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Some Properties of Eigenvalues and Generalized Eigenvectors of One Boundary Value Problem

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Abstract. We investigate a discontinuous boundary value problem which consists of a Sturm-Liouville equation with piecewise continuous potential together with eigenparameter dependent boundary conditions and supplementary transmission conditions. We establish some spectral properties of the considered problem. In particular, it is shown that the problem under consideration has precisely denumerable many eigenvalues $\lambda_1, \lambda_2, ...$, which are real and tends to $+\infty$. Moreover, it is proven that the generalized eigenvectors form a Riesz basis of the adequate Hilbert space.

1. Introduction

Sturm-Liouville problems with eigenparameter appearing in the boundary conditions have been studied by many authors (see, [4–6, 10, 20, 27] and corresponding references cited therein). The main goal of those papers is the analysis of the spectrum and justification of the eigenfunction expansion. The considerations of [10, 26] are based on the operator-theoretic formulation of such type Sturm-Liouville problems. In [6, 7], the residue calculus in a manner similar to [24] is employed.

In recent years, there has been increasing interest of Sturm-Liouville type problems together with eigenparameter dependent boundary conditions and supplementary transmission conditions (see, [2–4, 11–14, 18–22]). Such properties as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel basis property of a system of root functions, asymptotics of eigenvalues of some boundary value problems with transmission conditions and its applications to the corresponding initial boundary value problems for parabolic equations have been investigated in [11, 20–22]. The various physics applications of this kind of problems arise in heat and mass transfer problems [16, 17, 23], in vibrating string problems when the string loaded additionally with point masses [23], in diffraction problems [25], etc.

Basis properties and eigenfunction expansions for Sturm-Liouville problems involving eigenparameter in the boundary conditions have been considered in [1, 5, 9, 10, 14, 27–29]. Aliyev and Kerimov [1] studied basisness of root functions of Sturm-Liouville problems with a boundary condition depending quadratically on the spectral parameter. In [5], it is shown that the problem of eigenoscillations of various mechanical

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systems is formulated as the Sturm-Liouville problem with the eigenvalue parameter appearing in the boundary conditions. It is proven that the eigenfunctions form a Riesz basis of suitable Hilbert space. In [9], such properties as completeness, minimality and basis property for the eigenfunctions are investigated. In [29], for a non-linear eigenvalue problem similar to a linear Sturm-Liouville problem the properties of the eigenvalues and corresponding eigenfunctions are analysed and the system of eigenfunctions is shown to be a Riesz basis in L_2.

In this paper we shall investigate the Sturm-Liouville equation on two disjoint intervals $[-1,0)$ and $(0,1]$ given by

$$-f''(x)+q(x)f(x)=\lambda f(x) \quad x\in[-1,0)\cup(0,1], \tag{1}$$

together with boundary conditions at the end-points $x=-1,$ and $x=1$ given by

$$f'(-1) = 0, \tag{2}$$

$$(\ln f)'(1) = \frac{a\lambda}{b\lambda+c}, \tag{3}$$

and two supplementary transmission conditions at the point of discontinuity $x=0$ given by

$$f(0^+)-f(0^-)=0\,, \quad f'(0^+)-f'(0^-)=\delta f(0). \tag{4}$$

Here $a,b,c,\delta\in\mathbb{R},\ \delta>0,\ \theta=ac>0,\ \lambda\in\mathbb{C}$, the function $q(x)$ is positively definite, measurable and bounded on $[a,c)\ \cup\ (c,b]$.

The purpose of this paper is to reduce the Sturm-Liouville problem to the operator polynomial equation in an appropriate Hilbert space and to prove that the corresponding eigenfunctions form a Riesz basis of this space.

At first, we shall define some new Hilbert spaces and give some inequalities which is needed for further investigation. It is well-known that the standard Sobolev spaces play a fundamental role in studying various spectral properties of differential operators.

