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Hypernear-Rings with a Defect of Distributivity

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Abstract. Since in a near-ring the distributivity holds just on one side (left or right), it seems naturally to study the behaviour and properties of the set of elements that "correct" the lack of distributivity, in other words that elements that assure the validity of the distributivity. The normal subgroup of the additive structure of a near-ring generated by these elements is called a defect of distributivity of the near-ring. The purpose of this note is to initiate the study of the hypernear-rings (generalizations of near-rings, having the additive part a quasicanonical hypergroup) with a defect of distributivity, making a comparison with similar properties known for near-rings.

1. Introduction

The interest in near-rings and near-fields started at the beginning of the 20th century, when L. Dickson wanted to know whether the list of axioms for skew fields is redundant or not. He found in [11] that there do exist "near-fields" which fulfill all axioms for skew fields except one distributive law. Since 1950, the theory of near-rings had applications to several domains, for instance in the area of dynamical systems, graphs, homological algebra, universal algebra, category theory, geometry, and so on. A comprehensive review of the theory of near-rings and its applications appears in Pilz [26], Meldrun [22], Clay [1], Wähling [31], Scott [28], Ferrero-Ferrero [12], Vuković [29], or Satyanarayana and Prasad [27].

In [7] Dašić introduced the notion of *hypernear-ring* as a generalization of a zero symmetric near-ring, i.e. a near-ring in which any element x satysfies the relation $x \cdot 0 = 0 \cdot x = 0$. On the other hand, the notion of hypernear-ring can be viewed as a generalization of a Krasner hyperring. The additive structure $(R, +)$ of a hypernear-ring is a quasicanonical hypergroup [21, 23] (called also polygroup, by Comer [2]), i.e. a non commutative canonical hypergroup, while the multiplicative one is a semigroup, having 0 as an absorbing element. In Dašić pionering definition, 0 is a bilaterally absorbing element, that is $0 \cdot x = x \cdot 0 = 0$, for any $x \in R$. This particular case of hypernear-ring was called, later, by Gontineac [14] a *zero symmetric hypernear-ring*, and he introduced and studied the concept of hypernear-ring in a general case, where 0 is just a right absorbing element: $x \cdot 0 = 0$, for any $x \in R$, while for some $y \in R$, it takes $0 \cdot y \neq 0$. He called this new hyperstructure a *(right) hypernear-ring*.

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The aim of this paper is to extend to the case of hypernear-rings the notion of *defect of distributivity*, that was studied by Dašić [5] in 1978 for the zero-symmetric (left) near-rings. As Dašić said in his paper [5], a defect of distributivity helps to soften the non-linearity presented in the theory of near-rings, since it can be mathematically translated as the collection of all elements $d \in R$ satisfying the relation $(x+y)\cdot s = x\cdot s+y\cdot s+d$, for all $x, y \in R$ and $s \in S$, where $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$ whose elements generate $(R, +)$. More exactly, the normal subgroup D of $(R, +)$ generated by these elements $d \in R$ is called a *defect of distributivity* of the near-ring R [5]. In particular, if $D = 0$, then R is a *distributively generated near-ring* [13], while in the opposite extremal case, when $D = R$, the near-ring R is zero-symmetric. In all other cases, we say that R is a *near-ring with the defect D*. Following this idea, in this note we introduce the concept of *hypernear-ring with a defect of distributivity*, and present several properties of this class of hypernear-rings, in connection with their direct product, hyperhomomorphisms, or factor hypernear-rings. Since this note is the first research on the above mentioned argument, we will present the results with all the details needed for a better understanding of the topic.

2. Preliminaries

For the sake of completeness of the paper, we recall in this section the basic properties of the near-rings with a defect of distibutivity and those connected with hypernear-rings.

2.1. Near-rings with a defect of distributivity

We keep the notation in Dašić [5]. Let $(R, +, \cdot)$ be a left near-ring, i.e. $(R, +)$ is a group (not necessarily commutative) with the unit element 0, $(R, \cdot)$ is a semigroup and the left distributivity holds: $x \cdot (y + z) = x\cdot y+x\cdot z$, for any $x, y, z \in R$. It is clear that $x\cdot 0 = 0$, for any $x \in R$, while it might exist $y \in R$ such that $0\cdot y \neq 0$. If 0 is a bilaterally absorbing element, that is $0 \cdot x = x \cdot 0 = 0$, for any $x \in R$, then R is called a zero-symmetric near-ring. The classical example of near-ring, that suggested its definition, is represented by the set of the functions from an additive group into itself with pointwise addition and natural composition of functions.

Throughout this section, by a near-ring we intend a zero-symmetric left near-ring.

Definition 2.1. [5] Let R be a near-ring. A set S of generators of R is a multiplicative subsemigroup $(S, \cdot)$ of the semigroup $(R, \cdot)$, whose elements generate $(R, +)$. The normal subgroup D of the group $(R, +)$ which is generated by the set $D_S = \{d \in R \mid d = -(x \cdot s + y \cdot s) + (x + y) \cdot s,\ \ x, y \in R, s \in S\}$ is called the *defect of distributivity* of the near-ring R.

In other words, if $s \in S$, then for all $x, y \in R$, there exists $d \in D$ such that $(x + y) \cdot s = x \cdot s + y \cdot s + d$. This expresses the fact that the elements of S are distributive with the defect D.

When we want to stress the set S of generators, we will denote the near-ring by the couple (R, S).

The main properties of this kind of near-rings are sumarized in the following results.

Theorem 2.2. *[5]*

i) Every homomorphic image of a near-ring with the defect D is a near-ring with the defect $f(D)$, when f is a homomorphism of near-rings.

ii) Every direct sum of a family of near-rings R_i with the defects D_i, respectively, is a near-ring whose defect is a direct sum of the defects D_i, for $i \in I$.

iii) Let R be a near-ring with the defect D and A be an ideal of R. The quotient near-ring R/A has the defect $\overline{D} = \{d + A \mid d \in D\}$. Moreover, R/A is distributively generated if and only if $D \subseteq A$.

2.2. Hypernear-rings: terminology and basic results

At the beginning of this section, we insist on the terminology used for this topic, since along the years, several terms have been used by different researchers in connection with the distributive property. Referring to the history of ring-like hyperstructures, the first one who introduced them was Krasner [18], defining what we call now *Krasner hyperring* as a hyperstructure R endowed with a hyperaddition "$+$" (such that $(R, +)$ is a canonical hypergroup) and a multiplication "$\cdot$" operation (such that $(R, \cdot)$ is a semigroup) which is distributive over the hyperaddition, meaning that, for any $x, y, z \in R$, the following equalities are valid:

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$
$$(x + y) \cdot z = x \cdot z + y \cdot z.$$

In 1973 Mittas [24] defined the *superrings* as hyperstructures having both parts additive and multiplicative as hyperstructures. Later on, Vougiouklis [30] generalized Mittas'superrings, introducing the *hyperrings in the general sense*, where again the addition and multiplication are hyperoperations (called also hyperaddition and hypermultiplication), but only the "weak version" of distributivity holds, i.e. for any $x, y, z \in R$, the following inclusions are valid:

$$x \cdot (y + z) \subseteq x \cdot y + x \cdot z,$$
$$(x + y) \cdot z \subseteq x \cdot z + y \cdot z.$$

Moreover, if the hypermultiplication is distributive over the hyperaddition, then the hyperring is called *good (or strong) hyperring in the general sense*. If only the additive part is a hypergroup, while the multiplicative one is a semigroup, and the weak distributivity holds, then we call R a *additive hyperring* (in particular, Krasner hyperrings are additive hyperrings); when the multiplication is distributive over the hyperaddition, then we get *good (or strong) additive hyperrings*. The additive hyperrings (when the multiplication is weakly distributive over the hyperoperation) are also called *hyperrings with inclusive distributivity*, cf. Jančić-Rašović and Dašić [15–17]. On the other hand, the same term *weak distributivity* is used by Davvaz [8] (and later on by other researchers) to define the validity of the following relations on R: for any $x, y, z \in R$,

$$x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \emptyset,$$
$$(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset.$$

More details about the terminology and history of hyperrings can be read in Nakasis [25], or Cristea, et al. [3, 4].

Combining near-rings and hyperrings, Dašić [7] defined *hypernear-rings*, as an algebraic system $(R, +, \cdot)$, where R is a non-empty set endowed with a hyperoperation "$+$" $: R \times R \longrightarrow \mathcal{P}^*(R)$, and an operation "$\cdot$" $: R \times R \longrightarrow R$, satisfying the following axioms:

I) $(R, +)$ is a quasicanonical hypergroup, i.e. it satisfies the following axioms:

   i) $x + (y + z) = (x + y) + z$, for any $x, y, z \in R$

   ii) there exists $0 \in R$ such that, for any $x \in R$, $x + 0 = 0 + x = \{x\}$

   iii) for any $x \in R$, there exists a unique element $-x \in R$, such that $0 \in x + (-x) \cap (-x) + x$

   iv) for any $x, y, z \in R$, $z \in x + y$ implies that $x \in z + (-y)$, $y \in (-x) + z$.

