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Equiconvergence Property for Spectral Expansions Related to Perturbations of the Operator $-u''(-x)$ with Initial Data

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Abstract. Uniform equiconvergence of spectral expansions related to the second-order differential operators with involution: $-u''(-x)$ and $-u''(-x) + q(x)u(x)$ with the initial data $u(-1) = 0$, $u'(-1) = 0$ is obtained. Starting with the spectral analysis of the unperturbed operator, the estimates of the Green's functions are established and then applied via the contour integrating approach to the spectral expansions. As a corollary, it is proved that the root functions of the perturbed operator form the basis in $L_2(-1,1)$ for any complex-valued coefficient $q(x) \in L_2(-1,1)$.

1. Introduction

Let $q(x) \in L_2(-1,1)$ be a complex-valued function and $\mathcal{L}$ be the closed operator related to the functional-differential operation

$$l[u] = -u''(-x) + q(x)u(x), \quad -1 < x < 1, \tag{1}$$

equipped with the initial data at $x = -1$:

$$u(-1) = 0, \quad u'(-1) = 0. \tag{2}$$

If $q(x)$ is real-valued then the operator$\mathcal{L}$ is self-adjoint on the domain $\mathcal{D}(\mathcal{L}) = \left\{f(x) \in W_2^2(-1,1) \,|\, f(-1) = f'(-1) = 0\right\}$, the spectrum of $\mathcal{L}$ is discrete and, therefore, any function $f(x) \in L_2(-1,1)$ has the related eigenfunction expansion which converges in $L_2(-1,1)$.

In the general (non-self-adjoint) case the convergence of the corresponding biorthogonal series for a given $f(x)$ could be studied through the equiconvergence theorems. Namely, if $S_m(x, f)$ and $\sigma_m(x, f)$ are partial sums of the eigenfunction expansions for the unperturbed ($q(x) \equiv 0$) and general cases of $\mathcal{L}$ then the equiconvergence result means that

$$\sigma_m(x, f) - S_m(x, f) = o(1), \quad m \to \infty. \tag{3}$$

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For conventional ODE, $S_m(x, f)$ are the partial sums of the Fourier series which are widely studied. In particular, the relation (3) means that any test on convergence or divergence of Fourier series could be translated for the general eigenfunction expansion $\sigma_m(x, f)$.

In the case when the differential operator is defined by the operation with involution it is natural to compare $\sigma_m(x, f)$ with the sums $S_m(x, f)$ for the case $q(x) \equiv 0$ in (1) due to following reasons: (i) it is hard to find a competing boundary value problem for (1)–(2) as the ODE with initial data do not have the spectrum; (ii) the eigenfunctions for (1)–(2) with $q(x) \equiv 0$ are given explicitly and $S_m(x, f)$ could be studied easier; (iii) the asymptotics of eigenfunctions in the unperturbed case shows that its main terms form the "bad" system for constructing related expansions.

The central result of the paper is the equiconvergence property (3) that holds uniformly with respect to $x \in [-1, 1]$ for any integrable function $f(x)$. As a byproduct of this result, the basicity of root functions of $\mathcal{L}$ in $L_2(-1, 1)$ is obtained.

There are several approaches to attack the equiconvergence theorem.

The one that dates back to the early 1990's uses a representation of the partial sum of the eigenfunction expansion via the contour integral in the complex plane and the resolvent of the operator (see [5] and the recent research in [7, 12, 21, 25, 26]). It is worth also mentioning the approach by V. I. Il'in (see the survey in [9]) which is not connected directly with the form of the boundary conditions and could be applied to a variety of non-self-adjoint cases [8, 18–20]. The basis property of eigenfunctions for various types of differential operators was lately discussed in [10, 11, 23, 27].

Results on the spectral properties of one-dimensional differential operators with involution (we use the simplest one – with reflection $\nu(x) = -x$ on $[-1, 1]$) are eagerly applied in research of PDE. The recent papers by Aleroev, Kirane and Malik [1], Kirane and Al-Sati [13] give plausible examples. Various applications of differential operators with involutions could be found in [6].

