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On a Homogeneous Parabolic Problem in an Infinite Corner Domain

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Abstract. In addition to the trivial solution in the class of essentially bounded functions with a given weight for the Solonnikov-Fasano homogeneous parabolic problem in an infinite angular domain we establish the existence of the nontrivial solution, up to a constant factor.

1. Introduction and Statement of the Problem

We consider a homogeneous boundary value problem

$$\frac{\partial u(x,t)}{\partial t} - a^2\frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ \{x,t\} \in G = \{x,t:\ 0 < x < t,\ t > 0\}; \tag{1}$$

$$\frac{\partial u(x,t)}{\partial x}|_{x=0} = 0,\ \frac{\partial u(x,t)}{\partial x}|_{x=t} + \frac{d\tilde{u}(t)}{dt} = 0; \tag{2}$$

where $\tilde{u}(t) = u(t,t)$.

Note that the problem (1)–(2) is homogeneous case of the problem studied in the work [9], and, for simplicity, the coefficients from this work are taken equal: $k = b = 1$. These changes do not contradict the statement of the problem from [9]. It was noted in the work [9], the case of an nonhomogeneous boundary value problem "... is useful for study of some problems with free boundaries". For example, for single-phase problem "... Stefan under the following assumptions: the liquid phase with a positive temperature $u(x,t)$ occupies the segment $0 < x < s(t)$, at $x = 0$ a positive heat flow is given, and free boundary $x = s(t)$ starts at the solid boundary $x = 0$, i.e. the condition is satisfied $s(0) = 0$". In the work [9] the theorem on the unique solvability of the considered boundary value problem in weight Holder spaces is established.

In this paper, in addition to the trivial solution in the class of essentially bounded functions with a given weight we establish the existence of the nontrivial solution, up to a constant factor.

Let $G_T = \{x,t:\ 0 < x < t,\ 0 < t < T\}$ be an arbitrary bounded triangle, where T is a finite number. By using the weight functions

$$\theta(x,t) = \begin{cases} (x + t^{1/2})^{-1}, & \{x,t\} \in G_T, \\ \exp\left\{-\frac{2x+t}{4a^2}\right\}, & \{x,t\} \in G \setminus G_T, \end{cases} \qquad \theta_\varepsilon(x,t) = \begin{cases} (x^{1+\alpha} + t^{(1+\alpha)/2})^{-1}, & \{x,t\} \in G_T, \\ \exp\left\{-\frac{2x+t}{4a^2} + \frac{\varepsilon(2x+t)}{4a^2}\right\}, & \{x,t\} \in G \setminus G_T, \end{cases}$$

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$\alpha > 0,\ \varepsilon > 0$, we can determine the classes of functions

$$L_\infty(G,\ \theta(x,t)),\ L_\infty(G,\ \theta_\varepsilon(x,t)), \tag{3}$$

which mean, respectively, the following inclusions

$$\theta(x,t)\cdot u(x,t) \in L_\infty(G),\ \ \theta_\varepsilon(x,t)\cdot u(x,t) \in L_\infty(G).$$

2. Transformation of Problem (1)–(2) and Reducing it to an Integral Equation

We transform the problem (1)–(2). To do this, we introduce the function $v(x,t) = \frac{\partial u(x,t)}{\partial x}$. Further formally differentiating equation (1) with respect variable x, we get:

$$\frac{\partial v(x,t)}{\partial t} - a^2\frac{\partial^2 v(x,t)}{\partial x^2} = 0,\ \ 0 < x < t,\ \ t > 0; \tag{4}$$

$$v(x,t)|_{x=0} = 0,\ \left(\frac{\partial v(x,t)}{\partial x} + \frac{2}{a^2}v(x,t\right)|_{x=t} = 0. \tag{5}$$

Remark 2.1. First of all, we note the following: each solution to boundary value problem (4)–(5) determines a unique solution (up to a constant factor) to boundary value problem (1)–(2). Indeed, from the second boundary condition (2) and equation (1) for the function $u(x,t) = \int\limits_0^x v(\xi,t)d\,\xi + c(t)$ we obtain:

$$\left[2v(x,t) + u_t(x,t)\right]\Big|_{x=t} + c'(t) = \left[2v(x,t) + a^2 u_{xx}(x,t)\right]\Big|_{x=t} + c'(t) = 0,$$

or

$$a^2\left[v_x(x,t) + \frac{2}{a^2}v(x,t)\right]\Big|_{x=t} + c'(t) = 0.$$

Further from here, by virtue of second condition from (5) we have $c'(t) = 0$, i.e. $c(t) = \text{const}$. However, this constant does not belong to class (3). Therefore, it should be equal to zero. Which is the required result.