II) $(R, \cdot)$ is a semigroup endowed with a two-sided absorbing element 0, i.e. for any $x \in R$, $x \cdot 0 = 0 \cdot x = 0$.

III) The operation "$\cdot$" is distributive with respect to the hyperoperation "$+$" from the left side: for any $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

As we have already recalled in Introduction, this kind of hypernear-ring was called by Gontineac [14] a *zero-symmetric hypernear-ring*.

In this paper, we want to keep the initial terminology, and therefore, a *hypernear-ring* is meant to be a hyperstucture $(R, +, \cdot)$ satisfying the above mentioned axioms *I*) and *II*), and the new axiom

III') The operation $'' \cdot ''$ is inclusive distributive with respect to the hyperoperation $'' + ''$ from the left side: for any $x, y, z \in R$, $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$.

Accordingly, the Dašić' hypernear-ring is called in this note a *strongly distributive hypernear-ring*.

Next we recall two examples of strongly distributive hypernear-rings, then we present a new one of hypernear-ring, where the multiplication weakly distributes over the hyperaddition.

Example 2.3. [7, 14] *Let $(H, +)$ be a hypergroup (not necessarily commutative) and let $M(H)$ be the set of all mappings $f : H \longrightarrow H$. On the set $M(H)$ define the following hyperoperation:*

$$f \oplus g = \{h \in M(H) \mid h(x) \in f(x) + g(x), \forall x \in H\}.$$

Then $(M(H), \oplus, \circ)$ is a strongly distributive hypernear-ring, where $''\circ''$ is the composition of mappings.

Example 2.4. [19] *Consider the set $R = \{0, a, b, c\}$ endowed with the hyperaddition and multiplication defined by the tables below:*

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0, a, c\}$	$\{b, c\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, a, b\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Then $(R, +, \cdot)$ is a strongly distributive hypernear-ring. Moreover, notice that the additive part $(R, +)$ is a canonical hypergroup, since the hyperaddition is commutative.

In the first part of the following example, we will present a general method, from Davvaz' book [10], to construct a quasicanonical hypergroup (polygroup).

Example 2.5. *Let $(G, \cdot)$ be a group with the identity e, and set $P_G = G \cup \{a\}$, where a is an arbitrary element not belonging to G. Defining on P_G the hyperoperation $''\circ''$ as it follows:*

$$\begin{aligned}
&a \circ a = e \\
&e \circ x = x \circ e = x, \forall x \in P_G \\
&a \circ x = x \circ a = x, \forall x \in P_G \setminus \{e, a\} \\
&x \circ y = x \cdot y, \forall (x, y) \in G^2, y \neq x^{-1} \\
&x \circ x^{-1} = \{e, a\}, \forall x \in P_G \setminus \{e, a\},
\end{aligned}$$

one obtains that $(P_G, \circ)$ is a quasicanonical hypergroup [10]. *It is clear that, if G is a commutative group, then P_G is a canonical hypergroup.*

Take now $G = (\mathbb{Z}_3, +)$, the additive group of integers modulo 3, and $a = 3$. Then we endow the set $P_G = R = \{0, 1, 2, 3\}$, where for simplicity denote $\mathbb{Z}_3 = \{0, 1, 2\}$, with the hyperoperation defined above (i.e. it is represented by the following table):

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{2}	{0, 3}	{1}
2	{2}	{0, 3}	{1}	{2}
3	{3}	{1}	{2}	{0}

Then, define on R the multiplication, as follows: for any $y \in R$, $0 \cdot y = 0$, and for any $x \in R \setminus \{0\}$ and any $y \in R$, $x \cdot y = y$. Note this is the same multiplication used in Example 2.4. Then $(R, +, \cdot)$ is a strongly distributive hypernear-ring.

Lemma 2.6. *Let $(R, +, \cdot)$ be a hypernear-ring. For any $x, y \in R$, the following identities are fulfilled:*

i) $-(x + y) = (-y) + (-x)$

ii) $y \cdot (-x) = -(y \cdot x)$

Definition 2.7. [14, 21] Let $(R, +, \cdot)$ be a hypernear-ring.

i) A subhypergroup A of the hypergroup $(R, +)$ is called a *normal subhypergroup* if, for all $x \in R$, it holds: $x + A - x \subseteq A$.

ii) A normal subhypergroup A of the hypergroup $(R, +)$ is called a *left hyperideal* of R, if $x \cdot a \in A$, for all $x \in R, a \in A$.

iii) A normal subhypergroup A of the hypergroup $(R, +)$ is called a *right hyperideal* of R if $(x+A)\cdot y - x\cdot y \subseteq A$, for all $x, y \in R$.

iv) If A is a left and a right hyperideal of R, i.e. if $[(x + A) \cdot y - x \cdot y] \cup z \cdot A \subseteq A$, for all $x, y, z \in R$, then we say that A is a *hyperideal* of R.

Remark 2.8. i) If A is a normal subhypergroup of R, then $A = x + A - x$, or equivalently $x + A = A + x$, for any $x \in R$.

ii) It can be easily verified that the condition $(x + A) \cdot y - x \cdot y \subseteq A$ in the previous definition is equivalent to the condition $-(x \cdot y) + (x + A) \cdot y \subseteq A$, for any $x, y \in R$.

Definition 2.9. Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be two hypernear-rings. The map $f : R \longrightarrow R'$ is a *strong homomorphism* of the hypernear-rings R and R' if the following relations hold, for all $x, y \in R$:

i) $f(x + y) = f(x) +' f(y)$

ii) $f(x \cdot y) = f(x) \cdot' f(y)$

iii) $f(0) = 0$

Next we recall the construction and the basic properties of the quotient hypernear-ring [7, 14].

If A is a hyperideal of the hypernear-ring R, then we define the relation $x \cong y(modA)$ if and only if $(x - y) \cap A \neq \emptyset$. This is an equivalence relation on R and the class represented by x is $C(x) = x + A$.

Theorem 2.10. *Let $(R, +, \cdot)$ be a hypernear-ring. If A is a hyperideal of R, then on the set of classes $R/A = \{C(x) \mid x \in R\}$ we can define a hyperoperation " $\oplus$ " and an operation " $\odot$ " as follows:*

$$C(x) \oplus C(y) = \{C(z) \mid z \in x' + y', x' \in C(x), y' \in C(y)\}$$
$$C(x) \odot C(y) = C(x \cdot y)$$

The structure $(R/A, \oplus, \odot)$ is a hypernear-ring, called the factor hypernear-ring.

We conclude the introductory part with one result regarding the direct product of two hypernear-rings, that can be generalised to an arbitrary family of hypernear-rings.

Theorem 2.11. *Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ be two hypernear-rings. On the Cartesian product $R_1 \times R_2$ we can define the hyperoperation " $\oplus$ " and the operation " $\odot$ " as follows:*

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1) \times (x_2 + y_2) = \{(x, y) \mid x \in x_1 + y_1, y \in x_2 + y_2\}$$
$$(x_1, x_2) \odot (y_1, y_2) = (x_1 \cdot x_2, y_1 \cdot y_2)$$

The structure $(R_1 \times R_2, \oplus, \odot)$ is a hypernear-ring, called the direct product of the hypernear-rings R_1 and R_2.

3. Hypernear-rings with a defect of distributivity

In this section, we first introduce the concept of hypernear-ring with a defect of distributivity, illustrating it with some examples. Then, we characterize the defect D of a hypernear-ring and present several properties concerning the image of a strong homomorphism, the direct product of hypernear-rings and the factor hypernear-ring.

Definition 3.1. Let $(R, +, \cdot)$ be a hypernear-ring. If $(S, \cdot)$ is a multiplicative subsemigroup of the semigroup $(R, \cdot)$ such that the elements of S generate $(R, +)$, i.e. for every $r \in R$ there exists a finite sum $\sum_{i=1}^{n} \pm s_i$, where $s_i \in S$, for any $i \in \{1, 2, \ldots, n\}$, such that $r \in \sum_{i=1}^{n} \pm s_i$, then we say that S is a *set of generators* of the hypernear-ring R.

Note that in the sum $\sum_{i=1}^{n} \pm s_i$ we could have also some terms of the type $-s_i$ that are not elements is S, even if $s_i \in S$.

The hypernear-ring R with the set of generators S will be denoted by (R, S).

Definition 3.2. Let $(R, +, \cdot)$ be a hypernear-ring with the set of generators S and set

$$\begin{aligned} D_S &= \{d \mid d \in -(x \cdot s + y \cdot s) + (x + y) \cdot s, x, y \in R, s \in S\} = \\ &= \bigcup_{\substack{x,y \in R \\ s \in S}} [-(x \cdot s + y \cdot s) + (x + y) \cdot s]. \end{aligned}$$

The normal subhypergroup D of the hypergroup $(R, +)$ generated by D_S is called the *defect of distributivity* of the hypernear-ring (R, S). Moreover, we say that (R, S) is a *hypernear-ring with the defect D*.

