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Fixed Point Theorems for $(\varphi - \psi)$ Contractions in Partially Ordered Metric-Like Spaces Using New Auxiliary Functions

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Abstract. In this paper, we introduce a class of new auxiliary functions and establish certain fixed point theorems under $(\varphi-\psi)$ contractive conditions in partially ordered metric-like spaces. Our work generalizes some results in the literature and assumes some as particular case. Examples are provided to support the useability of our results.

1. Introduction and Mathematical Preliminaries

The extensions of fixed point theory to generalized structures such as cone-metric, partial-metric spaces and quasi-metric spaces have received much attention in the past years ([3, 5, 11, 14–17]). Fixed point theorems in partial metric spaces have their applications in computer science, engineering and many other fields ([10, 22, 23, 25]). Existence of fixed points in partially ordered metric spaces has been initiated by Ran and Reurings [21] and further studied by Nieto and Lopez [19]. Subsequently, several interesting and valuable results have appeared in the direction([1, 2, 4, 12, 13, 20, 24]). Recently, the notion of a metric-like space was first introduced by Amini-Harandi [6], and obviously it is a new generalization of a partial metric space [18].

Now we will recall some basic definitions and facts which will be used throughout the paper. Here we only list the notion of metric-like space.

Definition 1.1. ([6]) A mapping $\sigma : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is said to be metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:

$(\sigma 1)\sigma(x, y) = 0 \Rightarrow x = y$;

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$(\sigma 2)\sigma(x,y) = \sigma(y,x)$;
$(\sigma 3)\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$.
The pair (X,σ) is then called a metric-like space.

Every partial metric space is a metric-like space. Here, we give some examples of metric-like spaces, but not partial metric spaces.

Example 1.2. Let $X = [0,+\infty)$ and $\sigma : X \times X \mapsto [0,+\infty)$ be defined by

$$\sigma(x,y) = x + y,$$

for all $x,y \in [0,+\infty)$. Then (X,σ) is a metric-like space, but it is not a partial metric space.

Example 1.3. Let $X = \mathbb{R}$, then mappings $\sigma_i : X \times X \mapsto \mathbb{R}^+ (i = 1,2,3)$ defined by

$$\sigma_1(x,y) = |x| + |y| + a, \quad \sigma_2(x,y) = |x-b| + |y-b|, \quad \sigma_3(x,y) = x^2 + y^2,$$

are metric-like space on X, where $a \ge 0$ and $b \in \mathbb{R}$.

Proposition 1.4. *Let (σ,X) be a metric-like space, and suppose that $\{x_n\}$ is $\sigma-$ convergent to x. Then for any $y \in X$, one has*

$$\begin{aligned}\sigma(x,y) - \sigma(x,x) &\le \liminf_{n\to\infty} \sigma(x_n,y)\\ &\le \limsup_{n\to\infty} \sigma(x_n,y)\\ &\le \sigma(x,y) + \sigma(x,x).\end{aligned}$$

In particular, if $\sigma(x,x) = 0$, then one has $\lim_{n\to\infty} \sigma(x_n,y) = \sigma(x,y)$.

Proof. From the third condition of a metric-like, it follows that

$$\sigma(x_n,y) \le \sigma(x_n,x) + \sigma(x,y),$$
$$\sigma(x,y) \le \sigma(x_n,x) + \sigma(x_n,y).$$

Taking the upper limit as $n \to \infty$ in the first inequality and the lower limit as $n \to \infty$ in the second inequality, we can obtain the conclusion. □

Then we recall the notion of *C-class* and give some examples, for details see [7–9].

Definition 1.5. ([7]) A mapping $f : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following axioms:
(1) $f(s,t) \le s$;
(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,\infty)$.

Note that for some f we have that $f(0,0) = 0$. We denote C-class functions as $\mathcal{C}$.
Also note that for some f , f with respect to second variable is non-increasing.

Example 1.6. ([7]) The following functions $f : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:
(1)$f(s,t) = s - t$, $f(s,t) = s \Rightarrow t = 0$.
(2)$f(s,t) = ks$, $0<k<1$, $f(s,t) = s \Rightarrow s = 0$.
(3)$f(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $f(s,t) = s \Rightarrow s = 0$ or $t = 0$.
(4)$f(s,t) = \sqrt[n]{\ln(1+s^n)}$, $n \in N$, $f(s,t) = s \Rightarrow s = 0$.

In this paper, using a new auxiliary function called $C-$class functions (see Definition 1.5) introduced by Ansari [7], we establish some fixed and common fixed point theorems involving $(\psi - \phi)$ contractive mappings in the setting of partially ordered metric-like spaces. Our results extend, generalize, and improve some well-known results from literature. Some examples are given to support our main results.

2. Fixed Point Theorems

Throughout the rest of this paper, we denote a complete partially ordered metric-like space by $(X, \preceq, \sigma)$, i.e. $\preceq$ is a partial order on the set X and σ is a complete metric-like on X.

Theorem 2.1. *Let $(X, \preceq, \sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be a nondecreasing mapping such that for all comparable $x, y \in X$,*

$$\psi(\sigma(Fx, Fy)) \leq f(\psi(M(x, y)), \phi(M(x, y))), \tag{1}$$

where M is given by
$M(x, y) = \max\{\sigma(x, y), \sigma(x, Fx), \sigma(y, Fy), \frac{[\sigma(x,Fy)+\sigma(Fx,y)]}{4}\}$,
and

(1) $\psi : [0, \infty) \to [0, \infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$;

(2) $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$ or $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ if and only if $t > 0$,and $\phi(0) \geq 0$;

(3) $f : [0, \infty)^2 \to \mathbb{R}$ are elements of C such that f is non-increasing with respect to second variable.

(4) (a)F is continuous or (b)X has the following property:
if a non-decreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

(5) there exists $x_0 \in X$ with $x_0 \preceq Fx_0$.

Then F has a fixed point.

Proof. Let $x_0 \in X$. Then, we define a sequence $\{x_n\}$ in X such that $x_{n+1} = Fx_n$. Since $x_0 \preceq Fx_0$ and F is nondecreasing, we have

$$x_1 = Fx_0 \preceq x_2 = Fx_1 \preceq \cdots \preceq x_n = Fx_{n-1} \preceq x_{n+1} = Fx_n \cdots .$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = Fx_n$ and hence x_n is a fixed point of F. Then the conclusion holds. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By (1), we have

$$\psi(\sigma(x_{n+1}, x_n)) = \psi(\sigma(Fx_n, Fx_{n-1})) \leq f(\psi(M(x_n, x_{n-1})), \phi(M(x_n, x_{n-1}))), \tag{2}$$

which implies that $\psi(\sigma(x_{n+1}, x_n)) \leq \psi(M(x_n, x_{n-1}))$. Using the monotone property of the $\psi-$function, we get

$$\sigma(x_{n+1}, x_n) \leq M(x_n, x_{n-1}). \tag{3}$$

Now, from the triangle inequality for σ, we have

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{\sigma(x_n, x_{n-1}), \sigma(x_n, Fx_n), \sigma(x_{n-1}, Fx_{n-1}), \frac{[\sigma(x_n, Fx_{n-1}) + \sigma(Fx_n, x_{n-1})]}{4}\} \\ &= \max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n), \frac{[\sigma(x_n, x_n) + \sigma(x_{n+1}, x_{n-1})]}{4}\} \\ &\leq \max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \frac{[\sigma(x_n, x_{n+1}) + \sigma(x_n, x_{n-1})]}{2}\} \\ &= \max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\}. \end{aligned}$$