Recall that the Sobolev space $W_2^k(\Omega),(k=0,1,2,...)$ is the Hilbert space consisting of all functions $f\in L_2(\Omega)$ that have generalized derivatives $f^{(n)}\in L_2(\Omega)$ for $n=1,2,...,k$ with the inner product

$$\langle f,g\rangle_{W_2^k(\Omega)} := \int_\Omega (f(x)\overline{g}(x)+f'(x)\overline{g}'(x)+f''\overline{g}''+...+f^{(k)}(x)\overline{g}^{(k)}(x))dx,$$

where $\Omega\subset\mathbb{R}$ is any bounded interval.

Let $\Omega_1:=[-1,0),\Omega_2:=(0,1],\ \Omega=\Omega_1\cup\Omega_2$ and let f be any function defined on $\Omega=\Omega_1\cup\Omega_2$. Then by $f_{(i)}(x)$ we shall denote the restriction of $f(x)$ on the interval Ω_i. Below, by $\mathcal{H}^0(\Omega_1)\oplus\mathcal{H}^0(\Omega_2)$ we denote the Hilbert space $L_2(\Omega)\equiv L_2(\Omega_1)\oplus L_2(\Omega_2)$ with the inner product

$$\langle f,g\rangle_0 := \int_{\Omega_1} f_{(1)}(x)\,\overline{g_{(1)}}(x)\,dx+\int_{\Omega_2} f_{(2)}(x)\,\overline{g_{(2)}}(x)\,dx.$$

Since the embedding operators $J: W_2^1(\Omega_i)\ \hookrightarrow\ C(\Omega_i)$ $(i=1,2)$ are compact, we can show that the linear space

$$\mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)=\left\{f\in\mathcal{H}^0(\Omega_1)\oplus\mathcal{H}^0(\Omega_2)\mid f_{(i)}\in W_2^1(\Omega_i)\ (i=1,2), f_{(1)}(0^-)=f_{(2)}(0^+)\right\}$$

forms a Hilbert space with respect to the inner-product

$$\langle f,g\rangle_1 := \int_{\Omega_1}\left(f_{(1)}(x)\,\overline{g_{(1)}}(x)+f'_{(1)}(x)\,\overline{g_{(1)}}'(x)\right)dx+\int_{\Omega_2}\left(f_{(2)}(x)\,\overline{g_{(2)}}(x)+f'_{(2)}(x)\,\overline{g_{(2)}}'(x)\right)dx.$$

In the same linear space $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$, we shall introduce another inner-product as

$$< f, g >_2 := \langle f, qg \rangle_0 + \langle f', g' \rangle_0, \tag{5}$$

with corresponding norm

$$\|f\|_2^2 \ := \ \langle f, g \rangle_2. \tag{6}$$

The function $q(x)$ is positively definite and bounded, hence there exist positive constants $m,\ M$, independent of f, such that

$$m\,\|f\|_1 < \|f\|_2 < M\,\|f\|_1.$$

Consequently, the inner product space $\left\{\mathcal{H}_q^1(\Omega_1) \oplus \mathcal{H}_q^1(\Omega_2), \langle .,. \rangle_2\right\}$ is also Hilbert space.

It is clear that the functions in $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ are continuous on each $[-1, 0)$ and $(0, 1]$, but their generalized derivatives can only be assumed to be elements of $L_2(\Omega)$. From the well-known embedding theorems for Sobolev spaces (see [16]) we can derive easily following inequalities

$$|f_{(2)}(1)|^2 \leq \gamma_1\,\|f^{'}\|_0^2 + \frac{2}{\gamma_1}\,\|f\|_0^2, \tag{7}$$

$$|f_{(2)}(0^+)|^2 \leq \gamma_2\,\|f^{'}\|_0^2 + \frac{2}{\gamma_2}\,\|f\|_0^2, \tag{8}$$

$$|f_{(1)}(0^-)|^2 \leq \gamma_3\,\|f^{'}\|_0^2 + \frac{2}{\gamma_3}\,\|f\|_0^2, \tag{9}$$

$$|f(\xi)| \quad \leq \quad C(\xi)\,\|f\|_2 \ \text{ for any } \xi \in \Omega, \tag{10}$$

where γ_j $(j = 1, 2, 3)$ are any positive real numbers that are small enough and the constant $C(\xi)$ is independent of the function f, i.e. the constant $C(\xi)$ is dependent only of ξ. Moreover, the inequality

$$\|f\|_0 \quad \leq \quad C_1\,\|f\|_2 \tag{11}$$

follows directly from the definition of the norms (5)-(6).