Example 3.3. *Consider the strongly distributive hypernear-ring in Example 2.5. We take $S = \{2\}$ a system of generators of the canonical hypergroup $(R, +)$. We determine all elements in the set D_S and we get:*

$$\begin{aligned} D_S &= \{d \in R \mid d \in -(x \cdot 2 + y \cdot 2) + (x + y) \cdot 2, \forall x, y \in R\} \\ &= \{d \in R \mid d \in (-(2 + 2) + R \cdot 2) \cup (0 + R \cdot 2) \cup (-2 + R \cdot 2)\} \\ &= \{d \in R \mid d \in (-1 + \{0, 2\}) \cup \{0, 2\} \cup (1 + \{0, 2\})\} \\ &= \{d \in R \mid d \in \{1, 2\} \cup \{0, 2\} \cup \{0, 1, 3\}\} \\ &= R \end{aligned}$$

It is clear now that the normal subhypergroup D of $(R, +)$ generated by D_S is the entire suppport set R, i.e. the defect of distributivity of R is R.

Since we are more interested in hypernear-rings with the defect of distributivity not "too big", we consider another example. This is only a hypernear-ring (so not strongly distributive), having the additive part a canonical hypergroup obtained using the Davvaz' general method, presented in Example 2.5.

Example 3.4. *Take $G = (Z_6, +)$, the additive group of integers modulo 6, and set $R = P_G = G \cup \{6\}$. Then the additive part $(R, +)$ can be represented by following table:*

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	$\{0,6\}$	1
2	2	3	4	5	$\{0,6\}$	1	2
3	3	4	5	$\{0,6\}$	1	2	3
4	4	5	$\{0,6\}$	1	2	3	4
5	5	$\{0,6\}$	1	2	3	4	5
6	6	1	2	3	4	5	0

Define on R the multiplication as follows:

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	5	4	3	2	1	0
2	0	1	2	3	4	5	0
3	0	0	0	0	0	0	0
4	0	5	4	3	2	1	0
5	0	1	2	3	4	5	0
6	0	0	0	0	0	0	0

It is simple to check that the multiplication is associative, so $(R, \cdot)$ is a semigroup, having 0 as two-sided absorbing element. Moreover, the multiplication inclusive distributes over hyperaddition, so for any $x, y, z \in R$, we have $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ (we let this part to the reader as a simple exercise). The distributivity (so the equality in the above relation) is not satisfied, since, for example, $1 \cdot (4 + 2) = 1 \cdot \{0, 6\} = 0 \subsetneq \{0, 6\} = 2 + 4 = 1 \cdot 4 + 1 \cdot 2$.

Take $S = \{0, 2, 3\}$ a system of generators of the hypergroup $(R, +)$. We also notice that $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$. Now we determine the set D_S, using Definition 3.2:

$$\begin{aligned} D_S &= \bigcup_{\substack{x,y \in R \\ s \in S}} [-(x \cdot s + y \cdot s) + (x + y) \cdot s] \\ &= \{-(x \cdot 0 + y \cdot 0) + (x + y) \cdot 0 \mid x, y \in R\} \cup \\ &\cup \{-(x \cdot 2 + y \cdot 2) + (x + y) \cdot 2 \mid x, y \in R\} \cup \\ &\cup \{-(x \cdot 3 + y \cdot 3) + (x + y) \cdot 3 \mid x, y \in R\} \\ &= \{0\} \cup \{0, 6\} \cup \{0, 3, 6\} = \{0, 3, 6\}. \end{aligned}$$

Indeed, let $A = -(x \cdot 2 + y \cdot 2) + (x + y) \cdot 2$, for any $x, y \in R$. In the following tables, the values of x are written on the first column, while those for y in the first line. The table of the hypercomposition $x \cdot 2 + y \cdot 2$ is the following one:

	0	1	2	3	4	5	6
0	0	4	2	0	4	2	0
1	4	2	$\{0,6\}$	4	2	$\{0,6\}$	4
2	2	$\{0,6\}$	4	2	$\{0,6\}$	4	2
3	0	4	2	0	4	2	0
4	4	2	$\{0,6\}$	4	2	$\{0,6\}$	4
5	2	$\{0,6\}$	4	2	$\{0,6\}$	4	2
6	0	4	2	0	4	2	0

from which we obtain the table of $-(x\cdot 2+y\cdot 2)$*:*

	0	1	2	3	4	5	6
0	0	2	4	0	2	4	0
1	2	4	$\{0,6\}$	2	4	$\{0,6\}$	2
2	4	$\{0,6\}$	2	4	$\{0,6\}$	2	4
3	0	2	4	0	2	4	0
4	2	4	$\{0,6\}$	2	4	$\{0,6\}$	2
5	4	$\{0,6\}$	2	4	$\{0,6\}$	2	4
6	0	2	4	0	2	4	0

Similarly, the table of the hypercomposition $(x+y)\cdot 2$ *is:*

	0	1	2	3	4	5	6
0	0	4	2	0	4	2	0
1	4	2	0	4	2	0	4
2	2	0	4	2	0	4	2
3	0	4	2	0	4	2	0
4	4	2	0	4	2	0	4
5	2	0	4	2	0	4	2
6	0	4	2	0	4	2	0

It follows that $A=\{0\}\cup\{0,6\}=\{0,6\}$*. Similarly, we obtain that* $B=-(x\cdot 3+y\cdot 3)+(x+y)\cdot 3=\{0,3,6\}$*.*

Thereby, $D_S=\{0\}\cup A\cup B=\{0,3,6\}$ *and the hyperaddition on* D_S *has the following table:*

+	0	3	6
0	0	3	6
3	3	$\{0,6\}$	3
6	6	3	0

meaning that D_S *is a normal subhypergroup of* $(R,+)$*, so* $D_S=D$*. We conclude that the defect of distributivity of the hypernear-ring* R *is* $D=\{0,3,6\}$*.*

Lemma 3.5. *Let* (R,S) *be a hypernear-ring with the defect* D*.*

1. *Then the defect* D *may be characterized as*

$$D=\bigcup_{m,n\in\mathbb{N}}\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n}z_{ij}\pm d_i-\sum_{j=1}^{n}z_{ij}\right)\right],$$

where $z_{ij}\in R$ *and* $d_i\in D_S$*.*

2. *For all* $x,y\in R$ *and* $s\in S$*, it holds:*

$$(x+y)\cdot s\subseteq x\cdot s+y\cdot s+D.$$

Proof. 1. First we will prove that D is a subhypergroup of $(R,+)$. Let $u,v\in D$. Then $u\in\sum_{i=1}^{m}\left(\sum_{j=1}^{n}z_{ij}\pm d_i-\sum_{j=1}^{n}z_{ij}\right)$ and $v\in\sum_{i=1}^{k}\left(\sum_{j=1}^{l}z'_{ij}\pm d'_i-\sum_{j=1}^{l}z'_{ij}\right)$, for some $m,n,k,l\in\mathbb{N}$.

Thus, we can write

$$u+v\subseteq\sum_{i=1}^{m}\left[\underbrace{(z_{i1}+\ldots+z_{in}+0+\ldots+0)}_{n+l}\pm d_i-\underbrace{(z_{i1}+\ldots+z_{in}+0+\ldots+0)}_{n+l}\right]+$$

$$+\sum_{i=1}^{k}\left[\underbrace{(z'_{i1}+\ldots+z'_{il}+0+\ldots+0)}_{n+l}\pm d'_i-\underbrace{(z'_{i1}+\ldots+z'_{il}+0+\ldots+0)}_{n+l}\right]\subseteq D$$

and

$$-u \subseteq -\left(\sum_{i=1}^{m}\left(\underbrace{\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij}}_{A_i}\right)\right) = -\sum_{i=1}^{m} A_i = \sum_{i=m}^{1}(-A_i) =$$
$$= \sum_{i=m}^{1}\left(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij}\right) \subseteq D.$$

Also, for any $x \in R$, it holds:

$$\begin{aligned}
x + u - x \subseteq\ & x + z_{11} + \ldots + z_{1n} \pm d_1 - (z_{11} + \ldots + z_{1n}) + \\
& + z_{21} + \ldots + z_{2n} \pm d_2 - (z_{21} + \ldots + z_{2n}) + \\
& \vdots \\
& + z_{m1} + \ldots + z_{mn} \pm d_m - (z_{m1} + \ldots + z_{mn}) - x \subseteq \\
\subseteq\ & x + z_{11} + \ldots + z_{1n} \pm d_1 - (z_{11} + \ldots + z_{1n}) - x + \\
& + x + z_{21} + \ldots + z_{2n} \pm d_2 - (z_{21} + \ldots + z_{2n}) - x + \\
& + x + z_{31} + \ldots + z_{3n} \pm d_3 - (z_{31} + \ldots + z_{3n}) - x + \\
& \vdots \\
& + x + z_{m1} + \ldots + z_{mn} \pm d_m - (z_{m1} + \ldots + z_{mn}) - x = \\
& = \sum_{i=1}^{m}((x + \sum_{j=1}^{n} z_{ij}) \pm d_i - \sum_{j=1}^{n} z_{ij} - x)) = \\
& = \sum_{i=1}^{m}(x + \sum_{j=1}^{n} z_{ij}) \pm d_i - (x + \sum_{j=1}^{n} z_{ij}) \subseteq D.
\end{aligned}$$

Therefore, D is a normal subhypergroup of the hypergroup $(R, +)$. Obviously, $D_S \subseteq D$.

