int_{-\infty}^t Me^{-\delta(t-\sigma)}\cdot\left\|\int_{-\infty}^\sigma K_2(\sigma-s)[h(s,Y_1(s))-h(s,Y_2(s))]ds\right\|d\sigma\right]^p\\ &\le M^p\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}d\sigma\right)^{p-1}\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\cdot\mathbb{E}\left\|\int_{-\infty}^\sigma K_2(\sigma-s)h(s,Y_1(s))-h(s,Y_2(s))ds\right\|^p d\sigma\right)\end{aligned}$$

$$\leq M^p\delta^{1-p}\cdot\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\left[\int_{-\infty}^{\sigma}\|K_2(\sigma-s)\|ds\right]^{p-1}\left[\int_{-\infty}^{\sigma}\|K_2(\sigma-s)\|\cdot\mathbb{E}\|h(s,Y_1(s))-h(s,Y_2(s))\|^p ds\right]d\sigma\right)$$
$$\leq LM^p\delta^{-p}\|K_2\|_{L^1}^p\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_1(s)-Y_2(s)\|^p.$$

Hence, it follow that, for each $t\in\mathbb{R}$

$$\mathbb{E}\|(\mathcal{F}Y_1)(t)-(\mathcal{F}Y_2)(t)\|^p\leq\Theta\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_1(s)-Y_2(s)\|^p.$$

Since $\Theta<1$, we conclude that $\mathcal{F}$ is a contraction operator, hence there exists a unique mild solution to (1.1) in $SBC(\mathbb{R},\mathcal{L}^p(P,H))$.

(*ii*) Almost automorphy in p-distribution of mild solution.

Since $f,h\in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,H))$, $g\in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,L(H,H))$, thus for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a stochastic processes $\widetilde{f},\widetilde{h}:\mathbb{R}\times\mathcal{L}^p(P,H)\to\mathcal{L}^p(P,H)$, and $\widetilde{g}:\mathbb{R}\times\mathcal{L}^p(P,H)\to\mathcal{L}^p(P,L(H,H))$ such that

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n,Y)-\widetilde{f}(t,Y)\|^p=0 \quad\text{and}\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{f}(t-s_n,Y)-f(t,Y)\|^p=0. \tag{3.3}$$
$$\lim_{n\to\infty}\mathbb{E}\|g(t+s_n,Y)-\widetilde{g}(t,Y)\|^p_{L(H,H)}=0 \quad\text{and}\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{g}(t-s_n,Y)-g(t,Y)\|^p_{L(H,H)}=0. \tag{3.4}$$
$$\lim_{n\to\infty}\mathbb{E}\|h(t+s_n,Y)-\widetilde{h}(t,Y)\|^p=0 \quad\text{and}\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{h}(t-s_n,Y)-h(t,Y)\|^p=0. \tag{3.5}$$

for each $t\in\mathbb{R}$, $Y\in\mathbb{B}$.

Since $U(t,s)x\in bSAA(\mathbb{R}\times\mathbb{R},\mathcal{L}^p(P,H))$ uniformly for all $x\in\mathbb{B}$ by (H_3), hence for every sequence of real numbers $\{s'_n\}$, there exists a evolution family $\widetilde{U}(t,s)$ such that for each $t\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{E}\|U(t+s_n,s+s_n)x-\widetilde{U}(t,s)x\|^p=0,\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{U}(t-s_n,s-s_n)x-U(t,s)x\|^p=0 \tag{3.6}$$

hold for each $t,s\in\mathbb{R}$ and $x\in\mathbb{B}$. Furthermore, from (3.6) and (H_2), one has

$$\begin{aligned}\mathbb{E}\|\widetilde{U}(t,s)x\|^p&\leq 2^{p-1}\mathbb{E}\|\widetilde{U}(t,s)x-U(t+s_n,s+s_n)x\|^p+2^{p-1}\mathbb{E}\|U(t+s_n,s+s_n)x\|^p\\&\leq 2\cdot 2^{p-1}M^pe^{-\delta p(t-s)}\mathbb{E}\|x\|^p\end{aligned} \tag{3.7}$$

for all $t\geq s$ and $x\in\mathbb{B}$.

Let $\widetilde{W}_n(s):=W(s+s_n)-W(s_n)$, for each $s\in\mathbb{R}$. It is easy to show that $\widetilde{W}_n$ is a Q-Wiener process with the same distribution as W, then

$$\begin{aligned}&Y(t+s_n)\\&=\int_{-\infty}^{t+s_n}U(t+s_n,s)f(s,Y(s))ds+\int_{-\infty}^{t+s_n}U(t+s_n,\sigma)\int_{-\infty}^{\sigma}K_1(\sigma-s)g(s,Y(s))dW(s)d\sigma\\&\quad+\int_{-\infty}^{t+s_n}U(t+s_n,\sigma)\int_{-\infty}^{\sigma}K_2(\sigma-s)h(s,Y(s))dsd\sigma\\&=\int_{-\infty}^{t}U(t+s_n,s+s_n)f(s+s_n,Y(s+s_n))ds\\&\quad+\int_{-\infty}^{t}U(t+s_n,\sigma+s_n)\int_{-\infty}^{\sigma}K_1(\sigma-s)g(s+s_n,Y(s+s_n))d\widetilde{W}_n(s)d\sigma\\&\quad+\int_{-\infty}^{t}U(t+s_n,\sigma+s_n)\int_{-\infty}^{\sigma}K_2(\sigma-s)h(s+s_n,Y(s+s_n))dsd\sigma.\end{aligned}$$

Consider the process

$$\begin{aligned}
Y_n(t) = &\int_{-\infty}^{t} U(t+s_n, s+s_n) f(s+s_n, Y_n(s))ds \\
&+ \int_{-\infty}^{t} U(t+s_n, \sigma+s_n) \int_{-\infty}^{\sigma} K_1(\sigma-s) g(s+s_n, Y_n(s))dW(s)d\sigma \\
&+ \int_{-\infty}^{t} U(t+s_n, \sigma+s_n) \int_{-\infty}^{\sigma} K_2(\sigma-s) h(s+s_n, Y_n(s))dsd\sigma.
\end{aligned}$$

It is easy to see that $Y(t+s_n)$ has the same distribution as $Y_n(t)$ for each $t \in \mathbb{R}$.