If $\sigma(x_{n+1}, x_n) > \sigma(x_n, x_{n-1})$, then $M(x_n, x_{n-1}) \leq \sigma(x_{n+1}, x_n)$, combining with (3), we obtain that $M(x_n, x_{n-1}) = \sigma(x_n, x_{n+1}) > 0$. By (2), it further implies that

$$\begin{aligned} \psi(\sigma(x_{n+1}, x_n)) &\leq f(\psi(\sigma(x_{n+1}, x_n)), \phi(\sigma(x_{n+1}, x_n))) \\ &\leq \psi(\sigma(x_{n+1}, x_n)). \end{aligned}$$

It implies that

$$\psi(\sigma(x_{n+1},x_n)) = f(\psi(\sigma(x_{n+1},x_n)),\phi(\sigma(x_{n+1},x_n)))$$

Therefore, by (2) of Definition 1.5,

$$\psi(\sigma(x_{n+1},x_n)) = 0, \text{ or } \phi(\sigma(x_{n+1},x_n)) = 0.$$

It both yields that $\sigma(x_{n+1},x_n) = 0$ since $\psi(\sigma(x_{n+1},x_n)) = 0$ or $\phi(\sigma(x_{n+1},x_n)) = 0$, which is a contradiction with $\sigma(x_{n+1},x_n) > 0$. So $\sigma(x_{n+1},x_n) < \sigma(x_n,x_{n-1})$, then $M(x_n,x_{n-1}) \le \sigma(x_n,x_{n-1})$, combining with (3), thus we have

$$\sigma(x_{n+1},x_n) \le M(x_n,x_{n-1}) \le \sigma(x_n,x_{n-1}). \tag{4}$$

Therefore, the sequence $\{\sigma(x_n,x_{n+1})\}$ is monotone non-increasing and bounded. Thus, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \sigma(x_n,x_{n+1}) = \lim_{n\to\infty} M(x_n,x_{n-1}) = r. \tag{5}$$

Suppose $r > 0$. Then letting $n \to \infty$ in the inequality (2), we get

$$\psi(r) \le f(\psi(r)), \liminf_{n\to\infty} \phi(\sigma(x_{n+1},x_n))) \le f(\psi(r),\phi(r)) \le \psi(r),$$

where second inequality holds since f is non-increasing with respect second variable. It implies that

$$\psi(r) = f(\psi(r),\phi(r)),$$

which yields that

$$\psi(r) = 0 \text{ or } \phi(r) = 0.$$

The above equalities both hold when $r = 0$. Hence,

$$\lim_{n\to\infty} \sigma(x_n,x_{n+1}) = 0. \tag{6}$$

Next, we show that $\{x_n\}$ is a $\sigma-$Cauchy sequence in X. Suppose, to the contrary, that is, $\{x_n\}$ is not a $\sigma-$Cauchy sequence. Then there exists $\epsilon > 0$ for which we can choose two subsequences $\{x_{m(i)}\}$ and $\{x_{n(i)}\}$ of $\{x_n\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad \sigma(x_{m(i)},x_{n(i)}) \ge \epsilon. \tag{7}$$

This means that

$$\sigma(x_{m(i)},x_{n(i)-1}) < \epsilon. \tag{8}$$

Using (7), (8) and the triangle inequality, we have

$$\begin{aligned}\epsilon \le \sigma(x_{n(i)},x_{m(i)}) &\le \sigma(x_{m(i)-1},x_{m(i)}) + \sigma(x_{m(i)-1},x_{n(i)})\\ &\le \sigma(x_{m(i)-1},x_{m(i)}) + \sigma(x_{m(i)-1},x_{n(i)-1}) + \sigma(x_{n(i)-1},x_{n(i)})\\ &\le 2\sigma(x_{m(i)-1},x_{m(i)}) + \sigma(x_{m(i)},x_{n(i)-1}) + \sigma(x_{n(i)-1},x_{n(i)})\\ &< 2\sigma(x_{m(i)-1},x_{m(i)}) + \sigma(x_{n(i)-1},x_{n(i)}) + \epsilon.\end{aligned}$$

Using (6), (8) and letting $n \to \infty$, we get

$$\begin{aligned}\lim_{i\to\infty} \sigma(x_{m(i)},x_{n(i)}) &= \lim_{i\to\infty} \sigma(x_{m(i)-1},x_{n(i)})\\ &= \lim_{i\to\infty} \sigma(x_{m(i)-1},x_{n(i)-1})\\ &= \lim_{i\to\infty} \sigma(x_{m(i)},x_{n(i)-1})\\ &= \epsilon.\end{aligned} \tag{9}$$

As

$$M(x_{m(i)-1}, x_{n(i)-1}) = \max\{\sigma(x_{m(i)-1}, x_{n(i)-1}), \sigma(x_{m(i)-1}, x_{m(i)}), \sigma(x_{n(i)-1}, x_{n(i)}), \frac{[\sigma(x_{m(i)-1}, x_{n(i)}) + \sigma(x_{m(i)}, x_{n(i)-1})]}{4}\},$$

using (6) and (9), we have

$$\lim_{i\to\infty} M(x_{m(i)-1}, x_{n(i)-1}) = \max\{\epsilon, 0, 0, \frac{\epsilon}{2}\} = \epsilon. \tag{10}$$

As $n(i) > m(i)$ and $x_{n(i)}$, $x_{m(i)}$ are comparable, setting $x = x_{m(i)-1}$ and $y = x_{n(i)-1}$ in (1), we obtain

$$\begin{aligned}\psi(\sigma(x_{m(i)}, x_{n(i)})) &= \psi(\sigma(Fx_{m(i)-1}, Fx_{n(i)-1}))\\ &\leq f(\psi(M(x_{m(i)-1}, x_{n(i)-1})), \phi(M(x_{m(i)-1}, x_{n(i)-1}))).\end{aligned}$$

Letting $i \to \infty$ in the above inequality and using (9) and (10), we get

$$\psi(\epsilon) \leq f(\psi(\epsilon), \liminf_{i\to\infty} \phi(M(x_{m(i)-1}, x_{n(i)-1}))) \leq f(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon),$$

It implies that

$$\psi(\epsilon) = f(\psi(\epsilon), \phi(\epsilon)),$$

which yields that

$$\psi(\epsilon) = 0 \ \text{ or } \ \phi(\epsilon) = 0.$$

The above equalities both hold when $\epsilon = 0$, which is a contradiction with $\epsilon > 0$. Hence, $\{x_n\}$ is a $\sigma-$Cauchy sequence. By the completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$, that is

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{m,n\to\infty} \sigma(x_m, x_n) = 0. \tag{11}$$

Now consider the assumption 4(a) that F is continuous. The continuity of F implies that

$$\lim_{n\to\infty} \sigma(x_{n+1}, z) = \lim_{n\to\infty} \sigma(Fx_n, z) = \sigma(Fz, z) = 0,$$

It follows that $z = Fz$.