If A is a normal subhypergroup of $(R, +)$ such that $D_S \subseteq A$, then, for arbitrary $m, n \in \mathbb{N}$ and $z_{ij} \in R, d_i \in D_S$, (where $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$), it holds:

$$\begin{aligned}
& A_i = \sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij} = \\
& = z_{i1} + \ldots + (z_{in} \pm d_i - z_{in}) - z_{in-1} - \ldots - z_{i1} \subseteq \\
& \subseteq z_{i1} + \ldots + (z_{in-1} + A - z_{in-1}) - z_{in-2} - \ldots - z_{i1} \subseteq \\
& \vdots \\
& \subseteq z_{i1} + A - z_{i1} \subseteq A.
\end{aligned}$$

Thus, $\sum_{i=1}^{m} A_i \subseteq A$ and it follows that

$$D = \bigcup_{m,n \in \mathbb{N}}\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij}\right)\right] \subseteq A.$$

2. Let $x, y \in R$ and $s \in S$. Then:

$$(x + y) \cdot s = 0 + (x + y) \cdot s \subseteq (x \cdot s + y \cdot s) + (-(x \cdot s + y \cdot s)) + (x + y) \cdot s \subseteq x \cdot s + y \cdot s + D.$$

□

Definition 3.6. If (R, S) is a hypernear-ring with the defect $D = \{0\}$, then we say that (R, S) is a *distributively generated hypernear-ring* (by short, *d.g. hypernear-ring*).

Lemma 3.7. *If (R, S) is a hypernear-ring with the defect D, then:*

1. $(-x) \cdot s \in -(x \cdot s) + D$,
2. $(x - y) \cdot s \subseteq x \cdot s - y \cdot s + D$,

for all $x, y \in R$ and each $s \in S$.

Proof. 1. Since $0 = 0 \cdot s \in (x + (-x)) \cdot s \subseteq x \cdot s + (-x) \cdot s + D$, then there exists $d \in D$ such that $0 \in x \cdot s + (-x) \cdot s + d$ and therefore $-(x \cdot s) \in (-x) \cdot s + d$. By the axiom *I)iv)* of the definition of a hypernear-ring, it follows that $(-x) \cdot s \in -(x \cdot s) - d \subseteq -(x \cdot s) + D$.

2. Based on the previous point, it is clear that $(x - y) \cdot s \subseteq x \cdot s + (-y) \cdot s + D \subseteq x \cdot s - y \cdot s + D$. □

Moreover, we remark that in a hypernear-ring (R, S), generally it does not hold the equality $(-x) \cdot z = -(x \cdot z)$, for all $x, z \in R$, because the right distributivity generally doesn't hold.

Now we deal with some elementary properties of a hypernear-ring with the defect of distributivity. Next theorems are generalisations of the similar theorems in Dašić [5].

Theorem 3.8. *Let (R_1, S) be a hypernear-ring with the defect D and let R_2 be an arbitrary hypernear-ring. If $f : R_1 \longrightarrow R_2$ is a strong homomorphism, then Imf is a hypernear-ring $(f(R_1), f(S))$ with the defect $f(D)$.*

Proof. Obviously, $(f(R_1), +)$ is a hypergroup such that $f(-x) = -f(x)$, for all $x \in R_1$, and $(f(R_1), \cdot)$ is a semigroup with a two-sided absorbing element $f(0) = 0$. Moreover, for all $x, y, z \in R_1$, it holds: $f(x) \cdot (f(y) + f(z)) = f(x) \cdot f(y + z) = f(x \cdot (y + z)) \subseteq f(x \cdot y + x \cdot z) = f(x) \cdot f(y) + f(x) \cdot f(z)$. Thus, $(f(R_1), +, \cdot)$ is a hypernear-ring.

Besides, $(f(S), \cdot)$ is a subsemigroup of the semigroup $(f(R_1), \cdot)$ and if $x \in R_1$, then there exists $n \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} \pm s_i$, for some $s_1, s_2, \ldots, s_n \in S$ and so $f(x) \in \sum_{i=1}^{n} \pm f(s_i)$. Thus, $f(S)$ is a set of generators of the hypernear-ring $f(R_1)$.

The defect D' of the hypernear-ring $(f(R_1), f(S)$ is generated by the set

$$D_{f(S)} = \bigcup_{\substack{x,y \in R \\ s \in S}} [-(f(x) \cdot f(s) + f(y) \cdot f(s)) + (f(x) + f(y)) \cdot f(s)] =$$

$$= \bigcup_{\substack{x,y \in R \\ s \in S}} [f(-(x \cdot s + y \cdot s) + (x + y) \cdot s)] = f(D_S).$$

Thus,

$$D' = \bigcup_{m,n \in \mathbb{N}} \left[\sum_{i=1}^{m} \left(\sum_{j=1}^{n} f(z_{ij}) \pm f(d_i) - \sum_{j=1}^{n} f(z_{ij}) \right) \right],$$

where $z_{ij} \in R$ and $d_i \in D_S$. Thereby,

$$D' = \bigcup_{m,n \in \mathbb{N}} \left[\sum_{i=1}^{m} f\left(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij} \right) \right] =$$

$$= \bigcup_{m,n \in \mathbb{N}} f\left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij} \right) \right) = f(D).$$

□

Theorem 3.9. *The direct product of the hypernear-rings* (R_1, S_1) *with the defect* D_1 *and* (R_2, S_2) *with the defect* D_2 *is a hypernear-ring* $(R_1 \times R_2, S_1' \times S_2')$ *with the defect* $D_1 \times D_2$, *where* $S_1' = S_1 \cup \{0\}$ *and* $S_2' = S_2 \cup \{0\}$.

Proof. Let $(R_1 \times R_2, \oplus, \odot)$ be the direct product of the hypernear-rings R_1 and R_2. The element $(-x, -y)$ will be denoted by $\ominus(x, y)$. Obviously, $(0, 0)$ is the two-sided absorbing element in $(R_1 \times R_2, \odot)$. Using the fact that, for $i = 1, 2$, S_i is a subsemigroup of $(R_i, \cdot)$, while 0 is two-sided absorbing element of $(R_i, \cdot)$, we obtain that, for arbitrary $(x, y), (u, v) \in S_1' \times S_2'$, it holds: $(x, y) \odot (u, v) = (x \cdot u, y \cdot v) \in S_1' \times S_2'$, meaning that $(S_1' \times S_2', \odot)$ is a subsemigroup of $(R_1 \times R_2, \odot)$.

Next, we prove that the hypergroup $(R_1 \times R_2, \oplus)$ is generated by $S_1' \times S_2'$. For any $(x, y) \in R_1 \times R_2$, it holds: $(x, y) = (x, 0) \oplus (0, y)$. On the other side, there exist $n \in \mathbb{N}$ and $s_1^{(1)}, s_2^{(1)}, \ldots, s_n^{(1)} \in S_1$ such that $x \in \sum_{i=1}^{n} \pm s_i^{(1)}$. Thus, $(x, 0) \in \sum_{i=1}^{n} \oplus(s_i^{(1)}, 0)$. Similarly, there exist $k \in \mathbb{N}$ and $s_1^{(2)}, s_2^{(2)}, \ldots, s_k^{(2)} \in S_2$ such that $x \in \sum_{j=1}^{k} \pm s_j^{(2)}$. Thus, $(0, y) \in \sum_{j=1}^{k} \oplus(0, s_j^{(2)})$. It results that $(x, y) \in \sum_{i=1}^{n} \oplus(s_i^{(1)}, 0) \oplus \sum_{j=1}^{k} \oplus(0, s_j^{(2)})$. Therefore, $S_1' \times S_2'$ is a set of generators for the hypernear-ring $R_1 \times R_2$.