Spectral theory of differential operators with involution forms a specific niche in the study of ODE. Eigenfunction expansions for the first-order differential operators with involution are considered in [3, 14, 28]. Sample second-order differential operators with involution are discussed in [15, 29, 30]. A specific example of a boundary-value problem for the second-order differential operator with involution that produces an infinite number of associated functions is given in [16, 17]. There are also valuable results on the Green's function for the boundary value problems related to functional-differential operators with involution by Cabada and Tojo [4, 32] and new types of non-classical Sturm–Liouville problems by Aydemir, Mukhtarov et al. [2, 24].

The paper consists of three parts. Following this introduction the second section focuses on the unperturbed case. The eigenvalues, eigenfunctions, the Green's function are discussed. The third section proceeds with the general case of (1)–(2). The estimates of its Green's function are followed by the proof of the equiconvergence result and further by the statement on the basis property for root functions of $\mathcal{L}$ in $L_2(-1, 1)$.

2. The Unperturbed Case

Consider the self-adjoint operator $\mathcal{L}_0$ related to the problem

$$\begin{aligned} &-u''(-x) = \lambda u(x), \quad -1 < x < 1, \\ &u(-1) = u'(-1) = 0. \end{aligned} \tag{4}$$

Let $\lambda = \rho^2$. As $\mathcal{L}_0$ is self-adjoint, the parameter ρ is either real or pure imaginary. Since only the identical zero among the linear functions suits the initial data in (4), $\lambda = 0$ is not the eigenvalue of $\mathcal{L}_0$.

Let $\lambda \neq 0$. The general solution of the equation in (4),

$$u(x) = A \sinh \rho x + B \cos \rho x, \tag{5}$$

produces the following condition for the eigenvalues of (4):

$$\omega(\rho) \equiv \rho(\cosh \rho \cos \rho + \sinh \rho \sin \rho) = 0. \tag{6}$$

Suppose that ρ is a negative real number. Then (6) yields the equation

$$\tan\rho + \coth\rho = 0 \tag{7}$$

which has infinitely many roots $\{\rho_k\}_{k\le -1}$. The first root ρ_{-1} belongs to $(-3\pi/4, -\pi/2)$ and $\rho_k \to \infty$ as $k \to -\infty$. Since $\coth\rho_k \to -1$ we have $\tan\rho_k \to 1$ and it is reasonable to put $\rho_k = (\pi/4) + \pi k + \delta_k$. Then (7) transforms into the equation $\tan((\pi/4) + \delta_k) = (1 + \exp(2\rho_k))/(1 - \exp(2\rho_k))$ whence $\tan\delta_k = \exp(2\rho_k)$. Therefore δ_k is the infinitesimal which behaves like $\exp(2\rho_k) \sim \exp((\pi/2) - 2\pi|k|)$ as $k \to -\infty$.

The roots of (7) are simple since $(\tan\rho + \coth\rho)' = 2 + \tan^2\rho - \coth^2\rho$ which equals 2 when $\rho = \rho_k$.

The case when $\rho = i\nu$, $\nu \in \mathbb{R}_+$, is treated similarly. The condition (6) transforms into $\tan\nu - \coth\nu = 0$ whence $\nu = \nu_k = (\pi/4) + \pi k + \delta_k$, $k = 0, 1, 2, \dots$ (the first root ν_0 belongs to $(\pi/4, \pi/2)$). This easily implies the behavior $\delta_k \sim \exp(-(\pi/2) - 2\pi k)$ as $k \to +\infty$ and the simplicity of the roots ν_k.

Lemma 1. *The operator $\mathcal{L}_0$ is not bounded below and has simple eigenvalues $\lambda_k = \rho_k^2$, $k \in \mathbb{Z}$, which have the asymptotics*

$$\rho_k = (\pi/4) + \pi k + \delta_k,\ k = -1, -2, \dots, \quad \delta_k \sim \exp((\pi/2) - 2\pi|k|),\ k \to -\infty, \tag{8}$$

$$\rho_k = i((\pi/4) + \pi k + \delta_k),\ k = 0, 1, 2, \dots, \quad \delta_k \sim \exp(-(\pi/2) - 2\pi k),\ k \to +\infty. \tag{9}$$

Now the eigenfunctions $u_k(x)$ of $\mathcal{L}_0$ could be easily constructed from the general solution (5).