Let $\widetilde{Y}(t)$ satisfy the integral equation

$$\begin{aligned}
\widetilde{Y}(t) = &\int_{-\infty}^{t} \widetilde{U}(t,s)\widetilde{f}(s,\widetilde{Y}(s))ds + \int_{-\infty}^{t} \widetilde{U}(t,\sigma) \int_{-\infty}^{\sigma} K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma \\
&+ \int_{-\infty}^{t} \widetilde{U}(t,\sigma) \int_{-\infty}^{\sigma} K_2(\sigma-s)\widetilde{h}(s,\widetilde{Y}(s))dsd\sigma.
\end{aligned}$$

Similar as the proof of [26], it is not difficult to see that $\widetilde{Y}$ is $\mathcal{L}^p$-bounded.

Note that

$$\begin{aligned}
\mathbb{E}\|Y_n(t) - \widetilde{Y}(t)\|^p \le\ & 3^{p-1}\mathbb{E}\left\| \int_{-\infty}^{t} U(t+s_n, s+s_n) f(s+s_n, Y_n(s))ds - \int_{-\infty}^{t} \widetilde{U}(t,s)\widetilde{f}(s,\widetilde{Y}(s))ds \right\|^p \\
&+ 3^{p-1}\mathbb{E}\left\| \int_{-\infty}^{t} U(t+s_n, \sigma+s_n) \int_{-\infty}^{\sigma} K_1(\sigma-s) g(s+s_n, Y_n(s))dW(s)d\sigma \right. \\
&\qquad \left. - \int_{-\infty}^{t} \widetilde{U}(t,\sigma) \int_{-\infty}^{\sigma} K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma \right\|^p \\
&+ 3^{p-1}\mathbb{E}\left\| \int_{-\infty}^{t} U(t+s_n, \sigma+s_n) \int_{-\infty}^{\sigma} K_2(\sigma-s) h(s+s_n, Y_n(s))dsd\sigma \right. \\
&\qquad \left. - \int_{-\infty}^{t} \widetilde{U}(t,\sigma) \int_{-\infty}^{\sigma} K_2(\sigma-s)\widetilde{h}(s,\widetilde{Y}(s))dsd\sigma \right\|^p \\
:=\ & \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

- $\mathcal{J}_1$.

By Hölder inequality, we have

$$\begin{aligned}
\mathcal{J}_1 \le\ & 3^{2p-2}\mathbb{E}\left\| \int_{-\infty}^{t} U(t+s_n, s+s_n)[f(s+s_n, Y_n(s)) - f(s+s_n, \widetilde{Y}(s))]ds \right\|^p \\
&+ 3^{2p-2}\mathbb{E}\left\| \int_{-\infty}^{t} U(t+s_n, s+s_n)[f(s+s_n, \widetilde{Y}(s)) - \widetilde{f}(s, \widetilde{Y}(s))]ds \right\|^p \\
&+ 3^{2p-2}\mathbb{E}\left\| \int_{-\infty}^{t} [U(t+s_n, s+s_n) - \widetilde{U}(t,s)]\widetilde{f}(s,\widetilde{Y}(s))ds \right\|^p \\
\le\ & 3^{2p-2}M^p \left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds \right)^{p-1} \cdot \left(\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|f(s+s_n, Y_n(s)) - f(s+s_n, \widetilde{Y}(s))\|^p ds \right) \\
&+ 3^{2p-2}M^p \left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds \right)^{p-1} \cdot \left(\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|f(s+s_n, \widetilde{Y}(s)) - \widetilde{f}(s, \widetilde{Y}(s))\|^p ds \right)
\end{aligned}$$

$$
\begin{aligned}
&+ 3^{2p-2}\left(\int_{-\infty}^{t} e^{-\frac{b}{p-1}(t-s)}ds\right)^{p-1}\cdot\left(\int_{-\infty}^{t} e^{b(t-s)}\mathbb{E}\|[U(t+s_n,s+s_n)-\widetilde{U}(t,s)]\widetilde{f}(s,\widetilde{Y}(s))\|^p ds\right)\\
&\le 3^{2p-2}LM^p\delta^{-p}\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_n(s)-\widetilde{Y}(s)\|^2+\mathcal{E}_1^n(t),
\end{aligned}
$$

where $b\in(0,p\delta)$ is some constant and

$$
\begin{aligned}
\mathcal{E}_1^n(t) &= 3^{2p-2}M^p\delta^{1-p}\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|f(s+s_n,\widetilde{Y}(s))-\widetilde{f}(s,\widetilde{Y}(s))\|^p ds\\
&\quad + 3^{2p-2}[(p-1)b^{-1}]^{p-1}\int_{-\infty}^{t} e^{b(t-s)}\mathbb{E}\|[U(t+s_n,s+s_n)-\widetilde{U}(t,s)]\widetilde{f}(s,\widetilde{Y}(s))\|^p ds.
\end{aligned}
$$

By (3.3) and Lebesgue dominated convergence theorem, one has

$$
\lim_{n\to\infty}\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|f(s+s_n,\widetilde{Y}(s))-\widetilde{f}(s,\widetilde{Y}(s))\|^p ds = 0. \tag{3.8}
$$