Solution of the problem (4)–(5) we are looking as the sum of the double and the single-layer potentials ([10], p. 476–479):

$$v(x,t) = \frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \frac{x}{(t-\tau)^{3/2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau)d\tau + \frac{1}{2a\sqrt{\pi}}\int\limits_0^t \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau, \tag{6}$$

where the functions $\nu(t)$ and $\varphi(t)$ are unknown and should be defined.

We satisfy solution (6) to the first of the boundary conditions. We have:

$$v(x,t)_{|x=0} = \frac{\nu(t)}{2a^2} + \frac{1}{2a\sqrt{\pi}}\int\limits_0^t \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau = 0. \tag{7}$$

From here we express the function $\nu(t)$ in terms of $\varphi(t)$:

$$\nu(t) = -\frac{a}{\sqrt{\pi}}\int\limits_0^t \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau. \tag{8}$$

Using the representation (6) and equality (8), we obtain the following expression for the solution to problem (4)–(5):

$$v(x,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\left[-\exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\}+\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\right]\varphi(\tau)d\tau. \tag{9}$$

In order to satisfy the second boundary condition of (5) we find from (9) its derivative with respect to x:

$$\frac{\partial v(x,t)}{\partial x}=\frac{1}{4a^3\sqrt{\pi}}\int_0^t\left[\frac{x+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\}-\frac{x-\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\right]\varphi(\tau)d\tau. \tag{10}$$

According to the second boundary condition (5) we have:

$$\left(\frac{\partial v(x,t)}{\partial x}+\frac{2}{a^2}v(x,t)\right)\Big|_{x=t}=\frac{\varphi(t)}{2a^2}+\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau-$$

$$-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\tau-$$

$$-\frac{1}{a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau+\frac{1}{a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\tau=0. \tag{11}$$

Using equalities $t+\tau=2t-(t-\tau)$, $(t+\tau)^2=(t-\tau)^2+4t\tau$, from (11) we obtain an integral equation for the unknown function $\varphi(t)$, $t>0$:

$$\varphi(t)+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\tau-$$

$$-\frac{5}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\tau+\frac{3}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\tau=0. \tag{12}$$

The solution of the integral equation (12) will be sought in the class

$$\sqrt{t}\,\exp\left\{t/(4a^2)\right\}\varphi(t)\in L_\infty(G),\ \text{ i.e. }\ \varphi(t)\in L_\infty\left(G;\ \sqrt{t}\,\exp\left\{t/(4a^2)\right\}\right). \tag{13}$$

It should be noted that similar integral Volterra equation of the second kind we have been studied in [1, 2, 4].

3. Solving the Integral Equation (12)

If we introduce a new unknown function: $\varphi_1(t)=\varphi(t)\exp\left\{\frac{t}{4a^2}\right\}$, then from (12) it follows:

$$\varphi_1(t)+\frac{1}{a\sqrt{\pi}}\int_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\varphi_1(\tau)d\tau-$$

$$-\frac{5}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\varphi_1(\tau)d\tau+\frac{3}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\varphi_1(\tau)d\tau=0, \tag{14}$$

or in the operator form $\varphi+(I_1-I_2+I_3)\varphi=0$.

Remark 3.1. At first glance, in the integral equation (14) the main part is the first integral. However, due to the fact that the solutions to this equation are sought in the class of functions that for small values t have feature $t^{-1/2}$, in fact the main terms are the first and second integrals. The third integral is a weak perturbation for the two first integrals. Therefore, here, considering the first integral as the main one, the use of the Carleman-Vekua method of regularization is not justified. The above is confirmed by the following calculations norms of integral operators in the class of essentially bounded functions with weight $t^{1/2}$. At first, we calculate the following integrals

$$I_1(t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{t^{3/2}}{\tau^{1/2}(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\} d\tau = 1/2; \quad I_3 = \frac{3}{2a\sqrt{\pi}} \int\limits_0^t \frac{t^{1/2}}{\tau^{1/2}(t-\tau)^{1/2}} d\tau = \frac{3\sqrt{\pi}}{2a}\sqrt{t};$$

$$I_2(t) = \frac{5}{2a\sqrt{\pi}} \int\limits_0^t \frac{t^{1/2}}{\tau^{1/2}(t-\tau)^{1/2}} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\} d\tau = \frac{5\sqrt{\pi}}{2a} \exp\left\{\frac{t}{a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{a}\right).$$