For $k = -1, -2, \dots$, we put

$$u_k(x) = -2\exp(\rho_k)\Big(\sinh\rho_k \cos\rho_k x + \cos\rho_k \sinh\rho_k x\Big), \tag{10}$$

and, for $k = 0, 1, 2, \dots$,

$$u_k(x) = 2i\exp(\rho_k)\Big(\cos(i\rho_k)\sinh(i\rho_k x) + \sinh(i\rho_k)\cos(i\rho_k x)\Big). \tag{11}$$

The choice of constants in (10) and (11) becomes clear from the following lemma.

Lemma 2. *The eigenfunctions (10) satisfy the estimate (with $\alpha_k = (\pi/4) + \pi k$)*

$$u_k(x) = \cos\alpha_k x - (-1)^k\sqrt{2}\exp(-|\alpha_k|)\sinh\alpha_k x + O(\exp(-2\pi|k|)), \quad k \to -\infty, \tag{12}$$

and the eigenfunctions (11) — the estimate

$$u_k(x) = \sin\alpha_k x + (-1)^k\sqrt{2}\exp(-\alpha_k)\cosh\alpha_k x + O(\exp(-2\pi k)), \quad k \to +\infty. \tag{13}$$

The estimates (12) and (13) are proved by the direct substitution of (8) and (9) into (10) and (11).

It is clear from (12) and (13) that the eigenfunctions $u_k(x)$ of $\mathcal{L}_0$ are uniformly bounded with respect to $k \in \mathbb{Z}$ and $x \in [-1, 1]$. Moreover, their $L_2[-1, 1]$-norms are estimated as follows

– for $k \to -\infty$,

$$\begin{aligned}\|u_k\|_2^2 &= \|\cos\alpha_k x\|_2^2 + 2\exp(-2|\alpha_k|)\|\sinh\alpha_k x\|_2^2 + O(\exp(-2\pi|k|)) = \\ &= (1 + \alpha_k^{-1}/2) + \alpha_k^{-1}\exp(-2|\alpha_k|)\sinh(2\alpha_k) + O(\exp(-2\pi|k|)) = 1 + O(\exp(-2\pi|k|)),\end{aligned} \tag{14}$$

– for $k \to +\infty$,

$$\begin{aligned}\|u_k\|_2^2 &= \|\sin\alpha_k x\|_2^2 + 2\exp(-2|\alpha_k|)\|\cosh\alpha_k x\|_2^2 + O(\exp(-2\pi|k|)) = \\ &= (1 - \alpha_k^{-1}/2) + \alpha_k^{-1}\exp(-2\alpha_k)\sinh(2\alpha_k) + O(\exp(-2\pi|k|)) = 1 + O(\exp(-2\pi|k|)).\end{aligned} \tag{15}$$

It follows from (14) and (15) and the self-adjointness of $\mathcal{L}_0$ that the system $\{u_k(x)\}_{k\in\mathbb{Z}}$ forms the almost normalized orthogonal basis in $L_2(-1, 1)$.

The estimates (12) and (13) give the temptation to treat the eigenfunctions $u_k(x)$ of $\mathcal{L}_0$ as a perturbation of the system

$$u_k^0(x) = \begin{cases} \cos((\pi/4) + \pi k), & k = -1, -2, \dots, \\ \sin((\pi/4) + \pi k), & k = 0, 1, 2, \dots \end{cases} \tag{16}$$

The known properties of the system $\{u_k^0(x)\}_{k\in\mathbb{Z}}$ show that such an approach is faulty. The results in [22] and the symmetry of functions in (16) show that this system does not form the basis in $L_2(-1,1)$[1)] (though it is complete there).

Instead of the system (16) one could more likely consider the system $\{\widetilde{u}_k^0(x)\}_{k\in\mathbb{Z}}$ that differs from (12)–(13) by the absence of O-terms in the right-hand sides. Clearly $\sum_{k\in\mathbb{Z}} \left\| u_k - \widetilde{u}_k^0 \right\|_2 < \infty$ and it is possible to apply results on stability of bases in Banach spaces [31, Chapter I §10].