On the other hand, by (3.3) and the fact that $\sup_{s\in\mathbb{R}}\mathbb{E}\|f(s+s_n,\widetilde{Y}(s))\|^p<+\infty$, one has

$$
\mathbb{E}\|\widetilde{f}(s,\widetilde{Y}(s))\|^p \le 2^{p-1}\mathbb{E}\|f(s+s_n,\widetilde{Y}(s))-\widetilde{f}(s,\widetilde{Y}(s))\|^p+2^{p-1}\mathbb{E}\|f(s+s_n,\widetilde{Y}(s))\|^p\le M_f, \tag{3.9}
$$

where $M_f>0$ is a constant. By (H_2) and (3.7), one has

$$
\begin{aligned}
&\mathbb{E}\left\|[U(t+s_n,s+s_n)-\widetilde{U}(t,s)]\widetilde{f}(s,\widetilde{Y}(s))\right\|^p\\
&\le 2^{p-1}\mathbb{E}\left\|U(t+s_n,s+s_n)\widetilde{f}(s,\widetilde{Y}(s))\right\|^p+2^{p-1}\mathbb{E}\left\|\widetilde{U}(t,s)\widetilde{f}(s,\widetilde{Y}(s))\right\|^p\\
&\le 2^{p-1}M^p e^{-p\delta(t-s)}\mathbb{E}\|\widetilde{f}(s,\widetilde{Y}(s))\|^p+2^p\cdot 2^{p-1}M^p e^{-p\delta(t-s)}\mathbb{E}\|\widetilde{f}(s,\widetilde{Y}(s))\|^p\\
&\le 2^{p-1}M^p(1+2^p)M_f e^{-p\delta(t-s)},\quad \text{for } t\ge s,
\end{aligned}
$$

which implies that

$$
e^{b(t-s)}\mathbb{E}\left\|[U(t+s_n,s+s_n)-\widetilde{U}(t,s)]\widetilde{f}(s,\widetilde{Y}(s))\right\|^p\in L^1(-\infty,t).
$$

then, from the Lebesgue dominated convergence theorem, we obtain

$$
\lim_{n\to\infty}\int_{-\infty}^{t} e^{b(t-s)}\mathbb{E}\left\|[U(t+s_n,s+s_n)-\widetilde{U}(t,s)]\widetilde{f}(s,\widetilde{Y}(s))\right\|^p ds = 0. \tag{3.10}
$$

Hence, by (3.8) and (3.10), one has $\mathcal{E}_1^n(t)\to 0$ as $n\to\infty$.

- $\mathcal{J}_2$.

By Hölder inequality and [28, Lemma 2.2], one has

$$
\begin{aligned}
\mathcal{J}_2 &\le 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} U(t+s_n,\sigma+s_n)\int_{-\infty}^{\sigma} K_1(\sigma-s)[g(s+s_n,Y_n(s))-g(s+s_n,\widetilde{Y}(s))]dW(s)d\sigma\right\|^p\\
&\quad + 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} U(t+s_n,\sigma+s_n)\int_{-\infty}^{\sigma} K_1(\sigma-s)[g(s+s_n,\widetilde{Y}(s))-\widetilde{g}(s,\widetilde{Y}(s))]dW(s)d\sigma\right\|^p\\
&\quad + 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} [U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^{\sigma} K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma\right\|^p
\end{aligned}
$$

$$\leq 3^{2p-2}M^p\delta^{1-p}C_p\left(trQ\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\mathbb{E}\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)(g(s+s_n,Y_n(s))-g(s+s_n,\widetilde{Y}(s)))\|^2_{L(H,H)}ds\right]^{p/2}d\sigma\right)$$
$$+3^{2p-2}M^p\delta^{1-p}C_p\left(trQ\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\mathbb{E}\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)(g(s+s_n,\widetilde{Y}(s))-\widetilde{g}(s,\widetilde{Y}(s)))\|^2_{L(H,H)}ds\right]^{p/2}d\sigma\right)$$
$$+3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t}[U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma\right\|^p$$
$$\leq 3^{2p-2}M^p\delta^{1-p}C_p(trQ)^{p/2}\cdot\left(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2ds\right]^{p/2-1}\right.$$
$$\left.\times\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2\cdot\mathbb{E}\|(g(s+s_n,Y_n(s))-g(s+s_n,\widetilde{Y}(s)))\|^p_{L(H,H)}ds\right]d\sigma\right)$$
$$+3^{2p-2}M^p\delta^{1-p}C_p(trQ)^{p/2}\cdot\left(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2ds\right]^{p/2-1}\right.$$
$$\left.\times\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2\cdot\mathbb{E}\|(g(s+s_n,\widetilde{Y}(s))-\widetilde{g}(s,\widetilde{Y}(s)))\|^p_{L(H,H)}ds\right]d\sigma\right)$$
$$+3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t}[U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma\right\|^p$$
$$\leq 3^{2p-2}LM^p\delta^{-p}C_p(trQ)^{p/2}\|K_1\|^{p/2}_{L^2}\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_n(s)-\widetilde{Y}(s)\|^p+\mathcal{E}_2^n(t),$$

where

$$\mathcal{E}_2^n(t)=3^{2p-2}M^p\delta^{1-p}C_p(trQ)^{p/2}\|K_1\|^{p/2-1}_{L^2}$$
$$\times\left(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2\cdot\mathbb{E}\|(g(s+s_n,\widetilde{Y}(s))-\widetilde{g}(s,\widetilde{Y}(s)))\|^p_{L(H,H)}ds\right]d\sigma\right)$$
$$+3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t}[U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma\right\|^p.$$

Since $K_1\in L^2(0,+\infty)$ and $g\in SAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,L(H,H)))$, by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\left[\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2\cdot\mathbb{E}\|(g(s+s_n,\widetilde{Y}(s))-\widetilde{g}(s,\widetilde{Y}(s)))\|^p_{L(H,H)}ds\right]d\sigma=0. \tag{3.11}$$

On the other hand,

$$\mathbb{E}\left\|\int_{-\infty}^{t}[U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma\right\|^p$$
$$=\mathbb{E}\left\|\int_{-\infty}^{t}e^{-\frac{\omega}{p}(t-\sigma)}e^{\frac{\omega}{p}(t-\sigma)}\left(U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)\right)\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)d\sigma\right\|^p$$
$$\leq\left(\int_{-\infty}^{t}e^{-\frac{1}{p-1}\omega(t-\sigma)}d\sigma\right)^{p-1}\left(\int_{-\infty}^{t}e^{\omega(t-\sigma)}\mathbb{E}\left\|(U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma))\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)\right\|^pd\sigma\right)$$
$$\leq[(p-1)\omega^{-1}]^{p-1}\left(\int_{-\infty}^{t}e^{\omega(t-\sigma)}\mathbb{E}\left\|(U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma))\int_{-\infty}^{\sigma}K_1(\sigma-s)\widetilde{g}(s,\widetilde{Y}(s))dW(s)\right\|^pd\sigma\right)$$

where $\omega\in(0,p\delta)$ is a constant. Similar as the proof of (3.9), one has $\mathbb{E}\|\widetilde{g}(s,\widetilde{Y}(s))\|^p_{L(H,H)}\leq M_g$, where $M_g>0$

is a constant. Let

$$\eta(\sigma) = \int_{-\infty}^{\sigma} K_1(\sigma - s)\widetilde{g}(s, \widetilde{Y}(s))dW(s).$$