Hence, for the norm of the operator $I_2 - I_1$ we get:

$$\|I_2 - I_1\| = \sup_{t>0} \left| \frac{5\sqrt{\pi}}{2a} \exp\left\{\frac{t}{a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{a}\right) - 1/2 \right| = \frac{1}{2a\bar{t}^{3/2}},$$

where $\bar{t}$ is the unique solution to the equation with respect to variable t:

$$\operatorname{erfc}\left(\frac{\sqrt{\bar{t}}}{a}\right) = \frac{1}{2\sqrt{\pi}\bar{t}^{3/2}} \exp\left\{-\frac{\bar{t}^2}{a}\right\},$$

and the number $\bar{t}$ is bounded and strictly greater than zero.

In integral equation (14) we make the replacement of the independent variable, and introduce a new unknown function

$$t = \frac{1}{t_1}, \ \tau = \frac{1}{\tau_1}, \ \varphi_2(t_1) = \frac{1}{\sqrt{t_1}} \varphi_1(1/t_1),$$

as a result from (14) we obtain:

$$\varphi_2(t_1) + \frac{1}{a\sqrt{\pi}} \int\limits_{t_1}^{\infty} \frac{1}{(\tau_1 - t_1)^{3/2}} \exp\left\{-\frac{1}{a^2(\tau_1 - t_1)}\right\} \varphi_2(\tau_1) d\tau_1 -$$

$$-\frac{1}{2a\sqrt{\pi}} \int\limits_{t_1}^{\infty} \frac{1}{(\tau_1 - t_1)^{1/2}} \left[5\exp\left\{-\frac{1}{a^2(\tau_1 - t_1)}\right\} - 3\right] \frac{1}{\tau_1} \varphi_2(\tau_1) d\tau_1 = 0. \tag{15}$$

Note that from the solution of integral equation (15), returning to the initial independent variable and the initial unknown function, we can get the solution to the initial integral equation (12).

To solve equation (15) we use the Laplace transformation. We have:

$$\left[1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right] \hat{\varphi}_2(p) - \frac{1}{2a\sqrt{-p}} \left[5\exp\left\{-\frac{2}{a}\sqrt{-p}\right\} - 3\right] \int\limits_p^{\infty} \hat{\varphi}_2(q) dq = 0. \tag{16}$$

Here we have used the following formulas of Laplace transformation ([7], p. 472 and [6], p. 158):

$$\mathcal{L}\left[\int_{t_1}^{\infty} k(t_1 - \tau_1)\varphi_2(\tau_1) d\tau_1\right] = \hat{k}(-p) \cdot \hat{\varphi}_2(p), \quad \mathcal{L}\left[\frac{1}{t_1} \cdot \varphi_2(t_1)\right] = \int\limits_p^{\infty} \hat{\varphi}_2(q) dq,$$

We turn from integral equation (16) to the differential equation by introducing a new unknown function-image:

$$\hat{\psi}(p) = \int_p^\infty \hat{\varphi}_2(q)dq, \quad \text{i.e.,} \quad \hat{\varphi}_2(p) = -\frac{d\hat{\psi}(p)}{dp}, \tag{17}$$

$$\frac{d\hat{\psi}(p)}{\hat{\psi}(p)} = -\frac{5\exp\left\{-\frac{2}{a}\sqrt{-p}\right\} - 3}{2a\sqrt{-p}\left[1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right]}dp. \tag{18}$$

Integrating equation (18), we obtain

$$\ln\left(\frac{\hat{\psi}(p)}{C}\right) = -\int \frac{5\exp\left\{-\frac{2}{a}\sqrt{-p}\right\} - 3}{2a\sqrt{-p}\left[1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right]}dp = \frac{5}{a}\sqrt{-p} + \ln\left[\left(1 + \exp\left\{\frac{2}{a}\sqrt{-p}\right\}\right)^{-4}\right]. \tag{19}$$

From (19) we have:

$$\hat{\psi}(p) = C \cdot \frac{\exp\left\{\frac{5}{a}\sqrt{-p}\right\}}{\left(1 + \exp\left\{\frac{2}{a}\sqrt{-p}\right\}\right)^4} = C \cdot \frac{\exp\left\{-\frac{3}{a}\sqrt{-p}\right\}}{\left(1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right)^4}. \tag{20}$$