The defect D' of the hypernear-ring $(R_1 \times R_2, S_1' \times S_2')$ is generated by the set

$$
\begin{aligned}
&D_{S_1' \times S_2'} = \\
&= \bigcup_{\substack{(x,y),(u,v) \in R_1 \times R_2 \\ (s_1,s_2) \in S_1' \times S_2'}} [-((x, y) \odot (s_1, s_2) \oplus (u, v) \odot (s_1, s_2)) \oplus ((x, y) \oplus (u, v)) \odot (s_1, s_2)] = \\
&= \bigcup_{\substack{(x,y),(u,v) \in R_1 \times R_2 \\ (s_1,s_2) \in S_1' \times S_2'}} [-((x \cdot s_1 + u \cdot s_1) \times (y \cdot s_2 + v \cdot s_2)) \oplus (((x + u) \cdot s_1) \times ((y + v) \cdot s_2))] = \\
&= \bigcup_{\substack{(x,y),(u,v) \in R_1 \times R_2 \\ (s_1,s_2) \in S_1' \times S_2'}} [((-(x \cdot s_1 + u \cdot s_1) \times (-(y \cdot s_2 + v \cdot s_2))) \oplus ((x + u) \cdot s_1 \times (y + v) \cdot s_2))] = \\
&= \bigcup_{\substack{x,u \in R_1, y,v \in R_2 \\ s_1 \in S_1', s_2 \in S_2'}} [(-(x \cdot s_1 + u \cdot s_1) + (x + u) \cdot s_1) \times ((-(y \cdot s_2 + v \cdot s_2)) + (y + v) \cdot s_2)] = T
\end{aligned}
$$

It can be easily verified that, if $x, u \in R_1$ and $s_1 \in S_1$, then $-(x \cdot s_1 + u \cdot s_1) + (x + u) \cdot s_1 \in D_{S_1}$ and for $s_1 = 0$, it holds that $-(x \cdot s_1 + u \cdot s_1) + (x + u) \cdot s_1 = 0$. So, since $0 \in D_{S_1}$, it follows that $-(x \cdot s_1 + u \cdot s_1) + (x + u) \cdot s_1 \subseteq D_{S_1}$, for all $x, u \in R_1$ and each $s_1 \in S_1'$. Similarly, $-(y \cdot s_2 + v \cdot s_2) + (y + v) \cdot s_2 \subseteq D_{S_2}$, for all $y, v \in R_2$ and each $s_2 \in S_2'$. Thereby, $T = D_{S_1} \times D_{S_2}$, meaning that the defect D' of the hypernear-ring $(R_1 \times R_2, S_1' \times S_2')$ is generated by the set $D_{S_1} \times D_{S_2}$.

We can write now that

$$
D' = \bigcup_{m,n \in \mathbb{N}} \left[\sum_{i=1}^{m} \left(\sum_{j=1}^{n} (x_{ij}, y_{ij}) \oplus (d_i^{(1)}, d_i^{(2)}) \ominus \sum_{j=1}^{n} (x_{ij}, y_{ij}) \right) \right],
$$

where $(x_{ij}, y_{ij}) \in R_1 \times R_2$, $(d_i^{(1)}, d_i^{(2)}) \in D_{S_1} \times D_{S_2}$. Using the definition of the hyperoperation $\oplus$, we obtain:

$$
\begin{aligned}
D' &= \bigcup_{m,n \in \mathbb{N}} \left[\sum_{i=1}^{m} \left(\underbrace{\left(\sum_{j=1}^{n} x_{ij} \pm d_i^{(1)} - \sum_{j=1}^{n} x_{ij} \right)}_{A_i^{(1)}} \times \underbrace{\left(\sum_{j=1}^{n} y_{ij} \pm d_i^{(2)} - \sum_{j=1}^{n} y_{ij} \right)}_{A_i^{(2)}} \right) \right] = \\
&= \bigcup_{m \in \mathbb{N}} \sum_{i=1}^{m} (A_i^{(1)} \times A_i^{(2)}) = \bigcup_{m \in \mathbb{N}} \left(\sum_{i=1}^{m} (A_i^{(1)} \right) \times \left(\sum_{i=1}^{m} (A_i^{(2)} \right).
\end{aligned}
$$

Since $\sum_{i=1}^{m} A_i^{(1)} \subseteq D_1$ and $\sum_{i=1}^{m} A_i^{(2)} \subseteq D_2$, it follows that $D' \subseteq D_1 \times D_2$.

Conversely, if $(x, y) \in D_1 \times D_2$, then, for some $m, k, p, n \in \mathbb{N}$, it holds that $x \in \sum_{i=1}^{m}(\sum_{j=1}^{k} x_{ij} \pm d_i^{(1)} - \sum_{j=1}^{k} x_{ij})$ and $y \in \sum_{i=1}^{p}(\sum_{j=1}^{n} y_{ij} \pm d_i^{(2)} - \sum_{j=1}^{n} y_{ij})$. Without loss of generality, we can suppose that $n = k$ and $m = p$. Indeed, if for example $n > k$, then $\sum_{j=1}^{k} x_{ij} = \sum_{j=1}^{n} x'_{ij}$, where $x'_{ij} = x_{ij}$, for $j = 1, 2, \ldots k$ and $x'_{ij} = 0$, for $j = k + 1, k + 2, \ldots , n$. Also, if for example $m > p$, then

$$\sum_{i=1}^{p}(\sum_{j=1}^{n} y_{ij} \pm d_i^{(2)} - \sum_{j=1}^{n} y_{ij}) = \sum_{i=1}^{m}(\sum_{j=1}^{n} y'_{ij} \pm d_i^{'(2)} - \sum_{j=1}^{n} y'_{ij}),$$

where $y'_{ij} = y_{ij}$, for $i = 1, 2, \ldots , p$, $j = 1, 2, \ldots , n$, $d_i^{'(2)} = d_i^{(2)}$, for $i = 1, 2, \ldots , p$, and $y'_{ij} = 0$, for $i = p + 1, p + 2, \ldots , m$, $j = 1, 2, \ldots , n$ and $d_i^{'(2)} = 0$, for $i = p + 1, p + 2, \ldots , m$. Therefore,

$$\begin{aligned}(x, y) \in\ & \left[\sum_{i=1}^{m}(\sum_{j=1}^{n} x_{ij} \pm d_i^{(1)} - \sum_{j=1}^{n} x_{ij})\right] \times \left[\sum_{i=1}^{m}(\sum_{j=1}^{n} y_{ij} \pm d_i^{(2)} - \sum_{j=1}^{n} y_{ij})\right] = \\ =\ & \sum_{i=1}^{m}\left[(\sum_{j=1}^{n} x_{ij} \pm d_i^{(1)} - \sum_{j=1}^{n} x_{ij}) \times (\sum_{j=1}^{n} y_{ij} \pm d_i^{(2)} - \sum_{j=1}^{n} y_{ij})\right] = \\ =\ & \sum_{i=1}^{m}\left(\sum_{j=1}^{n}(x_{ij}, y_{ij}) \oplus (d_i^{(1)}, d_i^{(2)}) \ominus \sum_{j=1}^{n}(x_{ij}, y_{ij})\right) \subseteq D'.\end{aligned}$$

Thus, $D' = D_1 \times D_2$. □

Theorem 3.10. *Let (R, S) be a hypernear-ring with the defect D. If A is a normal subhypergroup of the hypergroup $(R, +)$ such that $D \subseteq A$, then A is a right hyperideal of the hypernear-ring R if and only if $A \cdot S \subset A$.*

Proof. If A is a right hyperideal of the hypernear-ring R, then $(0 + a) \cdot s - 0 \cdot s \subseteq A$, for all $a \in A$ and $s \in S$. Since $0 \cdot s = 0$, then $a \cdot s \in A$, and therefore $A \cdot S \subseteq S$.

Now suppose that $A \cdot S \subseteq A$. Let $x, y \in R$ and $a \in A$. There exist $s_1, s_2, \ldots , s_n \in S$ such that $y \in \sum_{i=1}^{n} \pm s_i$ and thus

$$\begin{aligned}(x + a) \cdot y - x \cdot y &\subseteq (x + a) \cdot \sum_{i=1}^{n} \pm s_i - x \cdot \sum_{i=1}^{n} \pm s_i = \\ &= \bigcup_{u \in x+a} u \cdot \sum_{i=1}^{n} \pm s_i - x \cdot \sum_{i=1}^{n} \pm s_i \subseteq \\ &\subseteq \bigcup_{u \in x+a} \sum_{i=1}^{n} \pm u \cdot s_i - \sum_{i=1}^{n} \pm(x \cdot s_i) = T\end{aligned}$$

If $u \in x + a$, then $u \cdot s_i \in (x + a) \cdot s_i \subseteq x \cdot s_i + a \cdot s_i + D$. Since $a \cdot s_i \in A$ and $D \subseteq A$, it follows that $u \cdot s_i \in x \cdot s_i + A$. Thus, $T \subseteq \sum_{i=1}^{n} \pm(x \cdot s_i + A) - \sum_{i=1}^{n} \pm(x \cdot s_i)$. Thereby, we have that $-(x \cdot s_i + A) = -A - x \cdot s_i = A - x \cdot s_i = -x \cdot s_i + A$ and so $\sum_{i=1}^{n} \pm(x \cdot s_i + A) = (\sum_{i=1}^{n} \pm x \cdot s_i) + A$. It follows that

$$\begin{aligned}&T \subseteq \pm x \cdot s_1 \pm \ldots \pm x \cdot s_n + A \mp x \cdot s_n \mp x \cdot s_{n-1} \mp \ldots \mp x \cdot s_1 \subseteq \\ &\subseteq \pm x \cdot s_1 \pm \ldots \pm x \cdot s_{n-1} + A \mp x \cdot s_{n-1} \mp x \cdot s_{n-2} \mp \ldots \mp x \cdot s_1 \subseteq \\ &\vdots \\ &\subseteq \pm x \cdot s_1 + A \mp x \cdot s_1 \subseteq A.\end{aligned}$$

It is clear now that A is a right hyperideal of the hypernear-ring R. □

Corollary 3.11. *If (R, S) is a hypernear-ring with the defect D, then:*

1. $R \cdot D \subseteq D$
2. $D \cdot S \subseteq D$
3. *D is a hyperideal of R.*

Proof. 1. Let $r \in R$ and $d \in D$. Then $d \in \sum_{i=1}^{m}(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij})$, for some $m, n \in \mathbb{N}$, $z_{ij} \in R$ and $d_i \in D_S$. So $r \cdot d \in \sum_{i=1}^{m}(\sum_{j=1}^{n} \pm(r \cdot z_{ij}) \pm r \cdot d_i - \sum_{j=1}^{n} r \cdot z_{ij})$. Let $i \in \{1, 2, \ldots, m\}$. Since $d_i \in D_S$, then there exist $x, y \in R$ and $s \in S$, such that $d_i \in -(x \cdot s + y \cdot s) + (x + y) \cdot s$ and thus, $r \cdot d_i \in r \cdot (-(x \cdot s + y \cdot s)) + r \cdot (x + y) \cdot s \subseteq -(r \cdot (x \cdot s + y \cdot s)) + (r \cdot x + r \cdot y) \cdot s \subseteq -(r \cdot x \cdot s + r \cdot y \cdot s) + (r \cdot x + r \cdot y) \cdot s$. So, $r \cdot d_i \in D_s$ and therefore $r \cdot d \in D$.