Let us introduce to the consideration the Hilbert space $\mathbb{H} := \left(\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)\right) \oplus \mathbb{C}$, with the inner product

$$\langle F, G \rangle_{\mathbb{H}} \ := \ \langle f, g \rangle_2 \ + \ z\,\overline{w}$$

for $F := \begin{pmatrix} f \\ z \end{pmatrix}, G := \begin{pmatrix} g \\ w \end{pmatrix} \in \mathbb{H}$ which is the main space for this study.

2. The Generalized Eigenvectors of the Problem

The concept of generalized eigenfunction for our problem (1) – (4) is the main object of this study. Note that this concept is based on the weak solutions of problem (1) – (4), which we shall define by the following procedure.

Multiplying both side of equation (1) by the complex conjugate of an arbitrary function $\mu \in \mathcal{H}_q^1(\Omega_1) \oplus \mathcal{H}_q^1(\Omega_2)$ integrate by parts over the intervals $[-1, 0)$ and $(0, 1]$ and applying the boundary and transmission conditions $(2) - (4)$ produce the following.

$$\begin{aligned}\int_{\Omega_1} \left(f'_{(1)}(x)\, \overline{\mu_{(1)}}'(x) \right. & + \left. q_1(x) f_{(1)}(x)\, \overline{\mu_{(1)}}(x)\right) dx + \int_{\Omega_2} \left(f'_{(2)}(x)\, \overline{\mu_{(2)}}'(x) + q_2(x) f_{(2)}(x)\, \overline{\mu_{(2)}}(x)\right) dx \\ & - \frac{a}{b} f_{(2)}(1)\overline{\mu_{(2)}}(1) + \delta f(0^-)\overline{\mu}(0^-) + \frac{\kappa}{b}\overline{\mu_{(2)}}(1) \\ & = \lambda \int_{\Omega_1} f_{(1)}(x)\, \overline{\mu_{(1)}}(x)\, dx + \lambda \int_{\Omega_2} f_{(2)}(x)\, \overline{\mu_{(2)}}(x)\, dx. \end{aligned} \tag{12}$$

By defining a new unknown parameter κ by $\kappa := af_2(1) - bf'_2(1)$ and using the boundary condition (2) we have

$$\frac{1}{b} f_{(2)}(1) - \frac{\kappa}{ab} = \lambda \frac{\kappa}{\theta}. \tag{13}$$

Thus, the following theorem is proven.

Theorem 2.1. *Let $q_{(i)} \in C(\Omega_i)$ $(i = 1, 2)$ and suppose that there exists a finite one-hand limits $q(0^\pm) := \lim_{x\to 0^\pm} q(x)$. If the function $f = f_0(x)$ is the classical eigenfunction of problem (1)-(4), then two-component vector function $\begin{pmatrix} f_0(x) \\ \kappa \end{pmatrix}$ satisfies equation (12) and equation (13) for any $\mu \in \mathcal{H}_q^1(\Omega_1) \oplus \mathcal{H}_q^1(\Omega_2)$.*

By applying the standard procedure [16], we have the following definition of a generalized solution of boundary value transmission problem (BVTP) $(1) - (4)$.

Definition 2.2. The two-component element $F = \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}$ of the Hilbert space $\mathbb{H}$ is said to be a generalized eigenvector of BVTP $(1) - (4)$, if this element satisfies equation (12) and equation (13) for any $\mu \in \mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$.

It is easy to see that the next theorem is true.