Now consider the assumption 4(b) holds. We have $x_n \preceq z$ for every $n \in \mathbb{N}$. By (1), we have

$$\psi(\sigma(Fz, x_{n+1})) = \psi(\sigma(Fz, Fx_n)) \leq f(\psi(M(z, x_n)), \phi(M(z, x_n))), \tag{12}$$

where

$$\begin{aligned}\sigma(Fz, x_{n+1}) &\leq M(z, x_n)\\ &= \max\{\sigma(z, x_n), \sigma(z, Fz), \sigma(x_n, x_{n+1}), \frac{[\sigma(z, x_{n+1}) + \sigma(Fz, x_n)]}{4}\}.\end{aligned}$$

Taking limit as $n \to \infty$, by (11), we obtain

$$\lim_{n\to\infty} M(z, x_n) = \sigma(Fz, z).$$

Therefore, letting $n \to \infty$ in (12), we get

$$\psi(\sigma(Fz, z)) \leq f(\psi(\sigma(Fz, z)), \phi(\sigma(Fz, z))),$$

which is a contradiction unless $\sigma(Fz, z) = 0$. Thus, $Fz = z$. The proof is completed. □

Remark 2.2. In the definition of $M(x, y)$, the set $\{\sigma(x, y), \sigma(x, Fx), \sigma(y, Fy), \frac{[\sigma(x,Fy)+\sigma(Fx,y)]}{4}\}$ is replaced by any of its subsets or $M_1(x; y) = \max\{\sigma(x, y), \frac{[\sigma(x,Fy)+\sigma(Fx,y)]}{2}\}$, Theorem 2.1 remains valid.

The following theorem gives a sufficient condition for the uniqueness of the fixed point.

Theorem 2.3. *Let all the conditions of Theorem 2.1 be fulfilled and let the pair (F, i_x) is weakly increasing. If the following condition is satisfied: For arbitrary two points $x, y \in X$, there exists $z \in X$ which is comparable with both x and y. Then the fixed point of F is unique.*

Proof. Suppose that there exist two fixed points $u, v \in X$, i.e. $Fu = u$ and $Fv = v$.
Consider the following two cases.
Case 1. If u and v are comparable, then we can apply contractive condition (1) and obtain that

$$\psi(\sigma(u, v)) = \psi(\sigma(Fu, Fv)) \leq f(\psi(M(u, v)), \phi(M(u, v))), \tag{13}$$

where

$$\begin{aligned} M(u, v) &= \max\{\sigma(u, v), \sigma(u, Fu), \sigma(v, Fv), \frac{[\sigma(Fu, v) + \sigma(u, Fv)]}{4}\} \\ &= \sigma(u, v). \end{aligned} \tag{14}$$

Using (13) and (14), we have

$$\psi(\sigma(u, v)) \leq f(\psi(\sigma(u, v)), \phi(\sigma(u, v))),$$

which is a contradiction unless $\sigma(u, v) = 0$. This implies that $u = v$.
Case 2. If u is not comparable to v, then there exists $y \in X$ which is comparable to u and v. The monotonicity of F implies that $F^n y$ is comparable to $F^n u = u$ and $F^n v = v$, for $n = 0, 1, 2, \cdots$.
Moreover,

$$\begin{aligned} \psi(\sigma(u, F^n y)) &= \psi(\sigma(F^n u, F^n y)) \\ &\leq f(\psi(M(F^{n-1}u, F^{n-1}y)), \phi(M(F^{n-1}u, F^{n-1}y))), \end{aligned} \tag{15}$$

where

$$\begin{aligned} M(F^{n-1}u, F^{n-1}y) = \max\{&\sigma(F^{n-1}u, F^{n-1}y), \sigma(F^{n-1}u, F^n u), \\ &\sigma(F^{n-1}y, F^n y), \frac{[\sigma(F^n u, F^{n-1}y) + \sigma(F^{n-1}u, F^n y)]}{4}\} \\ = \max\{&\sigma(u, F^{n-1}y), \sigma(u, u), \sigma(F^{n-1}y, F^n y), \frac{[\sigma(u, F^{n-1}y) + \sigma(u, F^n y)]}{4}\} \end{aligned}$$

for n sufficiently large, because $\sigma(F^{n-1}y, F^n y) \to 0$ when $n \to \infty$.
Similarly as in the proof of Theorem 2.1, it can be shown that

$$\sigma(u, F^n y) \leq M(u, F^{n-1}y) \leq \sigma(u, F^{n-1}y).$$

It follows that the sequence $\{\sigma(u, F^n y)\}$ is nonnegative decreasing. Then, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \sigma(u, F^n y) = \lim_{n\to\infty} M(u, F^n y) = r.$$

We suppose that $r > 0$. Then letting $n \to \infty$ in (15), we have

$$\psi(r) \leq f(\psi(r), \phi(r)),$$

which is a contradiction. Hence $r = 0$. Similarly, it can be proved that

$$\lim_{n\to\infty} \sigma(v, F^n y) = 0.$$

Now, passing the limit in $\sigma(u, v) \leq \sigma(u, F^n y) + \sigma(F^n y, v)$, as $n \to \infty$, it follows that $\sigma(u, v) = 0$, so $u = v$, and the uniqueness of the fixed point is proved. The proof is completed. □

Without the assumption of weakly increasing, we can get another version of uniqueness of fixed point theorem by some modification for $M(x,y)$.

Theorem 2.4. *Let all the conditions of Theorem 2.3 be fulfilled except that $M(x,y)$ defined in Theorem 2.1 is replaced by $M_1(x,y) = \max\{\sigma(x,y), \frac{[\sigma(y,Fx)+\sigma(x,Fy)]}{2}\}$. If the following additional condition is satisfied: For arbitrary two points $x, y \in X$, there exists $z \in X$ which is comparable with both x and y. Then the fixed point of F is unique.*

Proof. Following the similar arguments to those demonstrated in Theorem 2.3, one can obtain the result. □

Remark 2.5. In Theorem 2.1, Theorem 2.3, the condition $x_0 \preceq Fx_0$ can be replaced by $x_0 \succeq Fx_0$. Just as demonstrated in Theorem 2.3, the conclusion remains valid when the assumption is changed: from the pair (F, i_x) which is weakly increasing to that which is weakly decreasing.

Now, we present an example to support the useability of our result.