2. Let $d \in \sum_{i=1}^{m}(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij})$ and $s \in S$. Then

$$\begin{aligned} d \cdot s \in & \left[\sum_{i=1}^{m}(\sum_{j=1}^{n} z_{ij} \pm d_i - \sum_{j=1}^{n} z_{ij}) \cdot s\right] + D \\ \subseteq & \left[\sum_{i=1}^{m} \underbrace{(\sum_{j=1}^{n} z_{ij}) \cdot s \pm d_i \cdot s + (-\sum_{j=1}^{n} z_{ij}) \cdot s}_{A_i}\right] + D. \end{aligned} \tag{1}$$

Obviously, $(\sum_{j=1}^{n} z_{ij}) \cdot s \subseteq \sum_{j=1}^{n} z_{ij} \cdot s + D$ and, by Lemma 3.7, $(-\sum_{j=1}^{n} z_{ij}) \cdot s \subseteq -((\sum_{j=1}^{n} z_{ij}) \cdot s) + D \subseteq -\left[(\sum_{j=1}^{n} z_{ij} \cdot s) + D\right] + D = -D - \sum_{j=1}^{n} z_{ij} \cdot s + D = -\sum_{j=1}^{n} z_{ij} \cdot s + D$.

Let $d_i \in -(x \cdot s_i + y \cdot s_i) + (x + y) \cdot s_i$. Then,

$$\begin{aligned} d_i \cdot s \in\ & (-y \cdot s_i - x \cdot s_i) \cdot s + (x + y) \cdot s_i \cdot s + D \subseteq \\ \subseteq\ & (-y \cdot s_i) \cdot s + (-x \cdot s_i) \cdot s + D + (x + y) \cdot s_i \cdot s + D \subseteq \\ \subseteq\ & -(y \cdot s_i \cdot s) + D - (x \cdot s_i \cdot s) + D + D + (x + y) \cdot s_i \cdot s + D \subseteq \\ \subseteq\ & -y \cdot s_i \cdot s - x \cdot s_i \cdot s + (x + y) \cdot s_i \cdot s + D. \end{aligned}$$

Since $s_i \cdot s \in D_s$, it follows that $-y \cdot s_i \cdot s - x \cdot s_i \cdot s + (x + y) \cdot s_i \cdot s = -(x \cdot s_i \cdot s + y \cdot s_i \cdot s) + (x + y) \cdot s_i \cdot s \subseteq D_S$ and so there exists $d_i' \in D_S$ such that $d_i \cdot s \in d_i' + D$. Moreover, $-d_i \cdot s \in -(d_i' + D) = -D - d_i' = -d_i' + D$, since D is a normal subhypergroup. Therefore,

$$A_i \subseteq \sum_{j=1}^{n} z_{ij} \cdot s + D \pm (d_i' + D) - \sum_{j=1}^{n} z_{ij} \cdot s + D = \sum_{j=1}^{n} z_{ij} \cdot s \pm d_i' - \sum_{j=1}^{n} z_{ij} \cdot s + D \tag{2}$$

From (1) and (2), it follows that

$$\begin{aligned} d \cdot s \in & \left[\sum_{i=1}^{m}(\sum_{j=1}^{n} z_{ij} \cdot s \pm d_i' - \sum_{j=1}^{n} z_{ij} \cdot s) + D\right] + D = \\ = & \sum_{i=1}^{m}(\sum_{j=1}^{n} z_{ij} \cdot s \pm d_i' - \sum_{j=1}^{n} z_{ij} \cdot s) + D \subseteq D. \end{aligned}$$

3. It follows from 1. and 2., based on Theorem 3.10. □

Theorem 3.12. *Let (R, S) be a hypernear-ring with the defect D and let A be a hyperideal of R. Then the factor hypernear-ring $(\bar{R} = R/A, \oplus, \odot)$ has a set of generators $\bar{S} = \{C(s) = s + A \mid s \in S\}$ and the defect $\bar{D} = \{C(d) = d + A \mid d \in D\}$.*

Proof. For all $s_1, s_2 \in S$, it holds: $C(s_1) \odot C(s_2) = C(s_1 \cdot s_2) \in \bar{S}$. Thus, $\bar{S}$ is a subsemigroup of the semigroup $(\bar{R}, \odot)$. Moreover, $C(0)$ is a two-sided absorbing element of the semigroup $(\bar{R}, \odot)$. The hypergroup $(\bar{R}, \oplus)$ is generated by the set $\bar{S}$. Indeed, if $x \in R$, then $x \in \sum_{i=1}^{n} \pm s_i$, for some $s_1, s_2, \ldots, s_n \in S$. Thus, $C(x) \in C(\pm s_1) \oplus \ldots \oplus C(\pm s_n) = \sum_{i=1}^{n} \oplus C(s_i)$, since $C(-s_i) = -C(s_i)$, for each $i \in \{1, 2, \ldots, n\}$.

Besides,

$$\begin{aligned} D_{\bar{S}} &= \bigcup_{\substack{x,y\in R\\ s\in S}} [\ominus(C(x)\odot C(s)\oplus C(y)\odot C(s))\oplus (C(x)\oplus C(y))\odot C(s)] = \\ &= \bigcup_{\substack{x,y\in R\\ s\in S}} [(\ominus(C(x\cdot s)\oplus C(y\cdot s)))\oplus (C(x)\oplus C(y))\odot C(s)] = \\ &= \bigcup_{\substack{x,y\in R\\ s\in S}} \{C(r) \mid r\in -(x\cdot s+y\cdot s)+(x+y)\cdot s\} = \\ &= \{C(r)\mid r\in D_S\}. \end{aligned}$$

So

$$\bar{D} = \bigcup_{m,n\in\mathbb{N}} \sum_{i=1}^{m} (\sum_{j=1}^{n} C(z_{ij}\oplus C(d_i)\ominus \sum_{j=1}^{n} C(z_{ij})),$$

where $z_{ij}\in R$ and $d_i \in D_S$. By the definition of the hyperoperation $\oplus$, we obtain

$$\bar{D} = \{C(d)\mid d\in \left[\sum_{i=1}^{m}(\sum_{j=1}^{n} z_{ij}\pm d_i - \sum_{j=1}^{n} z_{ij})\right] = \{C(d)\mid d\in D\}.$$

□

Corollary 3.13. *Let (R,S) be a hypernear-ring with the defect D and let A be a hyperideal of R. Then the factor hypernear-ring $(\bar{R},\bar{S})$ is a d.g. hypernear-ring if and only if $D\subseteq A$.*

Proof. The factor hypernear-ring $(\bar{R},\bar{S})$ is a d.g. hypernear-ring if and only if its defect $\bar{D}=\{C(0)\}$, meaning that $C(d)=C(0)=A$, for all $d\in D$. This happens if and only if $d\in A$, for all $d\in D$, thus if and only if $D\subseteq A$. □

In [6] Dašić proved that in a near-ring R with the defect D, the set $A_D(R)=\{a\in R\mid a\cdot r\in D, r\in R\}$ is an ideal of R. Moreover, if R is D-distributive, i.e. for any $x,y,z\in R$ and any $d\in D$, it holds $(x+y)\cdot z = x\cdot z+y\cdot z+d$, then $R/A_D(R)$ is a ring. In the next corollary, we will present a generalisation of this result.

Corollary 3.14. *Let (R,S) be a hypernear-ring with the defect D.*

1. *$D\subseteq A_D(R)=\{a\in R\mid \forall r\in R, a\cdot r\in D\}$.*
2. *The factor hypernear-ring $(\bar{R}=R/A_D(R),\bar{S})$ is a d.g. hypernear-ring.*

Proof. 1. If $d\in D$, then, for all $r\in R$, it holds $r\cdot d\in D$, accordingly to Corollary 3.11 1. So $d\in A_D(R)$, meaning that $D\subseteq A_D(R)$.