The eigenfunction expansion of an arbitrary function $f(x) \in L_2(-1,1)$ could be rewritten using the resolvent of the considered operator. If $G_0(x,t;\lambda)$ is the integral kernel of the resolvent of $\mathcal{L}_0$ (i.e., it is the Green's function of $\mathcal{L}_0$) and $S_m(x,f)$ is the partial sum of the eigenfunction expansion for f:

$$S_m(x,f) = \sum_{|k|\le m} \|u_k\|_2^{-2} (f, u_k)\, u_k(x), \tag{17}$$

then the following relation holds

$$S_m(x,f) = -\frac{1}{2\pi i} \int_{L_m} \left(\int_{-1}^{1} G_0(x,t;\lambda) f(t)\, dt \right) d\lambda \tag{18}$$

where $L_m = \{\lambda \in \mathbb{C} \,|\, |\lambda| = R_m^2\}$ and this circle of radius R_m contains only the eigenvalues λ_k with $|k| \le m$. The asymptotics in Lemma 1 show that, for large m, the radius R_m could be taken as $(\pi/2) + \pi m$.

Let us construct the Green's function of $\mathcal{L}_0$. By its definition, for any function $f(x) \in L_1(-1,1)$, the Green's function $G_0(x,t;\lambda)$ gives the almost everywhere (a. e.) solution of the problem

$$\begin{aligned} &-u''(-x) = \lambda u(x) + f(x), \quad -1 < x < 1, \\ &u(-1) = u'(-1) = 0, \end{aligned} \tag{19}$$

in the form

$$u(x) = \int_{-1}^{1} G_0(x,t;\lambda) f(t)\, dt. \tag{20}$$

The straightforward calculation proves the following assertion.

[1)] The same property actually holds for any space $L_p(-1,1)$, $p > 1$.

Theorem 1. *If $\lambda = \rho^2$ is not the eigenvalue of $\mathcal{L}_0$ then the unique solution of (19) is given by the formula*[2)]

$$u(x) = \frac{1}{2\omega(\rho)}\Bigg[(\cosh\rho\sin\rho + \sinh\rho\cos\rho)\int_{-1}^{1} f(t)\sinh\rho t\,dt - \int_{-1}^{1} f(t)\cos\rho t\,dt\Bigg]\sinh\rho x +$$
$$+\frac{1}{2\omega(\rho)}\Bigg[(\cosh\rho\sin\rho - \sinh\rho\cos\rho)\int_{-1}^{1} f(t)\cos\rho t\,dt - \int_{-1}^{1} f(t)\sinh\rho t\,dt\Bigg]\cos\rho x +$$
$$+\frac{1}{2\rho}\Bigg[\int_{-1}^{-x} - \int_{x}^{1}\Bigg](\cos\rho x\sin\rho t + \sinh\rho x\cosh\rho t)f(t)\,dt - \frac{1}{2\rho}\int_{-x}^{x}(\sin\rho x\cos\rho t + \cosh\rho x\sinh\rho t)f(t)\,dt. \tag{21}$$

Hence the Green's function has the form

$$G_0(x,t;\lambda) = \frac{1}{2\omega(\rho)}\Bigg[(\cosh\rho\sin\rho + \sinh\rho\cos\rho)\sinh\rho t\sinh\rho x - \cos\rho t\sinh\rho x +$$
$$+ (\cosh\rho\sin\rho - \sinh\rho\cos\rho)\cos\rho t\cos\rho x - \sinh\rho t\cos\rho x\Bigg] - \frac{1}{2\rho}g_0(x,t;\lambda) \tag{22}$$

where

$$g_0(x,t;\lambda) = \begin{cases} \operatorname{sgn} x\,(\sin\rho x\cos\rho t + \cosh\rho x\sinh\rho t) & \text{if } |t| \le |x|, \\ \operatorname{sgn} t\,(\cos\rho x\sin\rho t + \sinh\rho x\cosh\rho t) & \text{if } |t| \ge |x|. \end{cases} \tag{23}$$

The properties of the Green's function $G_0(x,t;\lambda)$ can be easily extracted from its explicit form (22)–(23).

