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On Solvability of Some Boundary Value Problems for a Biharmonic Equation with Periodic Conditions

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Abstract. In the paper we study questions about solvability of some boundary value problems with periodic conditions for an inhomogeneous biharmonic equation. The exact conditions for solvability of the problems are found.

1. Introduction

For biharmonic equation the Dirichlet problem [8, 10, 12, 14] is well known. Recently other types of boundary value problems for the biharmonic equation such as the Neumann problem [3–5, 9, 13, 16, 17, 23**?** , 24], the spectral Steklov problem [6], the Robin problem [7], generalized Robin boundary value problem [15], as well as fractional analogous of Neumann problem [1, 2, 21, 22] are begun to investigate actively. In the paper, a new class of boundary value problems with periodic conditions is studied in the unit ball for an inhomogeneous biharmonic equation.

Let $\Omega = \{x \in R^n : |x| < 1\}$ be a unit ball, where $n \geq 2$ and let $\partial\Omega = \{x \in R^n : |x| = 1\}$ be a unit sphere. For any point $x \in \Omega$ we consider its "opposite" point $x* = (-x_1, -x_2, \ldots, -x_n) \in \Omega$ and denote

$$\partial\Omega_+ = \partial\Omega \cap \{x \in R^n : x_n \geq 0\},\ \partial\Omega_- = \partial\Omega \cap \{x \in R^n : x_n \leq 0\},\ I = \partial\Omega \cap \{x \in R^n : x_n = 0\}.$$

Let $D_\nu^m = \frac{\partial^m}{\partial \nu^m}$, $m \geq 1$, where ν is the unit vector of outer normal to the boundary of Ω. Consider the following problem in the domain Ω:

$$\Delta^2 u(x) = f(x),\ x \in \Omega, \tag{1}$$

$$D_\nu^m u(x) = g(x),\ x \in \partial\Omega, \tag{2}$$

$$D_\nu^{\ell_1} u(x) - (-1)^k D_\nu^{\ell_1} u(x*) = g_1(x),\ x \in \partial\Omega_+, \tag{3}$$

$$D_\nu^{\ell_2} u(x) + (-1)^k D_\nu^{\ell_2} u(x*) = g_2(x),\ x \in \partial\Omega_+, \tag{4}$$

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where $k = 1, 2, 1 \le m \le 3, 1 \le \ell_1 < \ell_2 \le 3, \ell_j \ne m, j = 1, 2$. We call the problem (1)-(4) homogeneous problem if $f = g = g_1 = g_2 = 0$. Solutions of the problem (1)-(4) are functions $u(x) \in C^4(\Omega) \cap C^3(\bar{\Omega})$, satisfying the conditions (1)-(4) in the classical sense.

Let $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_j \ge 0$ be a multi-index with $|\beta| = \beta_1 + \ldots + \beta_n$, $\partial^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}$, $\partial^\beta u(x) = u(x)$ if $|\beta| = 0$. Necessary existence conditions of a solution to the problem (1)-(4) from the class $C^3(\bar{\Omega})$ are the following conditions:

$$\partial^\beta g_1(0, \tilde{x}) + (-1)^k \partial^\beta g_1(0, \alpha\tilde{x}) = 0,\ (0, \tilde{x}) \in I,\ |\beta| \le 3, \tag{5}$$

and

$$\partial^\beta g_2(0, \tilde{x}) - (-1)^k \partial^\beta g_2(0, \alpha\tilde{x}) = 0,\ (0, \tilde{x}) \in I,\ |\beta| \le 2. \tag{6}$$

Furthermore, we assume that these conditions hold.

Note that analogous problems for elliptic equations of the second order were studied in [18–20].

2. Neumann Type Problems

In this section we study the following Neumann type problem:

$$\Delta^2 u(x) = f(x),\ x \in \Omega, \tag{7}$$

$$D_\nu^{m_1} u(x) = \varphi_1(x),\ x \in \partial\Omega, \tag{8}$$

$$D_\nu^{m_2} u(x) = \varphi_2(x),\ x \in \partial\Omega, \tag{9}$$

where $1 \le m_1 < m_2 \le 3$.