2. Let $x\in R$, $a\in A_D(R)$ and $y\in x+a-x$. For all $r\in R$ it holds:

$$r\cdot y\subseteq r\cdot(x+a-x)\subseteq r\cdot x+r\cdot a-r\cdot x\subseteq r\cdot x+D-r\cdot x\subseteq D.$$

Therefore, $x+a-x\subseteq A_D(R)$, i.e. $A_D(R)$ is a normal subhypergroup of $(R,+)$.

Now, we will prove that $(A_D(R))\cdot S\subseteq A_D(R)$. Let $a\in A_D(R)$ and $s\in S$. Then, for all $r\in R$, it holds $r\cdot(a\cdot s)=(r\cdot a)\cdot s\in D_S\subseteq D$, by Corollary 3.11 2. So $a\cdot s\in A_D(R)$. Thus, $(A_D(R))\cdot S\subseteq A_D(R)$, and accordingly to Theorem 3.10, the set $A_D(R)$ is a right hyperideal of R. Similarly, we obtain that $A_D(R)$ is a left hyperideal of R.

Finally, since $D\subseteq A_D(R)$, by Corollary 3.13, the factor hypernear-ring $(\bar{R}=R/A_D(R),\bar{S})$ is a d.g. hypernear-ring. □

Theorem 3.15. *If (R, S) is a strongly distributive hypernear-ring with the defect D, such that, for all $x, y, u, v \in R$, it holds $x \cdot y + u \cdot v \subseteq D + u \cdot v + x \cdot y$, then:*

$$(x + y) \cdot \sum_{i=1}^{n} \pm s_i \subseteq x \cdot (\sum_{i=1}^{n} \pm s_i) + y \cdot (\sum_{i=1}^{n} \pm s_i) + D, \tag{3}$$

for all $x, y \in R$ and $s_1, s_2, \ldots, s_n \in S$.

Proof. We prove the theorem by induction on n.

Let $n = 1$ and $s_1 = s \in S$. Obviously, $(x+y)\cdot s \subseteq x\cdot s+y\cdot s+D$. On the other hand, $(x+y)\cdot(-s) = -((x+y)\cdot s) \subseteq -(x \cdot s + y \cdot s + D) = -D - y \cdot s - x \cdot s = -D + y \cdot (-s) + x \cdot (-s)$. Since $y \cdot (-s) + x \cdot (-s) \subseteq D + x \cdot (-s) + y \cdot (-s)$, it follows that $(x + y) \cdot (-s) \subseteq -D + D + x \cdot (-s) + y \cdot (-s)$, i.e. $(x + y) \cdot (-s) \subseteq x \cdot (-s) + y \cdot (-s) + D$, as D is a normal subhypergroup of the hypergroup $(R, +)$.

Suppose now that inclusion (3) is valid, for all $x, y \in R$ and any summ of length $k < n$. Let $s_1, s_2, \ldots, s_k, s_{k+1} \in S$. Then, we get:

$$\begin{aligned} &(x + y) \cdot \sum_{i=1}^{k+1} \pm s_i = (x + y) \cdot (\sum_{i=1}^{k} \pm s_i \pm s_{k+1}) \subseteq \\ &\subseteq (x + y) \cdot \sum_{i=1}^{k} \pm s_i + (x + y) \cdot (\pm s_{k+1}) \subseteq \\ &\subseteq x \cdot (\sum_{i=1}^{k} \pm s_i) + y \cdot (\sum_{i=1}^{k} \pm s_i) + D + x \cdot (\pm s_{k+1}) + y \cdot (\pm s_{k+1}) + D = \\ &= x \cdot (\sum_{i=1}^{k} \pm s_i) + \underbrace{y \cdot (\sum_{i=1}^{k} \pm s_i) + x \cdot (\pm s_{k+1})}_{A} + y \cdot (\pm s_{k+1}) + D. \end{aligned} \tag{4}$$

If $z' \in \sum_{i=1}^{k} \pm s_i$, then $y\cdot z' + x\cdot(\pm s_{k+1}) \subseteq D + x\cdot(\pm s_{k+1}) + y\cdot z' = x\cdot(\pm s_{k+1}) + y\cdot z' + D \subseteq x\cdot(\pm s_{k+1}) + y\cdot(\sum_{i=1}^{k} \pm s_i) + D$. Therefore,

$$A = y \cdot (\sum_{i=1}^{k} \pm s_i) + x \cdot (\pm s_{k+1}) \subseteq x \cdot (\pm s_{k+1}) + y \cdot (\sum_{i=1}^{k} \pm s_i) + D, \tag{5}$$

and since $(x + y) \cdot \sum_{i=1}^{k+1} \pm s_i \subseteq x \cdot (\sum_{i=1}^{k} \pm s_i) + A + y \cdot (\pm s_{k+1}) + D$, it follows from (4) and (5) that

$$\begin{aligned} (x + y) \cdot \sum_{i=1}^{k+1} \pm s_i &\subseteq x \cdot (\sum_{i=1}^{k} \pm s_i) + x \cdot (\pm s_{k+1}) + y \cdot (\sum_{i=1}^{k} \pm s_i) + D + y \cdot (\pm s_{k+1}) + D \\ &= x \cdot \sum_{i=1}^{k+1} \pm s_i + y \cdot \sum_{i=1}^{k+1} \pm s_i + D, \end{aligned}$$

as D is a normal subhypergroup. Thus, condition (3) is valid for all $x, y \in R$ and all $s_1, s_2, \ldots, s_n \in S$. □

Remark 3.16. If (R, S) is a near-ring with the defect D, then the condition $x \cdot y + u \cdot v \subseteq D + u \cdot v + x \cdot y$, for $x, y, u, v \in R$, is equivalent with with the condition $x \cdot y + u \cdot v - x \cdot y - u \cdot v \subseteq D$, meaning that every additive commutator for R^2 is a subset of D. Moreover, relation (3) is equivalent with the following one: for any $x, y, z \in R$, there exists $d \in D$ such that $(x + y) \cdot z = x \cdot z + y \cdot z + d$. Thereby Theorem 3.15 is a generalisation of Proposition 2.7 in Dašić paper [5].

Example 3.17. *Let (R, S) be a near-ring with the defect D and let A be a normal subgroup of $(R, \cdot)$, i.e. $A \cdot x = x \cdot A$, for all $x \in R$. Let $R/A = \{\overline{x} = x \cdot A \mid x \in R\}$. The set R/A becomes a strongly distributive hypernear-ring $(R/A, \oplus, \cdot)$ if we define the hyperoperation "$\oplus$" by $\overline{x} \oplus \overline{y} = \{\overline{z} \mid z \in \overline{x} + \overline{y}\}$ and the operation "$\cdot$" by $\overline{x} \cdot \overline{y} = \overline{x \cdot y}$, for all $\overline{x}, \overline{y} \in R/A$ (see [14]).*

Obviously, $-(x \cdot A) = (-x) \cdot A$, for all $x \in R$ and $0 \cdot A = A \cdot 0 = 0$ is the bilateraly absorbing element in $(R/A, \cdot)$. Besides, for all $s_1, s_2 \in S$, it holds $(s_1 \cdot A) \cdot (s_2 \cdot A) = (s_1 \cdot s_2) \cdot A$ and if $x \in R$, then, since $x \in \sum_{i=1}^{n} \pm s_i$, for some $s_1, s_2, \ldots, s_n \in S$, it follows that $x \cdot A \in (\sum_{i=1}^{n} \pm s_i) \cdot A = A \cdot (\sum_{i=1}^{n} \pm s_i) \subseteq \sum_{i=1}^{n} A \cdot (\pm s_i) = \sum_{i=1}^{n} \pm(A \cdot s_i) = \sum_{i=1}^{n} \pm(s_i \cdot A)$. Therefore, the hypernear-ring $\bar{R}$ has a set of generators $\bar{S} = \{s \cdot A \mid s \in S\}$. In this case, $\bar{D}_S = \bigcup [\ominus((\overline{x} \cdot \overline{s}) \oplus (\overline{y} \cdot \overline{s})) \oplus (\overline{x} \oplus \overline{y}) \cdot \overline{s}]$, implying that

$$\begin{aligned}\bar{D}_S &= \bigcup_{\substack{x,y \in R\\ s \in S}} [\ominus(x \cdot s \cdot A \oplus y \cdot s \cdot A)) \oplus (x \cdot A \oplus y \cdot A) \cdot s \cdot A] = \\ &= \bigcup_{\substack{x,y \in R\\ s \in S}} \{(-z) \cdot A \oplus p \cdot s \cdot A \mid z \in x \cdot s \cdot A + y \cdot s \cdot A, p \in x \cdot A + y \cdot A\} = \\ &= \bigcup_{\substack{x,y \in R\\ s \in S}} \{\delta \cdot A \mid \delta \in (-z) \cdot A + p \cdot s \cdot A, z \in x \cdot s \cdot A + y \cdot s \cdot A, p \in x \cdot A + y \cdot A\} = \\ &= \bigcup_{\substack{x,y \in R\\ s \in S}} \{w \cdot A \mid w \in [(-(x \cdot A \cdot s + y \cdot A \cdot s)) + (x \cdot A + y \cdot A) \cdot s]\}.\end{aligned}$$