By Hölder inequality, one has

$$\begin{aligned}
\mathbb{E}\|\eta(\sigma)\|^p &\leq (trQ)^{p/2}\mathbb{E}\left[\int_{-\infty}^{\sigma} \|K_1(\sigma - s)\widetilde{g}(s, \widetilde{Y}(s))\|^2_{L(H,H)} ds\right]^{p/2} \\
&\leq (trQ)^{p/2}\|K_1\|_{L^2}^{p/2-1} \cdot \left[\int_{-\infty}^{\sigma} \|K_1(\sigma - s)\|^2 \cdot \mathbb{E}\|\widetilde{g}(s, \widetilde{Y}(s))\|^p_{L(H,H)} ds\right] \\
&\leq (trQ)^{p/2}\|K_1\|_{L^2}^{p/2} M_g.
\end{aligned}$$

Hence, by (3.7), one has

$$\begin{aligned}
&\mathbb{E}\left\|(U(t + s_n, \sigma + s_n) - \widetilde{U}(t, \sigma)) \int_{-\infty}^{\sigma} K_1(\sigma - s)\widetilde{g}(s, \widetilde{Y}(s))dW(s)\right\|^p \\
&\leq 2^{p-1}\mathbb{E}\left\|U(t + s_n, \sigma + s_n)\eta(\sigma)\right\|^p + 2^{p-1}\mathbb{E}\left\|\widetilde{U}(t, \sigma)\eta(\sigma)\right\|^p \\
&\leq 2^{p-1}(1 + 2^p)M^p(trQ)^{p/2}\|K_1\|_{L^2}^{p/2} M_g e^{-p\delta(t-\sigma)}, \quad \text{for } t \geq \sigma,
\end{aligned}$$

which implies that

$$e^{\omega(t-\sigma)}\mathbb{E}\left\|(U(t + s_n, \sigma + s_n) - \widetilde{U}(t, \sigma)) \int_{-\infty}^{\sigma} K_1(\sigma - s)\widetilde{g}(s, Y(s))dW(s)\right\|^p \in L^1(-\infty, t). \tag{3.12}$$

from the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{-\infty}^{t} e^{\omega(t-\sigma)}\mathbb{E}\left\|(U(t + s_n, \sigma + s_n) - \widetilde{U}(t, \sigma)) \int_{-\infty}^{\sigma} K_1(\sigma - s)\widetilde{g}(s, Y(s))dW(s)\right\|^p d\sigma = 0. \tag{3.13}$$

Hence, by (3.11) and (3.13), one has $\mathcal{E}_2^n(t) \to 0$ as $n \to \infty$.

- $\mathcal{J}_3$.

By Hölder inequality, one has

$$\begin{aligned}
\mathcal{J}_3 &\leq 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} U(t + s_n, \sigma + s_n) \int_{-\infty}^{\sigma} K_2(\sigma - s)[h(s + s_n, Y_n(s)) - h(s + s_n, \widetilde{Y}(s))]dsd\sigma\right\|^p \\
&\quad + 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} U(t + s_n, \sigma + s_n) \int_{-\infty}^{\sigma} K_2(\sigma - s)[h(s + s_n, \widetilde{Y}(s)) - \widetilde{h}(s, \widetilde{Y}(s))]dsd\sigma\right\|^p \\
&\quad + 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} [U(t + s_n, \sigma + s_n) - \widetilde{U}(t, \sigma)] \int_{-\infty}^{\sigma} K_2(\sigma - s)\widetilde{h}(s, \widetilde{Y}(s))dsd\sigma\right\|^p \\
&\leq 3^{2p-2}M^p\delta^{1-p}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^{\sigma} K_2(\sigma - s)[h(s + s_n, Y_n(s)) - h(s + s_n, \widetilde{Y}(s))]ds\right\|^p d\sigma\right) \\
&\quad + 3^{2p-2}M^p\delta^{1-p}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^{\sigma} K_2(\sigma - s)[h(s + s_n, \widetilde{Y}(s)) - \widetilde{h}(s, \widetilde{Y}(s))]ds\right\|^p d\sigma\right) \\
&\quad + 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^{t} [U(t + s_n, \sigma + s_n) - \widetilde{U}(t, \sigma)] \int_{-\infty}^{\sigma} K_2(\sigma - s)\widetilde{h}(s, \widetilde{Y}(s))dsd\sigma\right\|^p \\
&\leq 3^{2p-2}LM^p\delta^{-p}\|K_2\|_{L^1}^p \cdot \sup_{s\in\mathbb{R}} \mathbb{E}\|Y_n(s) - \widetilde{Y}(s)\|^p + \mathcal{E}_3^n(t),
\end{aligned}$$

where

$$\mathcal{E}_3^n(t) = 3^{2p-2}M^p\delta^{1-p}\|K_2\|_{L^1}^{p-1}\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\left[\int_{-\infty}^\sigma \|K_2(\sigma-s)\|\cdot\mathbb{E}\|h(s+s_n,\widetilde{Y}(s))-\widetilde{h}(s,\widetilde{Y}(s))\|^p ds\right]d\sigma\right)$$
$$+ 3^{2p-2}\mathbb{E}\left\|\int_{-\infty}^t [U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^\sigma K_2(\sigma-s)\widetilde{h}(s,\widetilde{Y}(s))dsd\sigma\right\|^p.$$