Using formula (17), from equation (20) we find the solution to integral equation (16)

$$\hat{\varphi}_2(p) = -\frac{3}{2a\sqrt{-p}} \cdot \frac{\exp\left\{-\frac{3}{a}\sqrt{-p}\right\}}{\left(1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right)^4} + 4\frac{\exp\left\{-\frac{3}{a}\sqrt{-p}\right\}}{\left(1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right)^5} \cdot \frac{\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}{a\sqrt{-p}} =$$

$$= \left[-\frac{3}{2} + \frac{5}{2}\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right] \cdot \frac{\exp\left\{-\frac{3}{a}\sqrt{-p}\right\}}{a\sqrt{-p}\left(1 + \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right)^5}. \tag{21}$$

Further, in order to find the original for the function $\hat{\varphi}_2(p)$ we will use the following decomposition:

$$\frac{1}{(1+z)^5} = \sum_{k=0}^\infty (-1)^k A_k z^k, \quad z = \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}, \quad |z| < 1, \quad \text{where } A_k = \frac{(k+1)(k+2)(k+3)(k+4)}{4!}. \tag{22}$$

Note that if $z = 1$, then the following equality holds $(1+z)^{-5}\big|_{z=1} = 1/32$.

Using the decomposition (22) from (21) we obtain the representation of the function $\hat{\varphi}_2(p)$ in the form of the series:

$$\hat{\varphi}_2(p) = \frac{1}{2a}\sum_{k=0}^\infty (-1)^k A_k \left[\frac{5}{\sqrt{-p}}\exp\left\{-\left(k + \frac{5}{2}\right)\frac{2}{a}\sqrt{-p}\right\} - \right.$$

$$\left. -\frac{3}{\sqrt{-p}}\exp\left\{-\left(k + \frac{3}{2}\right)\frac{2}{a}\sqrt{-p}\right\}\right], \quad \forall\, p \in \{p : \operatorname{Re}\{\sqrt{-p}\} > 0. \tag{23}$$

Since the inversion formula for the Laplace image is valid ([7], p. 497):

$$\mathcal{L}^{-1}\left[\frac{\exp\{-\alpha\sqrt{q}\}}{\sqrt{q}}\right] = \frac{\exp\{-\alpha^2/(4t_1)\}}{\sqrt{\pi t_1}}, \quad 0 < t_1 < \infty,$$

then from (23) we have the function-original $\varphi_2(t_1)$ for all $0 < t_1 < \infty$:

$$\varphi_2(t_1) = \frac{1}{2a\sqrt{\pi t_1}}\sum_{k=0}^\infty (-1)^k A_k \left[5\exp\left\{-\left(k + \frac{5}{2}\right)^2 \frac{1}{a^2 t_1}\right\} - 3\exp\left\{-\left(k + \frac{3}{2}\right)^2 \frac{1}{a^2 t_1}\right\}\right].$$

Further, from here, returning to the initial independent variable $0 < t < \infty$, we obtain:

$$\varphi_1(t) = \sum_{k=0}^{\infty} (-1)^k \left[\varphi_{1,k}^{(1)}(t) - \varphi_{1,k}^{(2)}(t)\right], \tag{24}$$

where

$$\varphi_{1,k}^{(1)}(t) = \frac{5A_k}{2a\sqrt{\pi}} \exp\left\{-\left(k+\frac{5}{2}\right)^2 \frac{t}{a^2}\right\}, \quad \varphi_{1,k}^{(2)}(t) = \frac{3A_k}{2a\sqrt{\pi}} \exp\left\{-\left(k+\frac{3}{2}\right)^2 \frac{t}{a^2}\right\}. \tag{25}$$

Thus, the desired solution to initial integral equation (12) is defined by the formula:

$$\varphi(t) = \sum_{k=0}^{\infty} (-1)^k \left[\varphi_{k}^{(1)}(t) - \varphi_{k}^{(2)}(t)\right], \ \ 0 < t < \infty, \tag{26}$$

where

$$\varphi_{k}^{(1)}(t) = \varphi_{1,k}^{(1)}(t) \exp\left\{-\frac{t}{4a^2}\right\}, \ \varphi_{k}^{(2)}(t) = \varphi_{1,k}^{(2)}(t) \exp\left\{-\frac{t}{4a^2}\right\}. \tag{27}$$

Solution (26) actually belongs to the class $L_\infty\left(G;\ \sqrt{t}\exp\left\{t/(4a^2)\right\}\right)$ (13).