Lemma 3. *If $\lambda = \rho^2$ is not the eigenvalue of $\mathcal{L}_0$ then*
a) $G_0(x,t;\lambda)$ is symmetric: $G_0(x,t;\lambda) = G_0(t,x;\lambda)$;
b) $G_0(x,t;\lambda)$ is continuous in the rectangle $-1 \le x,t \le 1$;
c) $G_0(x,t;\lambda)$ has the derivative $\partial_x G_0(x,t;\lambda)$ which is continuous in each triangle $\{(x,t)\,|\,-1 \le x \le 1, -1 \le t < -x\}$, $\{(x,t)\,|\,-1 \le x \le 1, -x < t \le 1\}$, has the limit values on the diagonal $x = -t$ and satisfies the relation

$$\partial_x G_0(x,t;\lambda)|_{t=-x+0} - \partial_x G_0(x,t;\lambda)|_{t=-x-0} = 1;$$

d) $G_0(x,t;\lambda)$ has the second derivative $\partial_x^2 G_0(x,t;\lambda)$ which is continuous inside the triangles of c), and satisfies there the equation

$$-\partial_x^2 G_0(-x,t;\lambda) = \lambda G_0(x,t;\lambda)$$

and the initial conditions $G_0(-1,t;\lambda) = \partial_0 G_0(-1,t;\lambda) = 0$.

The formula (20) readily implies that properties b)–d) could be taken as an alternative definition of the Green's function $G_0(x,t;\lambda)$ as these properties provide the integral in (20) gives the solution to the problem (19).

Our next step is obtaining the estimate of $G_0(x,t;\lambda)$ for $\lambda \in \mathbb{C}$ outside the spectrum of $\mathcal{L}_0$.

Let D_ε be the collection of the balls centered at ρ_k, $k \in \mathbb{Z}$, with sufficiently small radius ε.

[2)]The formula (21) also contains the solution of (19) in the case $\lambda = 0$: $u(x) = -\int_{-x}^{1}(t+x)f(t)\,dt$.

Lemma 4. *For $\lambda = \rho^2$ with ρ laying outside the balls D_ε and $|\rho| \geq 1$, the Green's function $G_0(x, t; \lambda)$ satisfies the uniform with respect to $-1 \leq x, t \leq 1$ estimate*

$$|G_0(x, t; \lambda)| \leq C_\varepsilon |\rho|^{-1} r(x, t, \rho) \tag{24}$$

where

$$r(x, t, \rho) = \exp\Big(-\rho_* \big||x| - |t|\big|\Big) + \exp\Big(-\rho_*(2 - |x| - |t|)\Big), \quad \rho_* = \min\Big(|\mathrm{Re}\,\rho|, |\mathrm{Im}\,\rho|\Big). \tag{25}$$

Proof. Due to the property a) in Lemma 3 it is sufficient to prove (24) only for $|t| \leq |x|$. Let us rewrite the relation (22) in the different form

$$G_0(x, t; \lambda) = \frac{1}{2\omega(\rho)} \Big[-\cos\rho t \sinh\rho x - \sinh\rho t \cos\rho x - \operatorname{sgn} x \sinh\rho t \Big(\sin\rho \sinh\rho(1 - |x|) + \\ + \cos\rho \cosh\rho(1 - |x|)\Big) + \cos\rho t \Big(\cosh\rho \sin\rho(1 - |x|) - \sinh\rho \cos\rho(1 - |x|)\Big)\Big]. \tag{26}$$

The estimate (24) readily follows from (26), the lower estimate

$$|\cosh\rho \cos\rho + \sinh\rho \sin\rho| \geq c_\varepsilon \exp\Big(|\mathrm{Re}\,\rho| + |\mathrm{Im}\,\rho|\Big)$$

and the usual upper estimates for trigonometric and hyperbolic functions. □

3. The General Case

Let us consider the general case of the complex-valued coefficient $q(x) \in L_2(-1, 1)$ in (1).

The operator $\mathcal{L}$ is not self-adjoint in general. If $\mathcal{L}$ is the closure in $L_2(-1, 1)$ of the minimal operator related to (1) and (2) then its adjoint $\mathcal{L}^*$ could be similarly defined by the operation

$$l^*[u] = -u''(-x) + \overline{q(x)}u(x) \tag{27}$$

with the same initial data (2).

In order to study the spectral properties of $\mathcal{L}$ let us construct the Green's function $G(x, t; \lambda)$ of $\mathcal{L}$.