Indeed, if $\delta \in (-z) \cdot A + p \cdot s \cdot A$, with $z \in x \cdot s \cdot A + y \cdot s \cdot A$, and $p \in x \cdot A + y \cdot A$, then

$$\begin{aligned}\delta \in\ & [-(x \cdot s \cdot A + y \cdot s \cdot A)] \cdot A + (x \cdot A + y \cdot A) \cdot s \cdot A = \\ =\ & A \cdot (-(x \cdot s \cdot A + y \cdot s \cdot A)) + A \cdot (x \cdot A + y \cdot A) \cdot s \subseteq \\ \subseteq\ & -[A \cdot (x \cdot s \cdot A + y \cdot s \cdot A)] + (A \cdot x \cdot A + A \cdot y \cdot A) \cdot s \subseteq \\ \subseteq\ & -(A \cdot x \cdot s \cdot A + A \cdot y \cdot s \cdot A) + (x \cdot A \cdot A + y \cdot A \cdot A) \cdot s = \\ =\ & -(A \cdot A \cdot x \cdot s + A \cdot A \cdot y \cdot s) + (x \cdot A + y \cdot A) \cdot s = \\ =\ & -(A \cdot x \cdot s + A \cdot y \cdot s) + (x \cdot A + y \cdot A) \cdot s = \\ =\ & -(x \cdot A \cdot s + y \cdot A \cdot s) + (x \cdot A + y \cdot A) \cdot s.\end{aligned}$$

Conversely, if $w \in -(x \cdot A \cdot s + y \cdot A \cdot s) + (x \cdot A + y \cdot A) \cdot s$, then $w \in -(A \cdot x \cdot s + A \cdot y \cdot s) + (A \cdot x + A \cdot y) \cdot s$. Let $a \in A$. Then we get:

$$\begin{aligned}w \in\ & -(a \cdot A \cdot x \cdot s + a \cdot A \cdot y \cdot s) + (a \cdot A \cdot x + a \cdot A \cdot y) \cdot s = \\ =\ & -[a \cdot (A \cdot x \cdot s + A \cdot y \cdot s)] + a \cdot (A \cdot x + A \cdot y) \cdot s = \\ =\ & a \cdot [-(A \cdot x \cdot s + A \cdot y \cdot s)] + a \cdot (A \cdot x + A \cdot y) \cdot s \subseteq \\ \subseteq\ & A \cdot (-(A \cdot x \cdot s + A \cdot y \cdot s)) + A \cdot (A \cdot x + A \cdot y) \cdot s = \\ =\ & -(x \cdot s \cdot A + y \cdot s \cdot A) \cdot A + (x \cdot A + y \cdot A) \cdot s \cdot A.\end{aligned}$$

Thus, there exist $z \in x \cdot s \cdot A + y \cdot s \cdot A$ and $p \in x \cdot A + y \cdot A$, such that $w \in (-z) \cdot A + p \cdot s \cdot A$. Therefore, the defect of the hypernear-ring $(\bar{R}, \bar{S})$ is the set

$$\bar{D} = \bigcup_{m,n \in \mathbb{N}} \sum_{i=1}^{m} (\sum_{j=1}^{n} \overline{z}_{ij} \oplus \overline{w} \ominus \sum_{j=1}^{n} \overline{z}_{ij}),$$

where $\overline{z}_{ij} = z_{ij} \cdot A$, $\overline{w} = w \cdot A \in D_{\bar{S}}$, meaning that

$$\bar{D} = \bigcup_{m,n \in \mathbb{N}} \{d \cdot A \mid d \in \sum_{i=1}^{m} (\sum_{j=1}^{n} z_{ij} \cdot A \pm w \cdot A - \sum_{j=1}^{n} z_{ij} \cdot A\},$$

where $z_{ij} \in R$ and $w \in \bigcup_{\substack{x,y \in R\\ s \in S}} [(-(x \cdot A \cdot s + y \cdot A \cdot s)) + (x \cdot A + y \cdot A) \cdot s]$.

4. Conclusions

This paper wants to open a new line of research in hypernear-ring theory, proposing the study of hypernear-rings with a defect of distributivity. These are near-ring like hyperstructures where the multiplicity inclusively distributives over the hyperaddition on just one side. We have concentrated on the characterisation of the set of elements that ”correct” the lack of right inclusive distributivity, called the defect of the distributivity of the hypernear-ring. Then we have presented several properties of the defect of distributivity of direct product of hypernear-rings, of the image of a homomorphism of hypernear-rings, and of the factor hypernear-ring. We have supported our study with several examples, and we remark that it was difficult to find examples of strongly distributive hypernear-rings with the defect of distributivity less than the entire support set. This remains an open problem for our future work.

References

[1] J. Clay, Nearrings: Geneses and Applications, Oxford Univ. Press, Oxford, 1992.
[2] S.D. Comer, Polygroups derived from cogroups, J. Algebra, 89(2) (1984), 397-405.
[3] I. Cristea, S. Jančić-Rašović, Composition hyperrings, An. Univ. ”Ovidius” Constanta, Seria Mat., 21(2) (2013), 81-94.
[4] I. Cristea, S. Jančić-Rašović, Operations on fuzzy relations: a tool to construct new hyperrings, J. Mult-valued Logic Soft Comput., 21(1-2) (2013), 183-200.
[5] V. Dašić, A defect of distributivity of the near-rings, Math. Balkanica 8 (1978), 63-75.
[6] V. Dašić, Some properties of D-distributive near-rings, Glas. Mat. Ser. III 18(38) (1983), no. 2, 237-242.
[7] V. Dašić, Hypernear-rings, Algebraic hyperstructures and applications (Xanthi, 1990), 75-85, World Sci. Publ., Teaneck, NJ, 1991.
[8] B. Davvaz, Remarks on weak hypermodules, Bull. Korean Math. Soc., 36(3) (1999), 599-608.
[9] B. Davvaz, V. Leoreanu-Fotea, Hyperring theory and applications, International Accademic Press, Palm Harbor, U.S.A., 2007.
[10] B. Davvaz, Polygroup theory and related systems, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.
[11] L. Dickson, Definitions of a group and a field by independent postulates, Trans. Amer. Math. Soc. 6 (1905), 198-204.
[12] G. Ferrero, C. Ferrero-Cotti, Nearrings. Some Developments Linked to Semigroups and Groups, Kluwer, Dordrecht, 2002.
[13] A. Frohlich, Distributively generated near-rings (I. Ideal Theory), Proc. London Math. Soc. (3) 8 (1958), 76-94.
[14] M. Gontineac, On hypernear-rings and H-hypergroups, Algebraic hyperstructures and applications (Iaşi, 1993), 171-179, Hadronic Press, Palm Harbor, FL, 1994.
[15] S. Jančić-Rašović, On a class of Chinese hyperrings, Italian J. Pure Appl. Math., 28(2011), 243-254.
[16] S. Jančić-Rašović, On hyperrings associated with binary relations on semihypergroups, Italian J. Pure Appl. Math., 30(2013), 279-288.
[17] S. Jančić-Rašović, V. Dašić, Some new classes of (m, n)-hyperrings, Filomat, 26(3) (2012), 585-596.
[18] M. Krasner, Approximations des corps values complets de caracteristique $p \neq 0$ par ceux de caracteristique 0, Coll. Alg. Sup. (Bruxelles, Decembres 1956), CBRM, Bruxelles, 1957.
[19] K. H. Kim, B. Davvaz, E.H. Roh, On hyper R-subgroups of hypernear-rings, Sci. Math. Jap. online, e-2007, 649-656.
[20] S. Ligh, On distributively generated near-rings, Proc. Edinburg Math. Soc. 16, Issue 3 (1969), 239-242.
[21] Ch.G. Massouros, Quasicanonical hypergroups, Algebraic hyperstructures and applications (Xanthi, 1990), 129-136, World Sci. Publ., Teaneck, NJ, 1991.
[22] J. Meldrum, Near-Rings and their Links with Groups, Pitman, London, 1985.
[23] J. Mittas, Hypergroupes canoniques, Math. Balkanica 2 (1972), 165-179.
[24] J. Mittas, Sur certaines classes de structures hypercompositionelles, Proc. Acad. Athens, 48(1973), 298-318.
[25] A. Nakasis, Recent results in hyperring and hyperfield theory, Internat. J. Math. Math. Sci., 11(2) (1988), 209-220.
[26] G. Pilz, Near-rings, North-Holland Publ.Co., rev.ed. 1983.
[27] B. Satyanarayana, K.S. Prasad, Near-Rings, Fuzzy Ideals, and Graph Theory, CRC Press, New York, 2013.
[28] S. Scott, Tame Theory, Amo Publishing, Auckland, 1983.
[29] V. Vuković, Nonassociative Near-Rings, Univ. of Kragujevac - Studio Plus, Belgrade, 1996.
[30] Th. Vougiouklis, On some representations of hypergroups, Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 95, série Mathematiques, 26(1990), 21-29.
[31] H. Wähling, Theorie der Fastkörper, Thales-Verlag, Essen, 1987.