Theorem 2.3. *Let $F = \begin{pmatrix} f(x) \\ \kappa \end{pmatrix} \in \mathbb{H}$ be generalized eigenvector of BVTP $(1) - (4)$. If $f_{(i)} \in C^2(\Omega_i), q_{(i)} \in C(\Omega_i)$ with the finite one-hand limits $f(0^\pm), f'(0^\pm), f''(0^\pm)$, then the first component $f(x)$ becomes a classical eigenfunction of BVTP $(1) - (4)$.*

Proof. Let $F = \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}$ be generalized eigenvector of BVTP $(1)-(4)$. Then for arbitrary $\mu \in \mathcal{H}_q^1(\Omega_1) \oplus \mathcal{H}_q^1(\Omega_2)$ the function $f_{(i)} \in C^2(\Omega_i)$ satisfy equalities (12)-(13). Integrating by parts over the interval Ω_i we have from (12) that the equality

$$\begin{aligned}\int_{\Omega_1} \left(-f''_{(1)}(x) + q_1(x) f_{(1)}(x)\right)\overline{\mu_{(1)}}(x) dx & + \int_{\Omega_2} \left(-f''_{(2)}(x) + q_2(x) f_{(2)}(x)\right)\overline{\mu_{(2)}}(x) dx \\ & = \lambda \int_{\Omega_1} f_{(1)}(x)\overline{\mu_{(1)}}(x) dx + \lambda \int_{\Omega_2} f_{(2)}(x)\overline{\mu_{(2)}}(x) dx \end{aligned}$$

is hold for arbitrary $\mu \in \mathcal{H}_q^1(\Omega_1) \oplus \mathcal{H}_q^1(\Omega_2)$.

The arbitrariness of the function μ shows that the function f satisfies equation (1). From the equalities (12)-(13) it follows immediately that the solution of (12)-(13) satisfies conditions (2)-(4). The proof is complete. □

Remark 2.4. If $q(x)$ is only measurable and bounded on Ω (i.e. are not necessarily continuous functions on Ω), then it is not possible to define the classical eigenfunction. Nevertheless, even in this situation all terms of integral equation (12) and equation (13) are defined in the space $\mathcal{H}_q^1(\Omega_1) \oplus \mathcal{H}_q^1(\Omega_2)$.

3. Main Results

Now we are ready to establish some important results. Particularly, based on the definitions and results of the previous section we shall reduce BVTP (1) − (4) into an operator-pencil equation with self-adjoint compact operators and then it will be proved that the generalized eigenvectors of BVTP (1) − (4) form a Riesz basis of the Hilbert space $\mathbb{H}$.

Let us introduce to consideration the following bilinear forms.

$$\ell_0(f,\mu) := \frac{-a}{b} f_{(2)}(1)\overline{\mu_{(2)}}(1) + \delta f(0^-)\overline{\mu}(0^-), \qquad \text{for all } f,\mu \in \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2), \tag{14}$$

$$\ell_1(f,\mu) := \langle f,\mu\rangle_{\oplus\mathcal{H}^0} = \int_{\Omega_1} f_{(1)}(x)\,\overline{\mu_{(1)}}(x)dx + \int_{\Omega_2} f_{(2)}(x)\,\overline{\mu_{(2)}}(x)dx, \text{ for all } f,\mu \in \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2), \tag{15}$$

$$\ell_2(\kappa,\mu) := \frac{\kappa\,\overline{\mu_{(2)}}(1)}{b}, \qquad \text{for all } \kappa\in\mathbb{C}, \mu \in \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2). \tag{16}$$

Theorem 3.1. *There are bounded linear operators $T_0, T_1 : \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2) \to \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)$ and $T_2 : \mathbb{C} \to \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)$ satisfying the following representations.*

$$\ell_j(f,\mu) = < T_j f,\mu >_2 \ (j = 0,1) \text{ and } \ell_2(\kappa,\mu) = < T_2\kappa,\mu >_2 . \tag{17}$$

Proof. We prove firstly that the bilinear forms ℓ_j $(j = 0,1)$ are continuous in $\mu \in \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)$ for any given $f \in \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)$. The proof is based on the following inequalities

$$|\ell_0(f,\mu)| \leq C_2\, \|f\|_2\, \|\mu\|_2,$$

and

$$|\ell_1(f,\mu)| \leq C_3\, \|f\|_2\, \|\mu\|_2. \tag{18}$$

Note that here, and below, the symbols C_n for $n = 1,2,...,$ are used to denote different constants which do not depend on the functions under consideration and whose exact values are not important for the proof.