Since $K_2 \in L^1(0,+\infty)$ by (H_4) and (3.5), by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_{-\infty}^t e^{-\delta(t-\sigma)}\left[\int_{-\infty}^\sigma \|K_2(\sigma-s)\|\cdot\mathbb{E}\|h(s+s_n,\widetilde{Y}(s))-\widetilde{h}(s,\widetilde{Y}(s))\|^p ds\right]d\sigma = 0. \tag{3.14}$$

On the other hand, by Hölder inequality,

$$\mathbb{E}\left\|\int_{-\infty}^t [U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma)]\int_{-\infty}^\sigma K_2(\sigma-s)\widetilde{h}(s,\widetilde{Y}(s))dsd\sigma\right\|^p$$
$$\le [(p-1)\omega^{-1}]^{p-1}\left(\int_{-\infty}^t e^{\omega(t-\sigma)}\mathbb{E}\left\|(U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma))\int_{-\infty}^\sigma K_2(\sigma-s)\widetilde{h}(s,\widetilde{Y}(s))ds\right\|^p d\sigma\right)$$

where $\omega \in (0, p\delta)$ is a constant. Similar as the proof of (3.13), one has

$$\lim_{n\to\infty}\int_{-\infty}^t e^{\omega(t-\sigma)}\mathbb{E}\left\|(U(t+s_n,\sigma+s_n)-\widetilde{U}(t,\sigma))\int_{-\infty}^\sigma K_2(\sigma-s)\widetilde{h}(s,\widetilde{Y}(s))ds\right\|^p d\sigma = 0. \tag{3.15}$$

Hence, by (3.14) and (3.15), one has $\mathcal{E}_3^n(t) \to 0$ as $n \to \infty$.

By the estimates of $\mathcal{J}_1$-$\mathcal{J}_3$, one has

$$\mathbb{E}\|Y_n(t)-\widetilde{Y}(t)\|^p \le \mathcal{E}^n(t) + \vartheta\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_n(s)-\widetilde{Y}(s)\|^p,$$

where $\mathcal{E}^n(t) = \sum_{i=1}^3 \mathcal{E}_i^n(t)$. Hence

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|Y_n(t)-\widetilde{Y}(t)\|^p \le \sup_{t\in\mathbb{R}}\mathcal{E}^n(t) + \vartheta\cdot\sup_{s\in\mathbb{R}}\mathbb{E}\|Y_n(s)-\widetilde{Y}(s)\|^p.$$

By $\vartheta < 1$ and $\lim\limits_{n\to\infty}\sup\limits_{t\in\mathbb{R}}\mathcal{E}^n(t) = 0$, it follows that

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|Y_n(t)-\widetilde{Y}(t)\|^p \to 0 \quad \text{as} \quad n\to\infty.$$

Since $Y(t+s_n)$ has the same distribution as $Y_n(t)$, by [22], one has $Y(t+s_n) \to \widetilde{Y}(t)$ in p-distribution as $n\to\infty$. Similarly, we have $\widetilde{Y}(t-s_n) \to Y(t)$ in p-distribution as $n\to\infty$. Hence Y is almost automorphic in one-dimensional p-distribution.

Note that the sequence $(\|Y_n(t)\|^p)$ is uniformly integrable, thus $(\|Y(t+s_n)\|^p)$ is also uniformly integrable, so the family $(\|Y(t)\|^p)_{t\in\mathbb{R}}$ is uniformly integrable. Next, we prove that Y is almost automorphic in p-distribution. For fixed $\tau\in\mathbb{R}$, let $\xi_n = Y(\tau+s_n)$, $f_n(t,Y) = f(t+s_n,Y)$, $g_n(t,Y) = g(t+s_n,Y)$, $h_n(t,Y) = h(t+s_n,Y)$. By the foregoing, (ξ_n) converges in p-distribution to some variable $Y(\tau)$. We deduce that (ξ_n) is tight, so (ξ_n, W) is tight also. We can thus choose a subsequence (still noted s_n for simplicity) such that (ξ_n, W) converges in p-distribution to $(Y(\tau), W)$. Similar as the proof of [12, Properties 3.1], for every $T\ge\tau$, $Y(\cdot+s_n)$ converges in p-distribution on $C([\tau,T],\mathcal{L}^p(P,H))$ to the (unique in p-distribution) solution to

$$Y(t) = U(t,\tau)Y(\tau) + \int_\tau^t U(t,s)f(s,Y(s))ds + \int_\tau^t U(t,\sigma)\int_\tau^\sigma K_1(\sigma-s)g(s,Y(s))dW(s)d\sigma$$

$$+\int_{\tau}^{t} U(t,\sigma)\int_{\tau}^{\sigma} K_2(\sigma-s)h(s,Y(s))dsd\sigma,$$

Note that Y does not depend on the chosen interval $[\tau,T]$, thus the convergence takes place on $C(\mathbb{R},\mathcal{L}^p(P,H))$. Similarly, one has that $Y_n := Y(\cdot - s_n)$ converges in p-distribution on $C(\mathbb{R},\mathcal{L}^p(P,H))$. Hence Y is almost automorphic in p-distribution. The proof is complete. □

3.2. (μ,ν)-pseudo almost automorphy in p-distribution

In this subsection, we assume that $g \in SPAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,L(H,H)),\mu,\nu)$, $f,h \in SPAA(\mathbb{R}\times\mathcal{L}^p(P,H),\mathcal{L}^p(P,H),\mu,\nu)$ and and study the existence, uniqueness of (μ,ν)-pseudo almost automorphic in p-distribution mild solution to (1.1).