Example 2.6. Let $f(s,t) = \frac{s}{1+t}$,$X = \{0,1,2\}$ and a partial order be defined as $x \preceq y$ whenever $y \leq x$ and define $\sigma : X \times X \to \mathbb{R}^+$ as follows:
$\sigma(0,0) = 10,\ \sigma(1,1) = 6,\ \sigma(2,2) = 0,\ \sigma(1,0) = \sigma(0,1) = 5,$
$\sigma(2,0) = \sigma(0,2) = 5,\ \sigma(1,2) = \sigma(2,1) = 3.$
Then $(X, \preceq, \sigma)$ is a complete partial ordered metric-like space.
Let $F : X \to X$ be defined by $F0 = 1, F1 = 2, F2 = 2$.
Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ and $\phi(t) = \frac{1}{2}$. We next verify that the function F satisfies the contractive condition (1). For that, given $x, y \in X$ with $x \preceq y$, so $y \leq x$. Then, we have the following cases.
Case 1. If $x = 1, y = 0$, then
$\sigma(F1, F0) = \sigma(2,1) = 3$
and

$$\begin{aligned} M(1,0) &= \max\{\sigma(1,0), \sigma(1,F1), \sigma(0,F0), \frac{[\sigma(F1,0)+\sigma(1,F0)]}{4}\} \\ &= \max\{5,3,5,\frac{(5+6)}{4}\} \\ &= 5. \end{aligned}$$

As $\psi(\sigma(F1,F0)) = 3 < \frac{5}{1+\frac{1}{2}} = \frac{\psi(M(1,0))}{1+\phi(M(1,0))}$, the contractive condition (1) is satisfied in this case.
Case 2. If $x = 2, y = 0$, then
$\sigma(F2, F0) = \sigma(2,1) = 3$
and

$$\begin{aligned} M(2,0) &= \max\{\sigma(2,0), \sigma(2,F2), \sigma(0,F0), \frac{[\sigma(F2,0)+\sigma(2,F0)]}{4}\} \\ &= \max\{5,0,5,\frac{(5+3)}{4}\} \\ &= 5. \end{aligned}$$

As $\psi(\sigma(F2,F0)) = 3 < \frac{5}{1+\frac{1}{2}} = \frac{\psi(M(2,0))}{1+\phi(M(2,0))}$, the contractive condition (1) is satisfied in this case.
Case 3. If $x = 2, y = 1$, then
$\sigma(F2, F1) = 0$
and

$$\begin{aligned} M(2,1) &= \max\{\sigma(2,1), \sigma(2,F2), \sigma(1,F1), \frac{[\sigma(F2,1)+\sigma(2,F1)]}{4}\} \\ &= \max\{3,0,3,\frac{(3+0)}{4}\} \\ &= 3. \end{aligned}$$

As $\psi(\sigma(F2,F1)) = 0 < \frac{3}{1+\frac{1}{2}} = \frac{\psi(M(2,1))}{1+\phi(M(2,1))}$, the contractive condition (1) is satisfied in this case.
Case 4. If $x = 0, y = 0$, then
$\sigma(F0,F0) = 6$
and

$$\begin{aligned} M(0,0) &= \max\{\sigma(0,0), \sigma(0,F0), \sigma(0,F0), \frac{[\sigma(F0,0)+\sigma(0,F0)]}{4}\} \\ &= \max\{10,5,5,\frac{(5+5)}{4}\} \\ &= 10. \end{aligned}$$

As $\psi(\sigma(F0,F0)) = 6 < \frac{10}{1+\frac{1}{2}} = \frac{\psi(M(0,0))}{1+\phi(M(0,0))}$, the contractive condition (1) is satisfied in this case.
Case 5. If $x = 1, y = 1$, then
$\sigma(F1,F1) = 0$
and

$$\begin{aligned} M(1,1) &= \max\{\sigma(1,1), \sigma(1,F1), \sigma(1,F1), \frac{[\sigma(F1,1)+\sigma(1,F1)]}{4}\} \\ &= \max\{6,3,3,\frac{(3+3)}{4}\} \\ &= 6. \end{aligned}$$

As $\psi(\sigma(F1,F1)) = 0 < \frac{6}{1+\frac{1}{2}} = \frac{\psi(M(1,1))}{1+\phi(M(1,1))}$, the contractive condition (1) is satisfied in this case.
Case 6. If $x = 2, y = 2$, then
$\sigma(F2,F2) = 0$
and

$$\begin{aligned} M(2,2) &= \max\{\sigma(2,2), \sigma(2,F2), \sigma(2,F2), \frac{[\sigma(F2,2)+\sigma(2,F2)]}{4}\} \\ &= \max\{0,0,0,\frac{(0+0)}{4}\} \\ &= 0. \end{aligned}$$

As $\psi(\sigma(F2,F2)) = 0 = \frac{6}{1+\frac{1}{2}} = \frac{\psi(M(2,2))}{1+\phi(M(2,2))}$, the contractive condition (1) is satisfied in this case.
So, F, ψ and ϕ satisfy all the hypotheses of Theorem 2.4 except that the pair (f, i_x) is weakly increasing. But according to Remark 2.5, we also obtain the uniqueness of fixed point. Indeed, here 2 is the unique fixed point of F.

Remark 2.7. Let $f(s,t) = s - t$ in Theorem 2.1-Theorem 2.4, then the conclusions coincide with Theorem 2.1, Theorem 2.2 and Theorem 2.3 in [26]. If we take $\psi(t) = t$, then the conclusions coincide with Corollary 2.1 in [26]. In addition, let $f(s,t) = ks, k \in [0,1), \psi(t) = t$ in Theorem 2.1-Theorem 2.4, then the conclusions coincide with Corollary 2.9 in [26].

Let $f(s,t) = \frac{s}{1+t}$, in Theorem 2.1-Theorem 2.4, we have the following corollary.

Corollary 2.8. *Let $(X, \leq, \sigma)$ Let $(X, \leq, \sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be nondecreasing mapping such that for all comparable $x, y \in X$,*

$$\psi(\sigma(Fx,Fy)) \leq \frac{\psi(M(x,y))}{1+\phi(M(x,y))},$$

where M is given by

$$M(x,y) = \max\{\sigma(x,y), \sigma(x,Fx), \sigma(y,Fy), \frac{[\sigma(x,Fy) + \sigma(Fx,y)]}{4}\}$$

And $\psi : [0,\infty) \to [0,\infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$, $\phi : [0,\infty) \to [0,\infty)$ *is lower semi-continuous and* $\phi(t) > 0$ *if* $t > 0$ *and* $\phi(0) \geq 0$ *.*
If there exists $x_0 \in X$ *with* $x_0 \preceq Fx_0$ *and in each of the following two cases, F has a fixed point:*

(a) *F is continuous in* $(X, \preceq, \sigma)$*,*
or

(b) $\{x_n\}$ *is a nondecreasing sequence in X such that* $x_n \to x \in X$ *implies* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

Moreover, if the additional conditions are satisfied:

(1) (F, i_x) *is weakly increasing or* $M(x,y)$ *is replaced by* $M_1(x,y)$*, and*

(2) *For arbitrary two points* $x, y \in X$*, then there exists* $z \in X$ *which is comparable with both x and y,*

then the fixed point of F is unique.

Let $f(s,t) = \log_a^{\frac{1+a^S}{2}}$, $a > 1$, in Theorem 2.1-Theorem 2.4, we have the following corollary.

Corollary 2.9. *Let* $(X, \preceq, \sigma)$ *Let* $(X, \preceq, \sigma)$ *be a complete partially ordered metric-like space. Let* $F : X \to X$ *be nondecreasing mapping such that for all comparable* $x, y \in X$*,*

$$\psi(\sigma(Fx,Fy)) \leq \log_a^{\frac{1+\psi(M(x,y))^S}{2}},$$

where M is given by

$$M(x,y) = \max\{\sigma(x,y), \sigma(x,Fx), \sigma(y,Fy), \frac{[\sigma(x,Fy) + \sigma(Fx,y)]}{4}\}.$$

And $\psi : [0,\infty) \to [0,\infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,
If there exists $x_0 \in X$ *with* $x_0 \preceq Fx_0$ *and in each of the following two cases, F has a fixed point:*

(a) *F is continuous in* $(X, \preceq, \sigma)$*,*
or

(b) $\{x_n\}$ *is a nondecreasing sequence in X such that* $x_n \to x \in X$ *implies* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

Moreover, if the additional conditions are satisfied:

(1) (F, i_x) *is weakly increasing or* $M(x,y)$ *is replaced by* $M_1(x,y)$*, and*

(2) *For arbitrary two points* $x, y \in X$*, then there exists* $z \in X$ *which is comparable with both x and y,*

then the fixed point of F is unique.