By using the well-known Schwarz inequality and (9), we get

$$|\ell_0(f,\mu)| \leq C_4\, \|f\|_0\, \|\mu\|_0 \leq C_4C_1\, \|f\|_0\, \|\mu\|_2 \leq C_4C_1^2\, \|f\|_2\, \|\mu\|_2.$$

The proof of inequality (18) is completely similar. Further, from (16), it follows immediately that

$$|\ell_2(\kappa,\mu)| \leq C_5|\kappa|\, |\mu_{(2)}(1)|.$$

By using interpolation inequalities (7)-(11), we have the following inequality

$$|\ell_2(\kappa,\mu)| \leq C_6|\kappa|\, |\overline{\mu}_{(2)}(1)| \leq C_7|\kappa|\, \|\mu\|_2.$$

To complete the proof it is enough to apply familiar Riesz Representation Theorem (see, for example [15]). □

Theorem 3.2. *We have the following assertions:*

i. *The operators $T_0, T_1 : \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2) \to \mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)$ are self-adjoint and compact,*

ii. *The operator T_1 is positive,*

iii. *The operator $T_2 : \mathbb{C} \to \mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ is compact.*

Proof. **i.** Firstly, let us show that the operator T_0 is self-adjoint. Let $f, \mu \in \mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ be arbitrary functions. By (14) and (17), we have that

$$\begin{aligned}
\langle f, T_0\mu\rangle_2 &= \overline{\langle T_0\mu, f\rangle_2} \\
&= -\frac{a}{b} f_{(2)}(1)\overline{\mu_{(2)}}(1) \ + \ \delta f(0^-)\overline{\mu}(0^-) \\
&= \overline{\ell_0(\mu, f)} \ = \ \ell_0(f, \mu) \\
&= \langle T_0 f, \mu\rangle_2.
\end{aligned}$$

So the operator T_0 is self-adjoint in the Hilbert space $\mathcal{H}^1(\Omega_1)\oplus\mathcal{H}^1(\Omega_2)$. The proof of the self-adjointness of the operator T_1 is completely similar.

Now, we shall prove that the operators T_0 and T_1 are compact operators in the Hilbert space $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$. For this it is sufficient to show that any weakly convergent sequence $\{f_k\}(k = 1, 2, ...)$ in $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ is transformed by $T_j(j = 0, 1)$ into a strongly convergent sequence $\{T_j f_k\}(j = 0, 1)$. The boundedness of $T_j(j = 0, 1)$ implies the weak convergence of $\{T_j f_k\}(j = 0, 1)$ to $\{T_j f\}(j = 0, 1)$ in $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$, where $f(x)$ is the weak limit of $\{f_k\}$. Moreover, the compactness of the embedding operator from $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ into $L_2(\Omega_1) \oplus L_2(\Omega_2)$ [16] implies the strong convergence of the sequences $\{f_k\}$ and $\{T_j f_k\}$ $(j = 0, 1)$ in $L_2(\Omega_1) \oplus L_2(\Omega_2)$ to f and $T_j f$ $(j = 0, 1)$, respectively. Further, the compactness of the embedding operator from $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ into $C(\Omega_1) \oplus C(\Omega_2)$ [16] implies the strong convergence of the sequences $\{f_k(d)\}$ and $\{T_j f_k(d)\}$ $(j = 0, 1)$ in $\mathbb{C}$ to $f(d)$ and $T_j f(d)$ $(j = 0, 1)$, respectively with $d_1 = -1$ or $d_2 = 0^{\mp}$ or $d_3 = +1$. If we use definition (17) of the operators T_j $(j = 0, 1)$, inequalities (7)-(11) and representations (14)-(17), then we have