4. Solving the Boundary Value Problem (1)–(2)

The solution $v(x,t)$ to boundary value problem (4)–(5) is determined according to formulas (9) and (26)–(27), and the solution to initial boundary value problem (1)–(2) has the form:

$$u(x,t) = C \int_0^x v(\xi, t) d\xi = C\tilde{u}(x,t), \tag{28}$$

since its solution is found up to a constant factor C, where $v(x,t)$ is defined according to formula (9).

To establish the boundedness of the obtained solution $u(x,t)$ (28) to problem (1)–(2) we need to study the properties of the solution $v(x,t)$ (9) to problem (4)–(5). Since $\varphi(t) \in L_\infty\left(\mathbb{R}_+,\ \sqrt{t}\exp\left\{t/(4a^2)\right\}\right)$ (13), we need to make an estimate and show the boundedness of the integral $I(x,t)$ for all $\{x,t\} \in G$:

$$I(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t I_1(x,t,\tau) \frac{\exp\left\{-\frac{\tau}{4a^2}\right\}}{\sqrt{\tau}} d\tau, \tag{29}$$

where

$$I_1(x,t,\tau) = \int_0^x \frac{1}{(t-\tau)^{1/2}} \left[-\exp\left\{-\frac{(x_1+\tau)^2}{4a^2(t-\tau)}\right\} + \exp\left\{-\frac{(x_1-\tau)^2}{4a^2(t-\tau)}\right\}\right] dx_1. \tag{30}$$

We calculate the integral (29). For this purpose, after the replacement

$$\left\| z = \frac{x_1 \pm \tau}{2a\sqrt{t-\tau}},\ dz = \frac{dx_1}{2a\sqrt{t-\tau}} \right\|,$$

from (30) we have

$$I_1(x,t,\tau) = a\sqrt{\pi}\left[-\mathrm{erf}\left(\frac{x+\tau}{2a\sqrt{t-\tau}}\right) + \mathrm{erf}\left(\frac{x-\tau}{2a\sqrt{t-\tau}}\right)\right]. \tag{31}$$

And further, substituting the value of the integral $I_1(x,t,\tau)$ (31) into (29), we get

$$I(x,t) = \frac{1}{2}\int_0^t \left[-\mathrm{erf}\left(\frac{x+\tau}{2a\sqrt{t-\tau}}\right) + \mathrm{erf}\left(\frac{x-\tau}{2a\sqrt{t-\tau}}\right)\right]\frac{\exp\left\{-\frac{\tau}{4a^2}\right\}}{\sqrt{\tau}}d\tau. \tag{32}$$

From here we immediately obtain an estimate for integral (32):

$$I(x,t) \leq 4a\int_0^{\sqrt{t}/(2a)} \exp\{-\xi^2\}d\xi = 2a\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{t}}{2a}\right). \tag{33}$$

Thus, we have established uniform boundedness of the integral (29) over the variables $\{x,t\} \in G_T$, that is we have shown that solution $\tilde{u}(x,t)$ (27) to boundary value problem (1)–(2) belongs to the class $L_\infty(G_T,\ (x+t^{1/2})^{-1})$. Note that the solution $\tilde{u}(x,t)$ is determined up to a constant factor C, i.e. the formula

$$u(x,t) = C\tilde{u}(x,t),\ u(x,t) \in L_\infty(G_T,\ (x+t^{1/2})^{-1}),$$

defines the general solution to boundary value problem (1)–(2). Estimate (33) also allows to get its order of smallness for all $\{x,t\} \in G_T$, i.e. the following inclusion holds:

$$\tilde{u}(x,t) \in L_\infty(G_T;\ \theta(x,t)).$$

This follows from the asymptotic behavior for the function $\mathrm{erf}\left(\frac{\sqrt{t}}{2a}\right)$ for small values of the variable t (having a place for small values x).

5. The Main Results

We formulate the main results of the work.

Theorem 5.1. *Boundary value problem (1)–(2) in addition to the trivial solution has the non-trivial solution* $u(x,t) = C\,\tilde{u}(x,t)$, *where* $\tilde{u}(x,t) \in L_\infty(G,\ \theta(x,t))$ *(3), and* $C = const$.

Theorem 5.2. *In the class of functions* $L_\infty(G,\ \theta_\varepsilon(x,t))$ *(3) boundary value problem (1)–(2) has only the trivial solution* $u(x,t) \equiv 0$.