Let $f(s,t) = \sqrt[n]{\ln(1+s^n)}$, $n \in N$, in Theorem 2.1-Theorem 2.4, we have the following corollary.

Corollary 2.10. Let $(X, \preceq, \sigma)$ Let $(X, \preceq, \sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be nondecreasing mapping such that for all comparable $x, y \in X$,

$$\psi(\sigma(Fx, Fy)) \leq \sqrt[n]{\ln(1 + \psi(M(x, y))^n)}, n \in N,$$

where M is given by

$$M(x, y) = \max\{\sigma(x, y), \sigma(x, Fx), \sigma(y, Fy), \frac{[\sigma(x, Fy) + \sigma(Fx, y)]}{4}\}$$

And $\psi : [0, \infty) \to [0, \infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$ and in each of the following two cases, F has a fixed point:

(a) F is continuous in $(X, \preceq, \sigma)$,
or

(b) $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ implies $x_n \preceq x$ for all $n \in \mathbb{N}$.

Moreover, if the additional conditions are satisfied:

(1) (F, i_x) is weakly increasing or $M(x, y)$ is replaced by $M_1(x, y)$, and

(2) For arbitrary two points $x, y \in X$, then there exists $z \in X$ which is comparable with both x and y,

then the fixed point of F is unique.

3. Common Fixed Point Theorems

In the following section, we present the common fixed point theorem of two self maps h, g in a complete partially ordered metric-like space. At the same time, we also present an example to support our result.

Theorem 3.1. *Let $(X, \preceq, \sigma)$ be a complete partially ordered metric-like space and let h, $g : X \to X$ be two weakly increasing mappings w.r.t. $\preceq$ such that for every two comparable elements $x, y \in X$,*

$$\psi(\sigma(hx, gy)) \leq f(\psi(M(x, y)), \phi(M(x, y))), \tag{16}$$

where M is given by

$$M(x, y) = \max\{\sigma(hx, gy), \sigma(x, hx), \sigma(y, gy), \frac{[\sigma(x, gy) + \sigma(hx, y)]}{4}\}$$

and

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$.*

(b) $\phi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$ or $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ if and only if $t > 0$,and $\phi(0) \geq 0$*

(c) $f : [0, \infty)^2 \to \mathbb{R}$ *are elements of $\mathcal{C}$ such that where f is non-increasing with respect second variable.*

Then in each of the following two cases the mappings h and g have a common fixed point:

(1) *h or g is continuous,*
or

(2) *if a nondecreasing sequence $\{x_n\}$ converges to $x^* \in X$, then $x_n \preceq x^*$ for all n.*

Proof. Let us divide the proof into two parts.
(I) We prove that u is a fixed point of h if and only if u is a fixed point of g.
Now, suppose that u is a fixed point of h, then $hu = u$. As $u \preceq u$, apply contractive condition (16) with $x = u$, $y = u$, we have

$$\psi(\sigma(u, gu)) = \psi(\sigma(hu, gu)) \leq f(\psi(M(u, u)), \phi(M(u, u))),$$

where

$$\begin{aligned} M(u, u) &= \max\{\sigma(hu, gu), \sigma(u, hu), \sigma(u, gu), \frac{[\sigma(u, gu) + \sigma(hu, u)]}{4}\} \\ &= \max\{\sigma(u, gu), \sigma(u, u), \frac{[\sigma(u, gu) + \sigma(u, u)]}{4}\} \\ &= \max\{\sigma(u, gu), \sigma(u, u)\} \\ &= \sigma(u, gu). \end{aligned}$$

Then, we have

$$\begin{aligned} \psi(\sigma(u, gu)) &\leq f(\psi(\sigma(u, gu)), \phi(\sigma(u, gu))) \\ &\leq \psi(\sigma(u, gu)). \end{aligned}$$

It follows that

$$\psi(\sigma(u, gu)) = f(\psi(\sigma(u, gu)), \phi(\sigma(u, gu))).$$

Therefore,

$$\psi(\sigma(u, gu)) = 0 \text{ or } \phi(\sigma(u, gu)) = 0$$

It both yields that $\sigma(u, gu) = 0$ since $\psi(\sigma(u, gu)) = 0$ or $\phi(\sigma(u, gu)) = 0$. Hence, $u = gu$.
Similarly, we show that if u is a fixed point of g, then u is a fixed point of h.
(II) Let $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = hx_{2n}, x_{2n+2} = gx_{2n+1}$, for all non-negative integers, i.e. $n \in \mathbb{N} \cup \{0\}$. As h and g are weakly increasing w.r.t. $\preceq$, we obtain that

$$x_1 = hx_0 \preceq ghx_0 = x_2 = gx_1 \preceq hgx_1 = x_3 \preceq \cdots \preceq x_{2n+1} = hx_{2n} \preceq ghx_{2n} \preceq x_{2n+2} \preceq \cdots .$$

If $x_{2n} = x_{2n+1}$, for some $n \in \mathbb{N}$, then $hx_{2n} = x_{2n}$. Thus x_{2n} is a fixed point of h. By the first part, we conclude that x_{2n} is also a fixed point of g. The conclusion holds.
If $x_{2n+1} = x_{2n+2}$, for some $n \in \mathbb{N}$, then $gx_{2n+1} = x_{2n+1}$. Thus x_{2n} is a fixed point of g. By the first part, we conclude that x_{2n} is also a fixed point of h. The conclusion holds.
Therefore, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now we complete the proof in the following steps:
Step 1. We will prove that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$.
As x_{2n+1} and x_{2n+2} are comparable, apply contractive condition (16) with $x = x_{2n+1}$, $y = x_{2n+2}$, we have

$$\begin{aligned} \psi(\sigma(x_{2n+1}, x_{2n+2})) &= \psi(\sigma(hx_{2n}, gx_{2n+1})) \\ &\leq f(\psi(M(x_{2n}, x_{2n+1})), \phi(M(x_{2n}, x_{2n+1}))). \end{aligned} \tag{17}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\{\sigma(hx_{2n}, gx_{2n+1}), \sigma(x_{2n}, hx_{2n}), \sigma(x_{2n+1}, gx_{2n+1}),\\ &\frac{[\sigma(hx_{2n}, x_{2n+1}) + \sigma(x_{2n}, gx_{2n+1})]}{4}\}\\ &= \max\{\sigma(x_{2n+1}, x_{2n+2}), \sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2}),\\ &\frac{[\sigma(x_{2n+1}, x_{2n+1}) + \sigma(x_{2n}, x_{2n+2})]}{4}\}\\ &\leq \max\{\sigma(x_{2n+1}, x_{2n+2}), \sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2}),\\ &\frac{[\sigma(x_{2n}, x_{2n+1}) + \sigma(x_{2n+1}, x_{2n+2})]}{2}\}\\ &\leq \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2})\}.\end{aligned}$$