Note that exact conditions on solvability of these problems in the case $m_1 = 1, m_2 = 2$ were established in [16], in the case $m_1 = 2$, $m_2 = 3$ in [24], and in the case $m_1 = 1$, $m_2 = 3$ in [13]. These conditions can be formulated in the form of the following theorems:

Theorem 2.1. *Let $m_1 = 1$, $m_2 = 2$, $f(x) \in C^1(\bar{\Omega})$, $\varphi_1(x) \in C^1(\partial\Omega)$, $\varphi_2(x) \in C^2(\partial\Omega)$. Then for solvability of the problem* (7)-(9) *the following condition is necessary and sufficient.*

$$\frac{1}{2}\int\limits_{\Omega} \left(1 - |x|^2\right) f(x)\, dx = \int\limits_{\partial\Omega} \left[\varphi_2(x) - \varphi_1(x)\right]\, dS_x. \tag{10}$$

If a solution of the problem exists, then it is unique up to an arbitrary constant.

Theorem 2.2. *Let $m_1 = 2$, $m_2 = 3$, $f(x) \in C^{\lambda+2}(\bar{\Omega})$, $\varphi_1(x) \in C^{\lambda+4}(\partial\Omega)$ and $\varphi_2(x) \in C^{\lambda+3}(\partial\Omega)$. Then for solvability of the problem* (7)-(9) *the following condition is necessary and sufficient:*

$$\frac{1}{2}\int\limits_{\Omega} \left[(n-1)|x|^2 - (n-2)\right] f(x)\, dx = \int\limits_{\partial\Omega} \varphi_2(x)\, dS_x, \tag{11}$$

$$\frac{1}{2}\int\limits_{\Omega} x_j \left[(n-1)|x|^2 - (n-2)\right] f(x)\, dx = \int\limits_{\partial\Omega} x_j \left[\varphi_2(x) - \varphi_1(x)\right]\, dS_x. \tag{12}$$

If a solution of the problem exists, then it is unique up to an arbitrary first order polynomial.

Theorem 2.3. *Let $m_1 = 1$, $m_2 = 3$, $f(x) \in C^2(\bar{\Omega})$, $\varphi_1(x) \in C^2(\partial\Omega)$, $\varphi_2(x) \in C(\partial\Omega)$. Then for solvability of the problem* (7)-(9) *the condition* (11) *is necessary and sufficient. If a solution of the problem exists, then it is unique up to an arbitrary constant.*

3. About Some Integrals over the Sphere and Ball

Denote

$$f^{\pm}(x) = \frac{f(x) \pm f(x*)}{2},\ x \in \bar{\Omega},\ g^{\pm}(x) = \frac{g(x) \pm g(x*)}{2},\ x \in \partial\Omega,\ \tilde{g}^{\pm}(x) = \begin{cases} g(x),\ x \in \partial\Omega_+ \\ \pm g(x*),\ x \in \partial\Omega_- \end{cases}.$$

Consider the following statements, related to the study of some integrals over ball and sphere, without proof.

Lemma 3.1. *Let $f(x) \in C(\bar{\Omega})$, $g(x) \in C(\partial\Omega)$. Then the following equalities hold:*

$$\int_{\Omega} f^{+}(x)\, dx = \int_{\Omega} f(x)\, dx,\ \int_{\Omega} f^{-}(x)\, dx = 0, \tag{13}$$

$$\int_{\partial\Omega} g^{+}(x)\, dS_x = \int_{\partial\Omega} g(x)\, dS_x,\ \int_{\partial\Omega} g^{-}(x)\, dS_x = 0, \tag{14}$$

$$\int_{\partial\Omega} \tilde{g}^{+}(x)\, dS_x = \int_{\partial\Omega_+} g(x)\, dS_x,\ \int_{\partial\Omega} \tilde{g}^{-}(x)\, dS_x = 0. \tag{15}$$

Lemma 3.2. *Let $f(x) \in C(\bar{\Omega})$, $g(x) \in C(\partial\Omega)$. Then the following equalities hold:*

$$\int_{\Omega} x_j f^{+}(x)\, dx = 0,\ \int_{\Omega} x_j f^{-}(x)\, dx = \int_{\Omega} x_j f(x)\, dx,\ j = 1, 2, \ldots, n, \tag{16}$$

$$\int_{\partial\Omega} x_j g^{+}(x)\, dS_x = 0,\ \int_{\partial\Omega} x_j g^{-}(x)\, dS_x = \int_{\Omega} x_j g(x)\, dS_x,\ j = 1, 2, \ldots, n, \tag{17}$$

$$\int_{\partial\Omega} x_j \tilde{g}^{+}(x*)\, dS_x = 0,\ \int_{\partial\Omega} x_j \tilde{g}^{-}(x*)\, dS_x = \int_{\partial\Omega_+} x_j g(x)\, dx,\ j = 1, 2, \ldots, n. \tag{18}$$

4. Uniqueness of a Solution of the Main Problem

In this section we consider the theorem on uniqueness of a solution of the problem with periodical conditions.