$$\begin{aligned}
&\| \quad T_0 f_k - T_0 f_m\|_2^2 = \langle T_0(f_k - f_m), T_0(f_k - f_m)\rangle_2 \\
&= \quad \ell_0(f_k - f_m \ , \ T_0(f_k - f_m)) \ \leq \ c_8 \sum_{\alpha=1}^{3} \left|(f_k - f_m)(d_\alpha)\right| \cdot \left|(T_0(f_k - f_m))(d_\alpha)\right| \\
&\leq \quad C_9\Big\{ \left|(f_k(+1) - f_m(+1))\right| \cdot \left|(T_0(f_k - f_m))(+1)\right| \ + \ \left|(f_k(0^-) - f_m(0^-))\right| \cdot \left|(T_0(f_k - f_m))(0^-)\right| \Big\},
\end{aligned}$$

$$\begin{aligned}
&\| \quad T_1 f_k - T_1 f_m\|_2^2 = \langle T_1(f_k - f_m), T_1(f_k - f_m)\rangle_2 \\
&= \quad \ell_1(f_k - f_m \ , \ T_1(f_k - f_m)) \\
&\leq \quad C_{10} \sum_{i=1}^{2} \left| \int_{\Omega_i} r_i(f_k(x) - f_m(x)) \cdot \overline{T_1(f_k - f_m)} dx \right| \\
&\leq \quad C_{11} \ \|f_k - f_m\|_0 \cdot \|T_1(f_k - f_m)\|_2.
\end{aligned}$$

Consequently, $\|T_j(f_k - f_m)\|_2 \ \to \ 0$ as $k, m \ \to \ \infty$. Hence, the sequence $\{T_j f_k\}$ $(j = 0, 1)$ converges strongly in the Hilbert space $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$, so the compactness of the operators T_j $(j = 0, 1)$ are proven.

ii. Since $r_1(x)$ and $r_2(x)$ are positive definite functions, the positivity of the operator T_1 is obvious.

iii. It is easy to verify that the adjoint operator of T_2 is defined on whole $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ with action law $T_2^*(f) \ = \ \frac{1}{b} \ f_2(1)$. From this representation it follows that, the operator T_2^* from $\mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$ to $\mathbb{C}$ is bounded, i.e there exists $C_{12} \ > \ 0$ such that, for all $f \ \in \ \mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2)$

$$|T_2^* f| \ \leq \ C_{12} \ \|f\|_2.$$

Consequently, the operator T_2^* is bounded linear operator with finite dimensional range and therefore is compact. The proof is complete.

□

Let us define two new operators R and S in the Hilbert space $\mathbb{H}$ by the equalities

$$R\begin{pmatrix} f(x) \\ \kappa \end{pmatrix} = \begin{pmatrix} f + T_0 f(x) + T_2\kappa \\ T_2^* f(x) - \frac{c}{b\theta}\,\kappa \end{pmatrix}$$

and

$$S\begin{pmatrix} f(x) \\ \kappa \end{pmatrix} = \begin{pmatrix} T_1 f(x) \\ \frac{\kappa}{\theta} \end{pmatrix}$$

respectively, in terms of which we shall define a linear operator-pencil $\mathcal{A}(\lambda)$ by the equality

$$\mathcal{A}(\lambda) = R + \lambda\, S.$$

By using (12), (13), (14), (15) and (16) we have the following result.

Lemma 3.3. *The generalized eigenfunctions of the BVTP (1)-(4) satisfy the operator-polynomial equation*

$$\mathcal{A}(\lambda)\begin{pmatrix} f(x) \\ \kappa \end{pmatrix} = 0 \tag{19}$$

in the Hilbert space $\mathbb{H}$.

The following result is very important for further consideration.

Lemma 3.4. *There exists $c > 0$, such that for all real $\lambda_0 > c$ the operator polynomial $\mathcal{A}(\lambda_0)$ is positive definite.*

Proof. Equations (14)-(17) and (19) give the formula

$$\begin{aligned} \left\langle \mathcal{A}(\lambda_0)\begin{pmatrix} f(x) \\ \kappa \end{pmatrix}, \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}\right\rangle_{\mathbb{H}} &= \|f\|_2^2 - \frac{a}{b}\,|f_{(2)}(1)|^2 + \delta|f(0)|^2 - \frac{c}{b\theta}\,|\kappa|^2 + \frac{\lambda_0}{\theta}\,|\kappa|^2 + \frac{2}{b}\,Re\Big(\kappa\overline{f_{(2)}}(1)\Big) \\ &+ \lambda_0\bigg(\int_{\Omega_1} r_1(x)\big|f_{(1)}(x)\big|^2 dx + \int_{\Omega_2} r_2(x)\big|f_{(2)}(x)\big|^2 dx\bigg). \end{aligned}$$