If $\sigma(x_{2n+1}, x_{2n+2}) \geq \sigma(x_{2n}, x_{2n+1}) > 0$, then it follows from the last inequality above, we have $M(x_{2n}, x_{2n+1}) \leq \sigma(x_{2n+1}, x_{2n+2})$. Combing (17) with the monotonicity of ψ, we have $\sigma(x_{2n+1}, x_{2n+2}) \leq M(x_{2n}, x_{2n+1})$. Therefore, $M(x_{2n}, x_{2n+1}) = \sigma(x_{2n+1}, x_{2n+2})$, and (17) implies that

$$\begin{aligned}\psi(\sigma(x_{2n+1}, x_{2n+2})) &\leq f(\psi(M(x_{2n}, x_{2n+1})), \phi(M(x_{2n}, x_{2n+1})))\\ &= f(\psi(\sigma(x_{2n+1}, x_{2n+2})), \phi(\sigma(x_{2n+1}, x_{2n+2}))), \end{aligned} \tag{18}$$

which is only possible when $\sigma(x_{2n+1}, x_{2n+2}) = 0$. We deduce that $x_{2n+1} = x_{2n+2}$. It is a contradiction with the assumption that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.
Hence, $\sigma(x_{2n+1}, x_{2n+2}) < \sigma(x_{2n}, x_{2n+1})$ and $M(x_{2n}, x_{2n+1}) \leq \sigma(x_{2n}, x_{2n+1})$.
Combing the above proof, we can obtain that

$$\sigma(x_{2n+1}, x_{2n+2}) \leq M(x_{2n}, x_{2n+1}) \leq \sigma(x_{2n}, x_{2n+1}).$$

In a similar way, we can obtain that

$$\sigma(x_{2n+2}, x_{2n+3}) \leq M(x_{2n+1}, x_{2n+2}) \leq \sigma(x_{2n+1}, x_{2n+2}).$$

Therefore, we conclude that for each $n = 0, 1, 2, \cdots$,

$$\sigma(x_n, x_{n+1}) \leq M(x_n, x_{n-1}) \leq \sigma(x_n, x_{n-1}).$$

It follows that the sequence $\{\sigma(x_n, x_{n+1})\}$ is nonnegative monotone non-increasing and bounded. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n-1}) = r.$$

Suppose $r > 0$. Then letting $n \to \infty$ in (18), we get

$$\psi(r) \leq f(\psi(r), \liminf_{n\to\infty} \phi(M(x_{2n}, x_{2n+1}))) \leq f(\psi(r), \phi(r)) \leq \psi(r),$$

which implies that

$$\psi(r) = f(\psi(r), \phi(r)).$$

With Definition 1.5, we have that

$$\psi(r) = 0, \ \text{ or, } \ \phi(r) = 0.$$

It both yields that $r = 0$ since $\psi(r) = 0$ or $\phi(r) = 0$.
So we have that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{19}$$

Step 2. We will prove that the sequence $\{x_n\}$ is a $\sigma-$Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a $\sigma-$Cauchy sequence. Suppose, to the contrary, that is, $\{x_{2n}\}$ is not a $\sigma-$Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences of positive integers $\{x_{2m(i)}\}$ and$\{x_{2n(i)}\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \ \ \sigma(x_{2m(i)}, x_{2n(i)}) \geq \epsilon. \tag{20}$$

This means that

$$\sigma(x_{2m(i)}, x_{2n(i)-2}) < \epsilon. \tag{21}$$

From (20) and (21) and the triangle inequality, we get

$$\begin{aligned} \epsilon &\leq \sigma(x_{2m(i)}, x_{2n(i)}) \\ &\leq \sigma(x_{2m(i)}, x_{2n(i)-2}) + \sigma(x_{2n(i)-2}, x_{2n(i)-1}) + \sigma(x_{2n(i)-1}, x_{2n(i)}) \\ &< \epsilon + \sigma(x_{2n(i)-2}, x_{2n(i)-1}) + \sigma(x_{2n(i)-1}, x_{2n(i)}). \end{aligned} \tag{22}$$

By letting $i \to \infty$ in the above inequality and using (19) and (22), we have that

$$\lim_{i\to\infty} \sigma(x_{2m(i)}, x_{2n(i)}) = \epsilon. \tag{23}$$

Moreover,

$$\begin{aligned} \epsilon &\leq \sigma(x_{2m(i)}, x_{2n(i)}) \\ &\leq \sigma(x_{2m(i)}, x_{2m(i)-1}) + \sigma(x_{2m(i)-1}, x_{2n(i)+1}) + \sigma(x_{2n(i)+1}, x_{2n(i)}) \\ &\leq \sigma(x_{2m(i)}, x_{2m(i)-1}) + \sigma(x_{2m(i)-1}, x_{2n(i)}) + \sigma(x_{2n(i)+1}, x_{2n(i)}) + \sigma(x_{2n(i)+1}, x_{2n(i)}) \\ &= \sigma(x_{2m(i)}, x_{2m(i)-1}) + \sigma(x_{2m(i)-1}, x_{2n(i)}) + 2\sigma(x_{2n(i)+1}, x_{2n(i)}) \\ &\leq \sigma(x_{2m(i)}, x_{2m(i)-1}) + \sigma(x_{2m(i)-1}, x_{2m(i)}) + \sigma(x_{2m(i)}, x_{2n(i)}) + 2\sigma(x_{2n(i)+1}, x_{2n(i)}) \\ &\leq 2\sigma(x_{2m(i)}, x_{2m(i)-1}) + \sigma(x_{2m(i)}, x_{2n(i)+1}) + 2\sigma(x_{2n(i)+1}, x_{2n(i)}). \end{aligned} \tag{24}$$

Using (21) and (24) and letting $i \to \infty$, we get

$$\begin{aligned} \lim_{i\to\infty} \sigma(x_{2m(i)}, x_{2n(i)}) &= \lim_{i\to\infty} \sigma(x_{2m(i)-1}, x_{2n(i)+1}) \\ &= \lim_{i\to\infty} \sigma(x_{2m(i)-1}, x_{2n(i)}) \\ &= \lim_{i\to\infty} \sigma(x_{2m(i)}, x_{2n(i)+1}) = \epsilon. \end{aligned}$$

Since $x_{2n(i)}$ and $x_{2m(i)-1}$ are comparable, so by the definition of $M(x, y)$ and using previous limits, we get that $\lim_{i\to\infty} M(x_{2n(i)}, x_{2m(i)-1}) = \epsilon$. Indeed,

$$\begin{aligned} M(x_{2n(i)}, x_{2m(i)-1}) &= \max\{\sigma(x_{2n(i)+1}, x_{2m(i)}), \sigma(x_{2n(i)}, x_{2n(i)+1}), \sigma(x_{2m(i)-1}, x_{2m(i)}) \\ &\quad \frac{[\sigma(x_{2n(i)+1}, x_{2m(i)-1}) + \sigma(x_{2m(i)}, x_{2n(i)})]}{4}\} \\ &\to \max\{\epsilon, 0, 0, \frac{\epsilon}{2}\} \\ &= \epsilon. \end{aligned}$$