Theorem 4.1. *Let a solution of the problem* (1)-(4) *exist. Then*
1) if $m = 1$, $\ell_1 = 2$, $\ell_2 = 3$, then for $k = 1, 2$ the solution is unique up to an arbitrary constant;
2) in the case $m = 2$, $\ell_1 = 1$, $\ell_2 = 3$ or $m = 3$, $\ell_1 = 1$, $\ell_2 = 2$ solution of the homogeneous problem for $k = 1$ is the function $u(x) = c_0 + \sum_{j=1}^{n} c_j x_j$, and for $k = 2$ is the function $u(x) = c_0$.

Proof. Let $u_1(x)$ and $u_2(x)$ be two solutions of the problem (1)-(4) then $u(x) = u_1(x) - u_2(x)$ is a solution of the corresponding homogeneous problem (1)-(4). So, to investigate the uniqueness of solutions of the nonhomogeneous problem, we investigate the solvability of the corresponding homogeneous problem. Let $u(x)$ be a solution of the homogeneous problem (1)-(4). Then $u(x)$ is a bi-harmonic function, satisfying the homogeneous conditions (2)-(4), i.e. $D_\nu^m u(x) = 0$, $x \in \partial\Omega$ and

$$D_\nu^{\ell_1} u(x) = (-1)^k D_\nu^{\ell_1} u(x*),\ D_\nu^{\ell_2} u(x) = -(-1)^k D_\nu^{\ell_2} u(x*),\ x \in \partial\Omega_+. \tag{19}$$

If $x \in \partial\Omega_-$, then $x* \in \partial\Omega_+$, and therefore, the condition (19) implies:

$$D_\nu^{\ell_1} u(x*) = (-1)^k D_\nu^{\ell_1} u(x),\ x \in \partial\Omega_-,\ D_\nu^{\ell_2} u(x*) = -(-1)^k D_\nu^{\ell_2} u(x),\ x \in \partial\Omega_-.$$

Then

$$D_\nu^{\ell_1} u(x) = (-1)^k D_\nu^{\ell_1} u(x*),\ D_\nu^{\ell_2} u(x) = -(-1)^k D_\nu^{\ell_2} u(x*),\ \forall x \in \partial\Omega.$$

On the other side, from the equality $D_\nu^{\ell_1} u(x) = (-1)^k D_\nu^{\ell_1} u(x*)$ it follows that

$$D_\nu^{\ell_2} u(x) = (-1)^k D_\nu^{\ell_2} u(x*),\ \forall x \in \partial\Omega.$$

Then we have $D_\nu^{\ell_2} u(x)|_{\partial\Omega} = 0$. Thus, the function $u(x)$ is a solution of the homogeneous problem

$$\Delta^2 u(x) = 0,\ x \in \Omega, \tag{20}$$

$$D_\nu^m u(x)\big|_{\partial\Omega} = 0,\ D_\nu^{\ell_2} u(x)\big|_{\partial\Omega} = 0. \tag{21}$$

Hence, if $m = 1$, $\ell_2 = 3$, then by Theorem 2.3 the function $u(x) = c_0$ is solution of the problem (20)-(21). It is obvious, that the function satisfies conditions of the problem (1) - (4) for $k = 1, 2$. Consequently, $u(x) = c_0$. If $m = 2$, $\ell_2 = 3$, then by Theorem 2.2 the function $u(x) = c_0 + \sum\limits_{j=1}^n c_j x_j$ is a solution of the problem (20)-(21). Moreover, in this case $\ell_1 = 1$ and

$$D_\nu^1 u(x*)\big|_{\partial\Omega} = \sum_{i=1}^n x_i \frac{\partial u(x*)}{\partial x_i}\Bigg|_{\partial\Omega} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\left[c_0 - \sum_{j=1}^n c_j x_j\right] = -\sum_{i=1}^n c_i x_i.$$

Then

$$D_\nu^1 u(x*) - (-1)^k D_\nu^1 u(x*)\big|_{\partial\Omega} = (1 + (-1)^k) \sum_{i=1}^n c_i x_i.$$

The last expression vanishes when $k = 1$ for any c_j, $j = 1, 2, \ldots, n$, and when $k = 2$ only in the case $c_j = 0, j = 1, 2, \ldots, n$. Therefore, solution of the homogeneous problem (1)-(4) ($f = g = g_1 = g_2 = 0$) when $k = 1$ is the function $u(x) = c_0 + \sum\limits_{j=1}^n c_j x_j$, and when $k = 2$ it is the function $u(x) = c_0$. Similarly, we can show that in the case $m = 3$, $\ell_2 = 2$ solution of the homogeneous problem (1)-(4) when $k = 1$ is the function $u(x) = c_0 + \sum\limits_{j=1}^n c_j x_j$, and when $k = 2$ it is the function $u(x) = c_0$. $\square$