6. Addition to the Definition of Class (3)

For the solution to boundary value problem (4)–(5) we have

$$v(x,t) = -v_1(x,t) + v_2(x,t), \tag{34}$$

where

$$v_1(x,t) \leq v_{10}(x,t) = \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\}\frac{\exp\left\{-\frac{\tau}{4a^2}\right\}}{\sqrt{\tau}}d\,\tau, \tag{35}$$

$$v_2(x,t) \leq v_{20}(x,t) = \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left\{\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\frac{\exp\left\{-\frac{\tau}{4a^2}\right\}}{\sqrt{\tau}}d\,\tau. \tag{36}$$

For function $v_{10}(x,t)$ (35) we have:

$$v_{10}(x,t) = \frac{1}{2a\sqrt{\pi}} \exp\left\{\frac{2x+t}{4a^2}\right\} \int_0^t \frac{1}{\sqrt{\tau(t-\tau)}} \exp\left\{-\frac{(x+t)^2}{4a^2(t-\tau)}\right\} d\tau. \tag{37}$$

We make the following replacements

$$z = \frac{x+t}{2a\sqrt{t-\tau}}, \ d\tau = \frac{(x+t)^2}{2a^2 z^3} dz.$$

As a result from (37) we get:

$$v_{10}(x,t) = \frac{1}{2a^2\sqrt{\pi}} \cdot \frac{x+t}{\sqrt{t}} \exp\left\{\frac{2x+t}{4a^2}\right\} \int_{\frac{x+t}{2a\sqrt{t}}}^{\infty} \frac{\exp\{-z^2\}}{\sqrt{z^2 - \left(\frac{x+t}{2a\sqrt{t}}\right)^2}} \cdot \frac{dz}{z} =$$

$$\left\|\xi^2 = z^2 - \frac{(x+t)^2}{4a^2 t}, \ \xi d\xi = z dz\right\|$$

$$= \frac{1}{2a^2\sqrt{\pi}} \cdot \frac{x+t}{\sqrt{t}} \exp\left\{\frac{2x+t}{4a^2} - \frac{(x+t)^2}{4a^2 t}\right\} \int_0^{\infty} \frac{1}{\xi^2 + \left(\frac{x+t}{2a\sqrt{t}}\right)^2} \exp\left\{-\xi^2\right\} d\xi,$$

i.e., we have

$$v_{10}(x,t) = \frac{\sqrt{\pi}}{2a} \exp\left\{\frac{2x+t}{4a^2}\right\} \operatorname{erfc}\left(\frac{x+t}{2a\sqrt{t}}\right). \tag{38}$$

In deriving the equality (38) for calculating the integral we have used the formula from ([3], 3.466.1).

Now, for obtaining the first part of the solution to boundary value problem (1)–(2) $u_1(x,t) = -\int_0^x v_1(\xi,t) d\xi$ we nave:

$$|u_1(x,t)| \le \frac{\sqrt{\pi}}{2a} \int_0^x \exp\left\{\frac{2\xi+t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\xi+t}{2a\sqrt{t}}\right) d\xi = \frac{1}{a} \int_0^x \exp\left\{\frac{2\xi+t}{4a^2}\right\} \int_{\frac{\xi+t}{2a\sqrt{t}}}^{\infty} \exp\left\{-z^2\right\} dz\, d\xi =$$

$$\|\text{we change the order of integration}\| = 2a \cdot \exp\left\{\frac{t}{4a^2}\right\} \left[\int_{\frac{\sqrt{t}}{2a}}^{\frac{x+t}{2a\sqrt{t}}} \exp\left\{-z^2\right\} \int_0^{2a\sqrt{t}z-t} \exp\left\{\frac{\xi}{2a^2}\right\} d\left(\frac{\xi}{2a^2}\right) dz + \right.$$

$$\left. + \int_{\frac{x+t}{2a\sqrt{t}}}^{\infty} \exp\left\{-z^2\right\} \int_0^x \exp\left\{\frac{\xi}{2a^2}\right\} d\left(\frac{\xi}{2a^2}\right) dz \right] = 2a \cdot \exp\left\{\frac{t}{4a^2}\right\} \left\{ \int_{\frac{\sqrt{t}}{2a}}^{\frac{x+t}{2a\sqrt{t}}} \exp\left\{-z^2\right\} \left[\exp\left\{\frac{\sqrt{t}}{a} z - \frac{t}{2a^2}\right\} - 1\right] dz + \right.$$

$$\left. + \int_{\frac{x+t}{2a\sqrt{t}}}^{\infty} \exp\left\{-z^2\right\} \left[\exp\left\{\frac{x}{2a^2}\right\} - 1\right] dz \right\} = 2a \cdot \exp\left\{-\frac{t}{4a^2}\right\} \int_{\frac{\sqrt{t}}{2a}}^{\frac{x+t}{2a\sqrt{t}}} \exp\left\{-z^2 + \frac{\sqrt{t}}{a} z\right\} dz -$$

$$-a\sqrt{\pi}\exp\left\{\frac{t}{4a^2}\right\}\left\{\mathrm{erf}\left(\frac{x+t}{2a\sqrt{t}}\right)-\mathrm{erf}\left(\frac{\sqrt{t}}{2a}\right)-\left[\exp\left\{\frac{x}{2a^2}\right\}-1\right]\mathrm{erfc}\left(\frac{x+t}{2a\sqrt{t}}\right)\right\}.$$