Lemma 3.1. *If (A_2), (H_4) hold and $\phi \in PAA_0(\mathbb{R},\mathbb{R},\mu,\nu)$, then the functions*

$$\mathcal{M}_1(t) = \int_{-\infty}^{t} e^{-\delta(t-s)}\phi(s)ds \in PAA_0(\mathbb{R},\mathbb{R},\mu,\nu),$$
$$\mathcal{M}_2(t) = \int_{-\infty}^{t} e^{-\delta(t-\sigma)}\int_{-\infty}^{\sigma} \|K_1(\sigma-s)\|\phi(s)dsd\sigma \in PAA_0(\mathbb{R},\mathbb{R},\mu,\nu),$$
$$\mathcal{M}_3(t) = \int_{-\infty}^{t} e^{-\delta(t-\sigma)}\int_{-\infty}^{\sigma} \|K_2(\sigma-s)\|\phi(s)dsd\sigma \in PAA_0(\mathbb{R},\mathbb{R},\mu,\nu).$$

Proof. By Fubini theorem, one has

$$\begin{aligned}&\frac{1}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\phi(s)ds\right)d\mu(t)\\ &= \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{0}^{+\infty} e^{-\delta s}\phi(t-s)dsd\mu(t) = \int_{0}^{+\infty} e^{-\delta s}\Phi_r(s)ds,\end{aligned}$$

where $\Phi_r(s) = \frac{1}{\nu([-r,r])}\int_{[-r,r]}\phi(t-s)d\mu(t)$. Since $\lim_{r\to+\infty}\Phi_r(s) = 0$ for all $s\in\mathbb{R}$ and using Lebesgue dominated convergence theorem, we have

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\phi(s)ds\right)d\mu(t) = 0.$$

So $\mathcal{M}_1 \in PAA_0(\mathbb{R},\mathbb{R},\mu,\nu)$. Similarly, one has $\mathcal{M}_i \in PAA_0(\mathbb{R},\mathbb{R},\mu,\nu)$, $i = 2,3$. □

Theorem 3.2. *Let (A_1)-(A_2) and (H_1)-(H_6) be satisfied, then (1.1) has a unique mild solution X being (μ,ν)-pseudo almost automorphic in p-distribution if $\vartheta < 1$. More precisely, let $Y \in SBC(\mathbb{R},\mathcal{L}^p(P,H))$ be the unique almost automorphic in p-distribution mild solution to*

$$\begin{aligned}Y'(t) = A(t)Y(t) + f_1(t,Y(t)) + + \int_{-\infty}^{t} K_1(t-s)g_1(s,Y(s))dW(s)&\\ + \int_{-\infty}^{t} K_2(t-s)h_1(s,Y(s))ds, \qquad t\in\mathbb{R},&\end{aligned} \tag{3.16}$$

then X has the decomposition:

$$X = Y + Z, \qquad Z \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu).$$

Proof. The proof of existence and properties of Y can be seen in the proof of Theorem 3.1. The existence and uniqueness of the mild solution X to (1.1) are proved as in Theorem 3.1, using the classical method of the fixed point theorem for the contractive operator $\mathcal{F}$ on $SBC(\mathbb{R}, \mathcal{L}^p(P,H))$ defined by

$$\begin{aligned}\mathcal{F}X(t) = &\int_{-\infty}^{t} U(t,s)f(s,X(s))ds + \int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_1(\sigma-s)g(s,X(s))dW(s)d\sigma \\ &+ \int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_2(\sigma-s)h(s,X(s))dsd\sigma. \end{aligned}\tag{3.17}$$

The solution X is the limit in $SBC(\mathbb{R}, \mathcal{L}^p(P,H))$ of a sequence (X_n) with arbitrary X_0 and for $n \in \mathbb{N}$, $X_{n+1} = \mathcal{F}(X_n)$. To prove that X is (μ,ν)-pseudo almost automorphic in p-distribution, we choose a special sequence. Set

$$X_0 = Y, \;\; X_{n+1} = \mathcal{F}(X_n), \;\; Z_n = X_n - Y, \;\; n \in \mathbb{N}.$$

Let us show that $Z_n \in SPAA_0(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$ for $n \in \mathbb{N}$, and prove it by induction. In fact, assume that $Z_n \in SPAA_0(\mathbb{R}, \mathcal{L}^p(P,H), \mu, \nu)$, then for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\begin{aligned} Z_{n+1} &= \mathcal{F}X_n(t) - Y(t) \\ &= \Big[\int_{-\infty}^{t} U(t,s)[f(s,X_n(s)) - f(s,Y(s))]ds \\ &\quad + \int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_1(\sigma-s)[g(s,X_n(s)) - g(s,Y(s))]dW(s)d\sigma \\ &\quad + \int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_2(\sigma-s)[h(s,X_n(s)) - h(s,Y(s))]dsd\sigma\Big] \\ &\quad + \Big[\int_{-\infty}^{t} U(t,s)f_2(s,Y(s))ds + \int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_1(\sigma-s)g_2(s,Y(s))dW(s)d\sigma \\ &\quad + \int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_2(\sigma-s)h_2(s,Y(s))dsd\sigma\Big] \\ &:= \mathcal{H}_1(t) + \mathcal{H}_2(t). \end{aligned}$$

By (H_5), one has $\mathbb{E}\|f(t,X_n(t)) - f(t,Y(t))\|^p \le L \cdot \mathbb{E}\|Z_n(t)\|^p$, so the mapping

$$\mathfrak{f} : t \to \mathbb{E}\|f(t,X_n(t)) - f(t,Y(t))\|^p$$

lies in $PAA_0(\mathbb{R}, \mathbb{R}, \mu, \nu)$. The same conclusions hold for

$$\begin{aligned} &\mathfrak{g} : t \to \mathbb{E}\|g(t,X_n(t)) - g(t,Y(t))\|^p_{L(H,H)}, \\ &\mathfrak{h} : t \to \mathbb{E}\|h(t,X_n(t)) - h(t,Y(t))\|^p. \end{aligned}$$