5. Existence of Solution of the Main Problem

Concerning to the problem (1)-(4) the following statement is true:

Theorem 5.1. *Let $k = 1$, $f(x)$, $g_j(x)$, $j = 1, 2, 3$ be smooth enough functions, and let the conditions* (5) *and* (6) *hold. Then the necessary and sufficiency conditions on solvability of the problem* (1)-(4) *have the form:*

1) if $m = 1$, $\ell_1 = 2$, $\ell_2 = 3$, then

$$\frac{1}{2}\int\limits_\Omega (1 - |x|^2) f(x)\, dx = \int\limits_{\partial\Omega_+} g_1(x)\, dS_x - \int\limits_{\partial\Omega} g(x)\, dS_x, \tag{22}$$

2) if $m = 2$, $\ell_1 = 1$, $\ell_2 = 3$, then

$$\frac{1}{2}\int\limits_\Omega \left[(n-1)|x|^2 - (n-3)\right] f(x)\, dx = \int\limits_{\partial\Omega_+} g_1(x)\, dS_x, \tag{23}$$

$$\frac{1}{2}\int\limits_{\Omega} x_j\left[(n-1)|x|^2-(n-3)\right]f(x)\,dx=\int\limits_{\partial\Omega_+} x_j g_2 dS_x-\int\limits_{\partial\Omega} x_j g(x)dS_x,\ j=1,2,\ldots,n \tag{24}$$

3) if $m=3$, $\ell_1=1$, $\ell_2=2$, *then*

$$\frac{1}{2}\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f(x)\,dx=\int\limits_{\partial\Omega} g(x)\,dS_x, \tag{25}$$

$$\frac{1}{2}\int\limits_{\Omega} x_j\left[(n-1)|x|^2-(n-3)\right]f(x)dx=\int\limits_{\partial\Omega} x_j g(x)dS_x-\int\limits_{\partial\Omega_-} x_j g_2 dS_x,\ j=1,2,\ldots,n. \tag{26}$$

Proof. Consider the auxiliary functions:

$$v(x)=\frac{1}{2}\left(u(x)+u(x*)\right),\ \ w(x)=\frac{1}{2}\left(u(x)-u(x*)\right).$$

It is easy to show that functions $v(x)$ and $w(x)$ are solutions of the following Neumann type problems:

$$\Delta^2 v(x)=f^+(x),x\in\Omega;\ D_\nu^m v(x)\big|_{\partial\Omega}=g^+(x),D_\nu^{\ell_1}v(x)|_{\partial\Omega}=\tilde{g}_1^+(x), \tag{27}$$

$$\Delta^2 w(x)=f^-(x),x\in\Omega;\ D_\nu^m w(x)|_{\partial\Omega}=g^-(x),D_\nu^{\ell_2}w(x)\big|_{\partial\Omega}=\tilde{g}_2^-(x). \tag{28}$$

Note that if the function $f(x)$ in the domain $\bar{\Omega}$ and the function $g(x)$ on the sphere $\partial\Omega$ are smooth enough, then it is obvious, that the functions $f^\pm(x)$, $g^\pm(x)$ have these properties. Moreover, if functions $g_1(x)$ and $g_2(x)$ are smooth on $\partial\Omega_+$, then when the matching conditions (5) and (6) hold the functions $\tilde{g}_1^\pm(x)$ and $\tilde{g}_2^\pm(x)$ will have the same properties.

To study solvability of the problem (27) and (28) we use Theorem 2.1- Theorem 2.3.

1) If $m=1$, $\ell_1=2$, $\ell_2=3$, then necessity and sufficiency conditions on solvability of the problems (27) and (28) are:

$$\frac{1}{2}\int\limits_{\Omega}(1-|x|^2)f^+(x)\,dx=\int\limits_{\partial\Omega}\left(\tilde{g}_1^+(x)-g^+(x)\right)dS_x, \tag{29}$$

and

$$\frac{1}{2}\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f^-(x)\,dx=\int\limits_{\partial\Omega}\tilde{g}_2^-(x)\,dS_x \tag{30}$$

respectively. Due to (13)-(15), it follows that

$$\int\limits_{\Omega}(1-|x|^2)f^+(x)\,dx=\int\limits_{\Omega}(1-|x|^2)f(x)dx,\int\limits_{\partial\Omega}\tilde{g}_1^+\,dS_x=\int\limits_{\partial\Omega_+}g_1(x)\,dS_x,\int\limits_{\partial\Omega}g^+(x)\,dS_x=\int\limits_{\partial\Omega}g(x)\,dS_x,$$

and

$$\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f^-(x)\,dx=0,\int\limits_{\partial\Omega}\tilde{g}_2^-(x)\,dS_x=0.$$

Then the condition (31) always holds, and it is possible to rewrite (30) as (22).