Since the integral in the last expression is calculated using the formula 1.3.17 from ([8], Vol. 1, p. 115), we obtain:

$$|u_1(x,t)|\le a\sqrt{\pi}\left\{\mathrm{erf}\left(\frac{x}{2a\sqrt{t}}\right)-\exp\left\{\frac{t}{4a^2}\right\}\mathrm{erfc}\left(\frac{\sqrt{t}}{2a}\right)+\exp\left\{\frac{2x+t}{4a^2}\right\}\mathrm{erfc}\left(\frac{x+t}{2a\sqrt{t}}\right)\right\}. \tag{39}$$

Further, similar to the previous one, for function $v_2(x,t)$ (36) we have:

$$v_2(x,t)\le v_{20}(x,t)=\frac{1}{2a\sqrt{\pi}}\exp\left\{-\frac{2x-t)}{4a^2}\right\}\int_0^t\frac{1}{\sqrt{\tau(t-\tau)}}\exp\left\{-\frac{(x-t)^2}{4a^2(t-\tau)}\right\}d\tau. \tag{40}$$

We make the following replacements $z=-\frac{x-t}{2a\sqrt{t-\tau}}$, $d\tau=\frac{(x-t)^2}{2a^2z^3}dz$, as a result from (40) we obtain:

$$v_{20}(x,t)=\frac{1}{2a^2\sqrt{\pi}}\cdot\frac{-(x-t)}{\sqrt{t}}\exp\left\{-\frac{2x-t}{4a^2}\right\}\int_{-\frac{x-t}{2a\sqrt{t}}}^{\infty}\frac{1}{\sqrt{z^2-\left(\frac{x-t}{2a\sqrt{t}}\right)^2}}\cdot\frac{dz}{z}=\left\|\xi^2=z^2-\frac{(x-t)^2}{4a^2t},\ \xi d\,\xi=z\,dz\right\|$$

$$=\frac{1}{2a^2\sqrt{\pi}}\cdot\frac{-(x-t)}{\sqrt{t}}\exp\left\{-\frac{2x-t}{4a^2}-\frac{(x-t)^2}{4a^2t}\right\}\int_0^{\infty}\frac{1}{\xi^2+\left(\frac{x-t}{2a\sqrt{t}}\right)^2}\exp\left\{-\xi^2\right\}d\,\xi,$$

i.e., we have

$$v_{20}(x,t)=\frac{\sqrt{\pi}}{2a}\exp\left\{-\frac{k(2x-kt)}{4a^2}\right\}\mathrm{erfc}\left(-\frac{x-kt}{2a\sqrt{t}}\right). \tag{41}$$

In deriving the equality (41) for calculating the integral we have used the formula from ([3], 3.466.1).

Now, for obtaining the second part of the solution to boundary value problem (1)–(2) $u_2(x,t)=\int_0^x v_2(\xi,t)d\xi$ we have:

$$|u_2(x,t)|\le\int_0^x v_{20}(\xi,t)d\xi=\frac{\sqrt{\pi}}{2a}\int_0^x\exp\left\{-\frac{2\xi-t}{4a^2}\right\}\mathrm{erfc}\left(-\frac{\xi-t}{2a\sqrt{t}}\right)d\xi=$$

$$=\frac{1}{a}\int_0^x\exp\left\{-\frac{2\xi-t}{4a^2}\right\}\int_{-\frac{\xi-t}{2a\sqrt{t}}}^{\infty}\exp\left\{-z^2\right\}dz\,d\,\xi=\left\|\text{we change the order of integration}\right\|$$

$$=2a\cdot\exp\left\{\frac{t}{4a^2}\right\}\left[\int_{-\frac{x-t}{2a\sqrt{t}}}^{\frac{\sqrt{t}}{2a}}\exp\left\{-z^2\right\}\int_{-2a\sqrt{t}z+t}^{x}\exp\left\{-\frac{\xi}{2a^2}\right\}d\left(\frac{\xi}{2a^2}\right)dz+\right.$$