By Lemma 3.1 and $\mathfrak{f}, \mathfrak{g}, \mathfrak{h} \in PAA_0(\mathbb{R}, \mathbb{R}, \mu, \nu)$, one has

$$\begin{aligned} &\frac{1}{\nu([-r,r])}\int_{[-r,r]} \mathbb{E}\left\|\int_{-\infty}^{t} U(t,s)[f(s,X_n(s)) - f(s,Y(s))]ds\right\|^p d\mu(t) \\ &\le \frac{M^p\delta^{1-p}}{\nu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t} e^{-\delta(t-s)}\mathfrak{f}(s)dsd\mu(t) \to 0 \quad \text{as} \;\; r \to +\infty, \end{aligned}$$

and

$$\frac{1}{\nu([-r,r])}\int_{[-r,r]} \mathbb{E}\left\|\int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_1(\sigma-s)[g(s,X_n(s)) - g(s,Y(s))]dW(s)d\sigma\right\|^p d\mu(t)$$

$$\leq \frac{M^p\delta^{1-p}}{\nu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^{\sigma} K_1(\sigma-s)[g(s,X_n(s))-g(s,Y(s))]dW(s)\right\|^p d\sigma d\mu(t)$$
$$\leq \frac{M^p\delta^{1-p}C_p(trQ)^{p/2}\|K_1\|_{L^2}^{p/2-1}}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\int_{-\infty}^{\sigma}\|K_1(\sigma-s)\|^2\mathfrak{g}(s)dsd\sigma\right)d\mu(t)\to 0 \text{ as } r\to+\infty,$$

and

$$\frac{1}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\left\|\int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_2(\sigma-s)[h(s,X_n(s))-h(s,Y(s))]dsd\sigma\right\|^p d\mu(t)$$
$$\leq \frac{M^p\delta^{1-p}\|K_2\|_{L^1}^{p-1}}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\int_{-\infty}^{\sigma}\|K_2(\sigma-s)\|\mathfrak{h}(s)dsd\sigma\right)d\mu(t)\to 0 \text{ as } r\to+\infty.$$

Thus, $\mathcal{H}_1 \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$.

Next, we show that $\mathcal{H}_2 \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$. Since Y is almost automorphic in p-distribution, $(\widetilde{Y}(t)) = (Y(t+\cdot))_{t\in\mathbb{R}}$ is uniformly tight in $C_k(\mathbb{R},\mathcal{L}^p(P,H))$, where $C_k(\mathbb{R},\mathcal{L}^p(P,H))$ is the space endowed with the topology of uniform convergence on compact subsets of $\mathcal{L}^p(P,H)$. Hence, for $\varepsilon>0$, there exists a compact subset K_ε of $C_k(\mathbb{R},\mathcal{L}^p(P,H))$ such that for $t\in\mathbb{R}$

$$P(\widetilde{Y}(t)\in K_\varepsilon)\geq 1-\varepsilon.$$

By the Arzelà-Ascoli theorem, for every $\varepsilon>0$ and for every compact interval I of $\mathbb{R}$, there exists a compact subset $K_{\varepsilon,I}$ of $\mathcal{L}^p(P,H)$ such that for $t\in\mathbb{R}$,

$$P(Y(t+s)\in K_{\varepsilon,I},\ \forall s\in I)\geq 1-\varepsilon.$$

In particular, $(Y(t))_{t\in\mathbb{R}}$ is tight, i.e., let $K_\varepsilon := K_{\varepsilon,\{0\}}$, for $t\in\mathbb{R}$, one has $P(Y(t)\in K_{\varepsilon,I})\geq 1-\varepsilon$. Since f_2 is uniform continuous on $\mathbb{R}\times K_\varepsilon$, there exists $\eta(\varepsilon)>0$ such that for all $x,y\in K_\varepsilon$,

$$\mathbb{E}\|x-y\|^p\leq\eta(\varepsilon)\Rightarrow\sup_{t\in\mathbb{R}}\mathbb{E}\|f_2(t,x)-f_2(t,y)\|^p\leq\varepsilon.$$

We can find a finite sequence $y_1,y_2,\ldots y_m$ such that $K_\varepsilon\subset\bigcup_{i=1}^{m}B(y_i,\eta(\varepsilon))$, where $B(y_i,\eta(\varepsilon))$ is the open balls with center y_i and radius less than $\eta(\varepsilon)$. Similar as the proof of [32], Y is bounded in $\mathcal{L}^p(P,H)$, then

$$\begin{aligned}\mathbb{E}\|f_2(t,Y(t))\|^p&\leq 2^{p-1}\mathbb{E}\|f_2(t,Y(t))-f_2(t,0))\|^p+2^{p-1}\mathbb{E}\|f_2(t,0))\|^p\\&\leq 2^{p-1}L\cdot\mathbb{E}\|Y(t)\|^p+2^{p-1}\mathbb{E}\|f_2(t,0))\|^p<+\infty,\end{aligned}$$

that is $f_2(\cdot,Y(\cdot))$ is bounded in $\mathcal{L}^p(P,H)$. Clearly, $f_2(\cdot,Y(\cdot))$ is in $SBC(\mathbb{R},\mathcal{L}^p(P,H))$. By the uniform integrability of $(\|Y(t)\|^p)_{t\in\mathbb{R}}$ and (H_5) holds, one has the uniform integrability of $(\|f_2(t,Y(t))\|^p)_{t\in\mathbb{R}}$. Hence, let $\kappa>0$, we can choose ε small enough such that, for any measurable A such that $P(A)<\varepsilon$ and $\sup_{t\in\mathbb{R}}\mathbb{E}(\mathbf{1}_A\|f_2(t,Y(t))\|^p)<\kappa$. Let Ω_ε be the measurable set on which $Y(t)\in K_\varepsilon$, one has

$$\begin{aligned}\mathbb{E}\|f_2(t,Y(t))\|^p&\leq 3^{p-1}\max_{1\leq i\leq m}(\mathbb{E}(\mathbf{1}_{\Omega_\varepsilon}\|f_2(t,Y(t))-f_2(t,y_i)\|^p))+3^{p-1}\max_{1\leq i\leq m}\mathbb{E}\|f_2(t,y_i)\|^p\\&\quad+3^{p-1}\mathbb{E}\left(\mathbf{1}_{\Omega_\varepsilon^c}\|f_2(t,Y(t))\|^p\right)\\&\leq 3^{p-1}\varepsilon+3^{p-1}\max_{1\leq i\leq m}\mathbb{E}\|f_2(t,y_i)\|^p+3^{p-1}\kappa.\end{aligned}$$