Applying the obvious inequality

$$\frac{2}{b}\,Re\Big(\kappa\overline{f_{(2)}}(1)\Big) \geq -\frac{1}{|b|}\left(\varepsilon\,|\kappa|^2 + \frac{1}{\varepsilon}\,|f_{(2)}(1)|^2\right)$$

with an arbitrary parameter $\varepsilon > 0$ yields

$$\begin{aligned} \left\langle \mathcal{A}(\lambda_0)\begin{pmatrix} f(x) \\ \kappa \end{pmatrix}, \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}\right\rangle_{\mathbb{H}} &\geq \|f\|_2^2 - \left(\Big|\frac{a}{b}\Big| + \frac{1}{\varepsilon\,|b|}\right)|f_{(2)}(1)|^2 + \left(-\frac{\varepsilon}{|b|} - \frac{c}{|b|\theta} + \frac{\lambda_0}{\theta}\right)|\kappa|^2 \\ &+ \delta|f(0)|^2 + \lambda_0\bigg(\int_{\Omega_1} r_1(x)\big|f_{(1)}(x)\big|^2 dx + \int_{\Omega_2} r_2(x)\big|f_{(2)}(x)\big|^2 dx\bigg). \end{aligned} \tag{20}$$

Let use define the following functionals

$$\begin{aligned} P(f) &:= \langle f', f'\rangle_0 = \int_{\Omega_1}\big|f'_{(1)}(x)\big|^2 dx + \int_{\Omega_2}\big|f'_{(2)}(x)\big|^2 dx, \\ Q(f) &:= \langle qf, f\rangle_0 = \int_{\Omega_1} q_1(x)\big|f_{(1)}(x)\big|^2 dx + \int_{\Omega_2} q_2(x)\big|f_{(2)}(x)\big|^2 dx, \\ R(f) &:= \langle rf, f\rangle_0 = \int_{\Omega_1} r_1(x)\big|f_{(1)}(x)\big|^2 dx + \int_{\Omega_2} r_2(x)\big|f_{(2)}(x)\big|^2 dx. \end{aligned}$$

Then we have

$$\|f\|_2^2 = P(f) + Q(f) \qquad \text{for all } f \in \mathcal{H}^1(\Omega_1) \oplus \mathcal{H}^1(\Omega_2). \tag{21}$$

Since the functions $q(x)$ and $r(x)$ are positive and bounded, we have

$$\|f'\|_0^2 \le C_{13} P(f), \quad \|f\|_0^2 \le C_{14} Q(f), \quad \langle rf, f \rangle_0 = R(f) \ge C_{15} Q(f). \tag{22}$$

By using inequalities (7)-(11), (20), (22) and equality (21) we can show easily that

$$\left\langle \mathcal{A}(\lambda_0) \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}, \begin{pmatrix} f(x) \\ \kappa \end{pmatrix} \right\rangle_{\mathbb{H}} \ge F_1 P(f) + F_2(\lambda_0) Q(f) + F_3(\lambda_0) |\kappa|^2,$$

where

$$\begin{aligned} F_1 &:= 1 - \left(\left|\frac{a}{b}\right| + \frac{1}{\varepsilon\, |b|} \right) \gamma_1 C_{13} + \delta \gamma_2 C_{13}, \\ F_2(\lambda_0) &:= 1 - \left(\left|\frac{a}{b}\right| + \frac{1}{\varepsilon\, |b|} \right) \frac{2}{\gamma_1} C_{14} + \delta \frac{2}{\gamma_2} C_{14} + \lambda_0 C_{15}, \\ F_3(\lambda_0) &:= -\frac{\varepsilon}{|b|} - \frac{c}{b\theta} + \frac{\lambda_0}{\theta}. \end{aligned}$$

Since $\theta > 0$, it is possible to choose the arbitrary positive parameters γ_1, γ_2 and ε so small and the positive parameter λ_0 so large that the inequalities $F_1 > 0$, $F_2(\lambda_0) > 0$ and $F_3(\lambda_0) > 0$ hold.