2) If $m=2$, $\ell_1=1$, $\ell_2=3$, then necessity and sufficiency condition on solvability of the problem (27) has the form:

$$\frac{1}{2}\int\limits_{\Omega}\left(1-|x|^2\right)f^+(x)\,dx=\int\limits_{\partial\Omega}\left(g^+(x)-\tilde{g}_1^+(x)\right)\,dS_x \tag{31}$$

and for the problem (28) we get the condition (30) and

$$\frac{1}{2}\int\limits_{\Omega} x_j\left[(n-1)|x|^2-(n-3)\right]f^-(x)\,dx=\int\limits_{\partial\Omega} x_j\left[\tilde{g}_2^-(x)-g^-(x)\right]dS_x,\ j=1,2,\ldots,n. \tag{32}$$

In this case, due to (16)-(18), we obtain

$$\int\limits_{\partial\Omega} x_j\tilde{g}_2^-(x)\,dS_x=\int\limits_{\partial\Omega_+} x_jg_2(x)\,dS_x,\ \int\limits_{\partial\Omega} x_jg^-(x)\,dS_x=\int\limits_{\partial\Omega} x_jg(x)\,dS_x,,\ j=1,2,\ldots,n.$$

Therefore, we can rewrite (31) and (32) as follows:

$$\frac{1}{2}\int\limits_{\Omega}\left(1-|x|^2\right)f(x)\,dx=\int\limits_{\partial\Omega} g(x)\,dS_x-\int\limits_{\partial\Omega_+} g_1\,dS_x,$$

$$\frac{1}{2}\int\limits_{\Omega} x_j\left[(n-1)|x|^2-(n-3)\right]f(x)\,dx=\int\limits_{\partial\Omega_+} x_jg_2\,dS_x-\int\limits_{\partial\Omega} x_jg(x)\,dS_x,\ j=1,2,\ldots,n.$$

3) If $m=3,\ell_1=1,\ell_2=2$, then in this case necessity and sufficiency condition on solvability of the problem (27) has the form

$$\frac{1}{2}\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f^+(x)\,dx=\int\limits_{\partial\Omega} g^+(x)\,dS_x, \tag{33}$$

and for the problem (28) has the form

$$\frac{1}{2}\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f^-(x)\,dx=\int\limits_{\partial\Omega} g^-(x)\,dS_x, \tag{34}$$

$$\frac{1}{2}\int\limits_{\Omega} x_j\left[(n-1)|x|^2-(n-3)\right]f^-(x)\,dx=\int\limits_{\partial\Omega} x_j\left[g^-(x)-\tilde{g}_2^-(x)\right]dS_x,\ j=1,2,\ldots,n. \tag{35}$$

In this case condition (34) on solvability of the problem always holds, and we can rewrite (33) and (35) as follows:

$$\frac{1}{2}\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f(x)\,dx=\int\limits_{\partial\Omega} g(x)\,dS_x,$$

$$\frac{1}{2}\int\limits_{\Omega} x_j\left[(n-1)|x|^2-(n-3)\right]f(x)\,dx=\int\limits_{\partial\Omega} x_jg(x)\,dS_x-\int\limits_{\partial\Omega_+} x_jg_2\,dS_x,\ j=1,2,\ldots,n.$$

□

Similarly we can prove the following statement.

Theorem 5.2. *Let $k=2$, $f(x)$, $g(x)$, $g_j(x)$, $j=1,2$ be smooth enough functions, and let the conditions* (5) *and* (6) *hold. Then necessity and sufficiency conditions on solvability of the problems* (1) - (4) *have the form:*

1) if $m=1$, $\ell_1=2$, $\ell_2=3$, then

$$\frac{1}{2}\int\limits_{\Omega}\left[(n-1)|x|^2-(n-3)\right]f(x)\,dx=\int\limits_{\partial\Omega_+} g_2(x)\,dS_x,$$

2) if $m=2$, $\ell_1=1$, $\ell_2=3$ or $m=3$, $\ell_1=1$, $\ell_2=2$, then

$$\frac{1}{2}\int\limits_{\Omega}\left(1-|x|^2\right)f(x)\,dx=\int\limits_{\partial\Omega_+} g(x)\,dS_x.$$

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