By the ergodicity of $f_2(t,y_i)$ for all $i\in\{1,2,\ldots,m\}$, hence $t\to\mathbb{E}\|f_2(t,Y(t))\|^p$ is in $SPAA_0(\mathbb{R},\mathbb{R},\mu,\nu)$. By Lemma 3.1, one has

$$\frac{1}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\left\|\int_{-\infty}^{t} U(t,s)f_2(s,Y(s))ds\right\|^p d\mu(t)$$

$$\leq \frac{M^p}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)^{p-1}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|f_2(s,Y(s))\|^p ds\right)d\mu(t)$$
$$\leq \frac{M^p\delta^{1-p}}{\nu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|f_2(s,Y(s))\|^p ds\right)d\mu(t) \to 0 \text{ as } r\to+\infty.$$

Similarly, one has $t \to \mathbb{E}\|g_2(t,Y(t))\|^p_{L(H,H)}$, $t \to \mathbb{E}\|h_2(t,Y(t))\|^p$ are in $SPAA_0(\mathbb{R},\mathbb{R},\mu,\nu)$ and

$$\frac{1}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\left\|\int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_1(\sigma-s)g_2(s,Y(s))dW(s)d\sigma\right\|^p d\mu(t) \to 0 \text{ as } r\to+\infty.$$
$$\frac{1}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\left\|\int_{-\infty}^{t} U(t,\sigma)\int_{-\infty}^{\sigma} K_2(\sigma-s)h_2(s,Y(s))dsd\sigma\right\|^p d\mu(t) \to 0 \text{ as } r\to+\infty.$$

Thus, $\mathcal{H}_2 \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$. So, we have proved $Z_{n+1} \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$. We deduce by induction that the sequence (Z_n) lies in $SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$.

Now, the sequence (X_n) convergences to X in $SBC(\mathbb{R},\mathcal{L}^p(P,H))$, hence (Z_n) convergences to $Z := X - Y$ in $SBC(\mathbb{R},\mathcal{L}^p(P,H))$, then let $\varepsilon > 0$ and n large enough such that $\sup_{t\in\mathbb{R}}(\mathbb{E}\|Z(t)-Z_n(t)\|^p) \leq \varepsilon$. Since $Z_n \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$ and

$$\begin{aligned}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\|Z(t)\|^p d\mu(t) &\leq \frac{2^{p-1}}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\|Z(t)-Z_n(t)\|^p d\mu(t)\\ &\quad+\frac{2^{p-1}}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\|Z_n(t)\|^p d\mu(t)\\ &\leq \frac{2^{p-1}\varepsilon\mu([-r,r])}{\nu([-r,r])}+\frac{2^{p-1}}{\nu([-r,r])}\int_{[-r,r]}\mathbb{E}\|Z_n(t)\|^p d\mu(t),\end{aligned}$$

which implies that $Z \in SPAA_0(\mathbb{R},\mathcal{L}^p(P,H),\mu,\nu)$. The proof is complete. □

4. Example

Consider the stochastic integro-differential equations:

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t} = \left[\dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)\sin\dfrac{1}{2+\sin t+\sin\pi t}\right] + f(t,u(t,x))\\ \qquad + \displaystyle\int_{-\infty}^{t} e^{-\omega(t-s)}g(s,u(s,x))dW(s) + \int_{-\infty}^{t} e^{-\omega(t-s)}h(s,u(s,x))ds, \quad (t,x)\in\mathbb{R}\times(0,1),\\ u(t,0)=u(t,1)=0, \quad t\in\mathbb{R},\end{cases} \tag{4.1}$$

where $W(t)$ is a standard two-sided and one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}},P)$. Let

$$\mathcal{D}(A) = \{x \in C^1[0,1] : x'(r) \text{ is absolutely continuous on } [0,1], x''(r) \in \mathcal{L}^2[0,1], x(0)=x(1)=0\},$$

then A generates a C_0-semigroup $(T(t))_{t\geq 0}$, which is given by $[T(t)x](r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x,e_n\rangle_{\mathcal{L}^2}e_n(r)$ where $e_n(r) = \sqrt{2}\sin(n\pi r)$ for $n = 1,2,\ldots$ and $\|T(t)\| \leq e^{-\pi^2 t}$ for $t \geq 0$.

Define a family of linear operators $A(t)$ by

$$\begin{cases}\mathcal{D}(A(t)) = \mathcal{D}(A), \quad t\in\mathbb{R}\\ A(t)x = \left(A + \sin\dfrac{1}{2+\sin t+\sin\pi t}\right)x, \quad x\in\mathcal{D}(A(t)).\end{cases}$$

Hence $A(t)$ generates an evolution family $(U(t,s))_{t\geq s}$ such that $U(t,s)x = T(t-s)e^{\int_s^t \sin\frac{1}{2+\sin\tau+\sin\pi\tau}d\tau}x$, therefore $\|U(t,s)\| \leq e^{-(\pi^2-1)(t-s)}$ for $t \geq s$, i.e., (H_1), (H_2) hold with $M = 1$, $\delta = \pi^2 - 1$.

Note that

$$\begin{aligned} U(t+s_n, s+s_n)y &= T(t-s)e^{\int_{s+s_n}^{t+s_n} \sin\frac{1}{2+\sin\tau+\sin\pi\tau}d\tau}y \\ &= T(t-s)e^{\int_s^t \sin\frac{1}{2+\sin(\tau+s_n)+\sin\pi(\tau+s_n)}d\tau}y \end{aligned}$$

uniformly for all y in any bounded subset of $\mathcal{L}^2(P, \mathcal{L}^2[0,1])$, so (H_3) holds.

Let $\mu = \nu$ and suppose that its Radon-Nikodym derivative is given by

$$\rho(t) = \begin{cases} e^t, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0, \end{cases}$$

then $\mu, \nu \in \mathcal{M}$ and satisfy (A_1), (A_2) [6]. It is not difficult to see that (H_4) holds. Assume that (H_5)-(H_6) hold and $\vartheta = 9L(\pi^2-1)^{-2}\left(1 + \omega^{-1}trQ + \omega^{-2}\right) < 1$, by theorem 3.1, (4.1) has a unique solution which is (μ,ν)-pseudo almost automorphic in 2-distribution.

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