Thus, we have that

$$\left\langle \mathcal{A}(\lambda_0) \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}, \begin{pmatrix} f(x) \\ \kappa \end{pmatrix} \right\rangle_{\mathbb{H}} \ge \min \left(F_1, F_2(\lambda_0), F_3(\lambda_0) \right) \|\Phi\|_{\mathbb{H}}^2$$

for all $\begin{pmatrix} f(x) \\ \kappa \end{pmatrix} := \Phi \in \mathbb{H}$ and hence the quadratic form $\left\langle \mathcal{A}(\lambda_0) \begin{pmatrix} f(x) \\ \kappa \end{pmatrix}, \begin{pmatrix} f(x) \\ \kappa \end{pmatrix} \right\rangle_{\mathbb{H}}$ is positive definite for sufficiently large positive values of λ_0. Thus the operator pencil $\mathcal{A}(\lambda_0)$ is positive definite for sufficiently large $\lambda_0 > 0$. The proof is complete. □

By virtue of the Lemma 3.4 there exists $c > 0$, such that for all real $\lambda_0 > c$ the operator polynomial $\mathcal{A}(\lambda_0)$ is positive definite. Moreover, we know that the operator $\mathcal{A}(\lambda_0)$ is also self-adjoint. Therefore there exists positive square root $\sqrt{\mathcal{A}(\lambda_0)}$ which is invertible. Consequently, we can introduce to consideration a new operator $\chi(\lambda_0)$ defined by

$$\chi(\lambda_0) := \left(\sqrt{\mathcal{A}(\lambda_0)} \right)^{-1} S \left(\sqrt{\mathcal{A}(\lambda_0)} \right)^{-1}$$

in the Hilbert space $\mathbb{H}$.

Hence we have the following result.

Theorem 3.5. *The operator $\chi(\lambda_0)$ is positive, self-adjoint and compact in the Hilbert space $\mathbb{H}$ for sufficiently large positive λ_0.*

Theorem 3.6. *Let λ_0 be any real positive number, such that $\mathcal{A}(\lambda_0)$ is positive defined operator in the Hilbert space $\mathbb{H}$. Then the operator $\chi(\lambda_0) := \left(\sqrt{\mathcal{A}(\lambda_0)} \right)^{-1} S \left(\sqrt{\mathcal{A}(\lambda_0)} \right)^{-1}$ has precisely denumerable many positive eigenvalues $\{\eta_n\}$ which tends to $+\infty$. The corresponding eigenvectors form an orthogonal basis of $\mathbb{H}$.*

Proof. The proof of this Theorem follows from the well-known Fredholm theorems for compact self-adjoint operators in a Hilbert spaces (see, for example, [15]). □

Now, by using the well-known fact, that every bounded invertible operator transforms any orthonormal basis of a Hilbert space $\mathbb{H}$ into Riesz basis of $\mathbb{H}$ (see, for example, [8]) we get the following main result.

Theorem 3.7. *Let $q(x)$ be positive defined, bounded and measurable function on $\Omega = \Omega_1 \cup \Omega_2$ and suppose that $ab > 0$ and $\delta > 0$. Then, BVTP (1)-(4) has precisely denumerable many eigenvalues $\lambda_1, \lambda_2, ...,$ which are real and tends to $+\infty$. Moreover, the corresponding system of generalized eigenvectors forms a Riesz basis of $\mathbb{H}$.*

4. Conclusion

The following important conclusions can be really drawn:

i. There exist a real λ_0 such that the operator $\chi(\lambda_0)$ is positive, self-adjoint and compact in the Hilbert space $\mathbb{H}$,

ii. BVTP (1)-(4) has only point spectrum,

iii. The eigenvalues form a real sequences with the only point of accumulation at $+\infty$,

iv. The generalized eigenvectors of BVTP (1)-(4) forms a Riesz basis of $\mathbb{H}$.

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