Now since the terms of the sequence $\{x_{2n}\}$ are mutually comparable, we can apply (16) to obtain

$$\begin{aligned} \psi(\sigma(x_{2n(i)+1}, x_{2m(i)}) &= \psi(\sigma(hx_{2n(i)}, gx_{2m(i)-1})) \\ &\leq f(\psi(M(x_{2n(i)}, x_{2m(i)-1})), \phi(M(x_{2n(i)}, x_{2m(i)-1}))). \end{aligned}$$

Passing to the limit when $i \to \infty$, we obtain that

$$\psi(\epsilon) \leq f(\psi(\epsilon), \phi(\epsilon)),$$

which is a contradiction unless $\epsilon = 0$. Hence, $\{x_{2n}\}$ is a $\sigma-$Cauchy sequence.
By the completeness of X, there is $z \in X$ such that $\lim_{i\to\infty} x_n = z$, that is,

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{m,n\to\infty} \sigma(x_m, x_n) = 0.$$

Step 3. We have to prove that z is a common fixed point of h and g. We shall distinguish the cases (1) and (2) of the theorem.
(1) Suppose that the mapping h is continuous. Since $x_{2n} \to z$, we obtain that $x_{2n+1} = hx_{2n} \to hz$. On the other hand, $x_{2n+1} \to z$ (as the subsequence of $\{x_n\}$). It follows that $hz = z$. To prove that $gz = z$, using $z \preceq z$, we can put $x = y = z$ in (16) and obtain that

$$\psi(\sigma(hz, gz)) \leq f(\psi(M(z, z)), \phi(M(z, z))),$$

where

$$\begin{aligned} M(z,z) &= \max\{\sigma(hz, gz), \sigma(z, hz), \sigma(z, gz), \frac{[\sigma(hz, z) + \sigma(z, gz)]}{4}\} \\ &= \max\{\sigma(z, gz), \sigma(z, z), \sigma(z, gz), \frac{[\sigma(z, z) + \sigma(z, gz)]}{4}\} \\ &= \sigma(z, gz). \end{aligned}$$

Hence, $\psi(\sigma(z, gz)) \leq f(\psi(\sigma(z, gz)), \phi(\sigma(z, gz)))$ and it follows that $z = gz$.
The proof is similar if g is continuous.
(2) Suppose now that the condition (2) of the theorem holds.
The sequence $\{x_n\}$ is nondecreasing w.r.t. $\preceq$ and it follows that $x_n \preceq x^*$.
Taking $x_{2n} = x, x^* = y$ in (16), we get that

$$\psi(\sigma(hx_{2n}, gx^*) \leq f(\psi(M(x_{2n}, x^*)), \phi(M(x_{2n}, x^*))), \tag{25}$$

where

$$\begin{aligned} M(x_{2n}, x^*)) &= \max\{\sigma(x_{2n+1}, gx^*), \sigma(x^*, gx^*), \sigma(x_{2n}, x_{2n+1}) \\ &\quad \frac{[\sigma(x_{2n+1}, x^*) + \sigma(x_{2n}, gx^*)]}{4}\} \\ &\quad \to \sigma(x^*, gx^*). \end{aligned}$$

Now passing the limits when $n \to \infty$ in (25) , we have

$$\psi(\sigma(x^*, gx^*)) \leq f(\psi(\sigma(x^*, gx^*)), \phi(\sigma(x^*, gx^*))) \leq \psi(\sigma(x^*, gx^*)).$$

It follows that

$$\psi(\sigma(x^*, gx^*)) = f(\psi(\sigma(x^*, gx^*)), \phi(\sigma(x^*, gx^*)))$$

With Definition 1.5, we obtain that

$$\psi(\sigma(x^*, gx^*)) = 0, \text{ or, } \phi(\sigma(x^*, gx^*)) = 0.$$

It yields that $\sigma(x^*, gx^*) = 0$ and hence $gx^* = x^*$.
The fact that $hx^* = x^*$ is now derived in the same way in the case (2). The proof is completed. □

Remark 3.2. Theorem 3.1 remains valid if the condition that (h, g) is weakly increasing is replaced by (h, g) is weakly decreasing, i.e., $hx \geq ghx$ and $gx \geq hgx$ for each $x \in X$.

Referring to Theorem 2.3 and Theorem 2.4, we present two theorems for uniqueness of common fixed point theorem which give sufficient conditions for the uniqueness of the common fixed point.

Theorem 3.3. *Let all the conditions of Theorem 3.1 be satisfied. If the following additional condition is satisfied: For arbitrary two points $x, y \in X$, there exists $z \in X$ which is comparable with both x and y. Then the common fixed point of h and g is unique.*

Proof. Following the similar arguments to those presented in Theorem 2.3, one can get the result. The proof is completed. □

Theorem 3.4. *Let all the conditions of Theorem 3.1 be satisfied except that $M(x, y)$ defined in Theorem 3.3 is replaced by $M_2(x, y) = \max\{\sigma(hx, gy), \frac{[\sigma(hx,y)+\sigma(x,gy)]}{2}\}$. If the following additional condition is satisfied: For arbitrary two points $x, y \in X$, there exists $z \in X$ which is comparable with both x and y. Then the common fixed point of f and g is unique.*

Proof. Following the similar arguments to those presented in Theorem 2.4, one can get the result. The proof is completed. □

Now, we present an example to support the useability of our result.

Example 3.5. Let $X = \{0, 1, 2, \cdots\}$. Define the function $h, g : X \to X$ by

$$hx=\begin{cases} 0, & \text{if} \quad x = 0, \\ x-1, & \text{if} \quad x \neq 0, \end{cases}$$

and

$$gx=\begin{cases} 0, & \text{if} \quad x \in \{0,1\}, \\ x-2, & \text{if} \quad x \geq 2. \end{cases}$$

Let $\sigma : X \times X \to \mathbb{R}^+$ be given by

$$\sigma(x, y)=\begin{cases} x, & \text{if} \quad x = y, \\ \max\{\frac{x}{2}, \frac{y}{2}\} & \text{if} \quad x \neq y. \end{cases}$$

Define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2$ and $\phi(t) = \frac{1}{|2t-1|+\frac{1}{1000}}$. Define a partial order $\preceq$ on X by $x \preceq y$ if and only if $y \leq x$. Then we have the following conclusions:

(1) $(X, \preceq, \sigma)$ is a complete partially ordered metric-like space,

(2) h and g are weakly increasing mappings w.r.t. $\preceq$,

(3) h is continuous,

(4) For every two comparable elements $x, y \in X$, (16) holds.

Proof. The proof of (1) holds obviously.

To prove (2), let $x \in X$. If $x \in \{0, 1, 2\}$, then $hgx = 0 \leq gx = 0$ and $ghx = 0 \leq hx$. So, $gx \preceq hgx$ and $hx \preceq ghx$. While, if $x \geq 3$, then $hgx = x - 3 \leq x - 2 = gx$ and $ghx = x - 3 \leq x - 1 = hx$. So, $gx \preceq hgx$ and $hx \preceq ghx$. Hence, h and g are weakly increasing mappings w.r.t.$\preceq$.

To prove h is continuous, let $\{x_n\}$ be a sequence in X such that $x_n \to x^* \in X$, i.e. there exists $k \in \mathbb{N}$ such that $x_n = x^*$ for all $n \geq k$. So $hx_n = hx^*$ for all $n \geq k$. Hence, $hx_n \to hx^*$, that is, h is continuous.