$$\left.+\int_{\frac{\sqrt{t}}{2a}}^{\infty}\exp\left\{-z^2\right\}\int_0^x\exp\left\{-\frac{\xi}{2a^2}\right\}d\left(\frac{\xi}{2a^2}\right)dz\right]=2a\cdot\exp\left\{\frac{t}{4a^2}\right\}\left\{\int_{-\frac{x-t}{2a\sqrt{t}}}^{\frac{\sqrt{t}}{2a}}\exp\left\{-z^2\right\}\left[\exp\left\{\frac{\sqrt{t}}{a}z-\frac{t}{2a^2}\right\}-\right.\right.$$

$$-\exp\left\{-\frac{x}{2a^2}\right\}\Bigg] dz + \int_{\frac{\sqrt{t}}{2a}}^{\infty} \exp\left\{-z^2\right\}\left[1-\exp\left\{-\frac{x}{2a^2}\right\}\right] dz \Bigg\} = 2a \cdot \exp\left\{-\frac{t}{4a^2}\right\} \int_{-\frac{x-t}{2a\sqrt{t}}}^{\frac{\sqrt{t}}{2a}} \exp\left\{-z^2 + \frac{\sqrt{t}}{a} z\right\} dz -$$

$$-a\sqrt{\pi}\exp\left\{-\frac{2x-t}{4a^2}\right\}\left[\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right) - \operatorname{erf}\left(-\frac{x-t}{2a\sqrt{t}}\right)\right] + a\sqrt{\pi}\exp\left\{\frac{t}{4a^2}\right\}\left[1-\exp\left\{-\frac{x}{2a^2}\right\}\right]\operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right).$$

Since the integral in the last expression is calculated using the formula 1.3.17 from ([8], Vol. 1, p. 115), we get:

$$|u_2(x,t)| \le a\sqrt{\pi}\left[\operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) + \exp\left\{\frac{t}{4a^2}\right\}\operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right) - \exp\left\{-\frac{2x-t}{4a^2}\right\}\operatorname{erfc}\left(-\frac{x-t}{2a\sqrt{t}}\right)\right]. \tag{42}$$

From the expressions for the components of solution $u_1(x,t)$ (39) and $u_2(x,t)$ (42) we finally obtain

$$|u(x,t)| \le |u_1(x,t)| + |u_2(x,t)| \le$$

$$\le a\sqrt{\pi}\left[2\operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) + \exp\left\{\frac{2x+t}{4a^2}\right\}\operatorname{erfc}\left(\frac{x+t}{2a\sqrt{t}}\right) - \exp\left\{-\frac{2x-t}{4a^2}\right\}\operatorname{erfc}\left(-\frac{x-t}{2a\sqrt{t}}\right)\right]. \tag{43}$$

Thus, from (43) we get an admissible order of growth for the solution $u(x,t)$ to boundary value problem (1)–(2):

$$|u(x,t)| \le C \cdot \exp\left\{\frac{2x+t}{4a^2}\right\}, \ \{x,t\} \in G \setminus G_T, \tag{44}$$

where $G_T = \{x, t: \ 0 < x < t, \ 0 < t < T < +\infty\}$ is an arbitrary bounded triangle.

With the help of the weight functions

$$\theta(x,t) = \begin{cases} (x+t^{1/2})^{-1}, & \{x,t\} \in G_T, \\ \exp\left\{-\frac{2x+t)}{4a^2}\right\}, & \{x,t\} \in G \setminus G_T, \end{cases} \qquad \theta_\varepsilon(x,t) = \begin{cases} (x^{1+\alpha} + t^{(1+\alpha)/2})^{-1}, & \{x,t\} \in G_T, \\ \exp\left\{-\frac{2x+t}{4a^2} + \frac{\varepsilon(2x+t)}{4a^2}\right\}, & \{x,t\} \in G \setminus G_T, \end{cases}$$

$\alpha > 0, \ \varepsilon > 0$, we can define the classes of functions

$$L_\infty(G, \ \theta(x,t)), \ L_\infty(G, \ \theta_\varepsilon(x,t)),$$

which mean, respectively, the following inclusions

$$\theta(x,t) \cdot u(x,t) \in L_\infty(G), \ \theta_\varepsilon(x,t) \cdot u(x,t) \in L_\infty(G).$$

Remark 6.1. The short version of this paper was published in extended abstracts of the Third International Conference on Analysis and Applied Mathematics, Almaty, Kazakhstan (September 07–10, 2016) [5].

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