To prove (4), given $x, y \in X$ with $x \preceq y$, so $y \preceq x$. Thus, we have the following cases:
Case 1. If $x = 0, y = 0$, then as $\sigma(h0, g0) = 0$,

$$\begin{aligned} M(0,0) &= \max\{\sigma(h0,g0),\sigma(0,h0),\sigma(0,g0),\frac{[\sigma(0,g0)+\sigma(h0,0)]}{4}\} \\ &= \max\{0,0,0,\frac{(0+0)}{4}\} \\ &= 0, \end{aligned}$$

and $\psi(\sigma(f0, g0)) = 0 \leq \frac{\psi(M(0,0))}{1+\phi(M(0,0))}$, the contractive condition (16) is satisfied in this case.
Case 2. If $x = 1, y = 0$, then as $\sigma(f1, g0) = 0$,

$$\begin{aligned} M(1,0) &= \max\{\sigma(h1,g0),\sigma(1,h1),\sigma(0,g0),\frac{[\sigma(1,g0)+\sigma(h1,0)]}{4}\} \\ &= \max\{0,\frac{1}{2},0,\frac{1}{8}\} \\ &= \frac{1}{2}, \end{aligned}$$

and $\psi(\sigma(f1, g0)) = 0 \leq \frac{\psi(M(1,0))}{1+\phi(M(1,0))}$, the contractive condition (16) is satisfied in this case.
Case 3. If $x = 2, y = 0$, then as $\sigma(f2, g0) = \frac{1}{2}$,

$$\begin{aligned} M(2,0) &= \max\{\sigma(f2,g0),\sigma(2,f2),\sigma(0,g0),\frac{[\sigma(2,g0)+\sigma(f2,0)]}{4}\} \\ &= \max\{\frac{1}{2},1,0,\frac{(1+\frac{1}{2})}{4}\} \\ &= 1, \end{aligned}$$

and $\psi(\sigma(h2, g0)) = \frac{1}{4} < \frac{[1]^2}{1+\frac{1}{1+\frac{1}{1000}}} = \frac{\psi(M(2,0))}{1+\phi(M(2,0))}$, the contractive condition (16) is satisfied in this case.
Case 4. If $x = 1, y = 1$, then as $\sigma(f1, g1) = 0$,

$$\begin{aligned} M(1,1) &= \max\{\sigma(f1,g1),\sigma(1,f1),\sigma(1,g1),\frac{[\sigma(1,g1)+\sigma(f1,1)]}{4}\} \\ &= \max\{0,\frac{1}{2},\frac{1}{2},\frac{(\frac{1}{2}+\frac{1}{2})}{4}\} \\ &= \frac{1}{2}, \end{aligned}$$

and $\psi(\sigma(f1, g1)) = 0 \leq \frac{\psi(M(1,1))}{1+\phi(M(1,1))}$, the contractive condition (16) is satisfied in this case.
Case 5. If $x = 2, y = 1$, then as $\sigma(h2, g1) = \frac{1}{2}$,

$$\begin{aligned} M(2,1) &= \max\{\sigma(h2,g1),\sigma(2,h2),\sigma(1,g1),\frac{[\sigma(2,g1)+\sigma(h2,1)]}{4}\} \\ &= \max\{\frac{1}{2},1,\frac{1}{2},\frac{(1+1)}{4}\} \\ &= 1, \end{aligned}$$

and $\psi(\sigma(h2, g1)) = (\frac{1}{2})^2 = \frac{1}{4} < \frac{[1]^2}{1+\frac{1}{1+\frac{1}{1000}}} = \frac{\psi(M(2,1))}{1+\phi(M(2,1))}$, the contractive condition (16) is satisfied in this case.

Case 6. If $x, y \geq 2$, with $x = y$, then as $\sigma(hx, gy) = \frac{x-1}{2}$,

$$\begin{aligned} M(x,y) &= \max\{\sigma(fx,gy), \sigma(x,fx), \sigma(y,gy), \frac{[\sigma(x,gy)+\sigma(fx,y)]}{4}\} \\ &= \max\{\frac{x-1}{2}, \frac{x}{2}, \frac{y}{2}, \frac{[\frac{x}{2}+\frac{y}{2}]}{4}\} \\ &= \frac{x}{2} = \frac{y}{2}, \end{aligned}$$

and $\psi(\sigma(fx, gy)) = (\frac{x-1}{2})^2 \leq \frac{(\frac{x}{2})^2}{1+\frac{1}{|x-1|+\frac{1}{1000}}} = \frac{\psi(\frac{x}{2})}{1+\phi(\frac{x}{2})}$, the contractive condition (16) is satisfied in this case.

Case 7. If $x > y \geq 2$, with $x = y + 1$, then as $\sigma(fx, gy) = \frac{y}{2}$,

$$\begin{aligned} M(x,y) &= \max\{\sigma(hx,gy), \sigma(x,hx), \sigma(y,gy), \frac{[\sigma(x,gy)+\sigma(hx,y)]}{4}\} \\ &= \max\{\frac{y}{2}, \frac{y+1}{2}, \frac{y}{2}, \frac{\frac{(y+1)}{2}+y}{4}\} \\ &= \frac{y+1}{2}, \end{aligned}$$

and $\psi(\sigma(hx, gy)) = (\frac{y}{2})^2 \leq \frac{(\frac{y+1}{2})^2}{1+\frac{1}{|y|+\frac{1}{1000}}} = \frac{\psi(\frac{y+1}{2})}{1+\phi(\frac{y+1}{2})}$, the contractive condition (16) is satisfied in this case.

Case 8. If $x > y \geq 2$, with $x > y + 1$, then as $\sigma(hx, gy) = \frac{x-1}{2}$,

$$\begin{aligned} M(x,y) &= \max\{\sigma(hx,gy), \sigma(x,hx), \sigma(y,gy), \frac{[\sigma(x,gy)+\sigma(hx,y)]}{4}\} \\ &= \max\{\frac{x-1}{2}, \frac{x}{2}, \frac{y}{2}, \frac{\frac{x}{2}+\frac{x-1}{2}}{4}\} \\ &= \frac{x}{2}, \end{aligned}$$

and $\psi(\sigma(hx, gy)) = (\frac{x-1}{2})^2 \leq \frac{(\frac{x}{2})^2}{1+\frac{1}{|x-1|+\frac{1}{1000}}} = \frac{\psi(\frac{x}{2})}{1+\phi(\frac{x}{2})}$, the contractive condition (16) is satisfied in this case.

Thus, h, g, ψ and ϕ satisfy the hypotheses of Theorem 3.3 in the case (1) and hence f and g have a common fixed point. Indeed, 0 is the common fixed point of h and g. □

Remark 3.6. **(1)** If we take $f(s, t) = s - t$ in the Theorem 3.1-Theorem 3.4, the conclusion coincides with Theorem 3.1.- Theorem 3.3 in [26].

(2) If we take $f(s, t) = s - t$ and $\psi(t) = t$ in the Theorem 3.1-Theorem 3.4, the conclusion coincides with Corollary 3.1.-Corollary 3.2 in [26].

(3) If we take $f(s, t) = ks$ and $\psi(t) = t$ in the Theorem 3.1-Theorem 3.4, the conclusion coincides with Corollary 3.3.- Corollary 3.4 in [26].

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