The Green's function $G(x, t; \lambda)$ should deliver the solution to the problem

$$\begin{aligned} &l[u] = \lambda u(x) + f(x), \quad -1 < x < 1, \\ &u(-1) = u'(-1) = 0, \end{aligned} \tag{28}$$

in the form

$$u(x) = \int_{-1}^{1} G(x, t; \lambda) f(t)\, dt. \tag{29}$$

Mimicking the considerations in the unperturbed case it is easy to show that if

– the function $G(x, t; \lambda)$ is absolutely continuous in the rectangle $-1 \leq x, t \leq 1$,

– the derivative $\partial_x G(x, t; \lambda)$ is absolutely continuous in each triangle $\{(x, t) \mid -1 \leq x \leq 1, -1 \leq t < -x\}$, $\{(x, t) \mid -1 \leq x \leq 1, -x < t \leq 1\}$ and has the limit values on the diagonal $x = -t$ which match the condition

$$\partial_x G(x, t; \lambda)|_{t=-x+0} - \partial_x G(x, t; \lambda)|_{t=-x-0} = 1;$$

– $G(x, t; \lambda)$ satisfies the equation

$$-\partial_x^2 G(-x, t; \lambda) + q(x) G(x, t; \lambda) = \lambda G(x, t; \lambda)$$

a. e. in each above mentioned triangle and matches the initial conditions $G(-1,t;\lambda) = \partial_x G(-1,t;\lambda) = 0$, then the integral in (29) is the solution to (28).

In other words, it means that these three conditions could be taken as the definition of the Green's function $G(x,t;\lambda)$ for $\mathcal{L}$.

The symmetric property a) for $G_0(x,t;\lambda)$ in Lemma 3 transforms here into the condition $G(t,x;\overline{\lambda}) = \overline{G(x,t;\lambda)}$ that could be derived from the definition of the adjoint operator $\mathcal{L}^*$ by (27).

The properties of $G_0(x,t;\lambda)$ and $G(x,t;\lambda)$ yield that the difference $G(x,t;\lambda) - G_0(x,t;\lambda)$ is the a.e. solution (with respect to x) of the equation

$$-w''(-x) = \lambda w(x) + q(x)G(x,t;\lambda) \tag{30}$$

which satisfies the initial data (2).

Using (20) we rewrite (30) in the form

$$G(x,t;\lambda) - G_0(x,t;\lambda) = -\int_{-1}^{1} G_0(x,s;\lambda)q(s)G(s,t;\lambda)\,ds \tag{31}$$

and obtain the existence of its solution.

Lemma 5. *For sufficiently large ρ laying outside D_ε (with any small $\varepsilon > 0$), the equation (31) has a unique absolutely continuous solution $G(x,t,\lambda)$.*

Proof. We apply the method of successive approximations. Let us introduce the functions

$$G^{(0)}(x,t;\lambda) \equiv 0, \quad G^{(p+1)}(x,t;\lambda) = G_0(x,t;\lambda) - \int_{-1}^{1} G_0(x,s;\lambda)q(s)G^{(p)}(s,t;\lambda)\,ds$$

and, for any $t \in [-1,1]$, the related constants

$$\gamma_0 = \max_{-1\le x\le 1}\left|G^{(1)}(x,t;\lambda)\right| |\rho| r^{-1}(x,t,\rho), \quad \gamma_p = \max_{-1\le x\le 1}\left|G^{(p+1)}(x,t;\lambda) - G^{(p)}(x,t;\lambda)\right| |\rho| r^{-1}(x,t,\rho).$$

We prove that, for sufficiently large ρ laying outside D_ε, the estimate

$$\gamma_p \le 2^{-p}C_\varepsilon \tag{32}$$

holds with the constant C_ε in the estimate (24) of Lemma 4.

In fact, the estimate (32) with $p = 0$ repeats (24). Then, by induction, we deduce the relations:

$$\begin{aligned}\gamma_{p+1} &\le \max_{-1\le x\le 1}\left\{|\rho| r^{-1}(x,t,\rho)\int_{-1}^{1} |G_0(x,s;\lambda)|\,|q(s)|\left|G^{(p+1)}(s,t;\lambda) - G^{(p)}(s,t;\lambda)\right| ds\right\} \le \\ &\le C_\varepsilon \gamma_p |\rho|^{-1} \max_{-1\le x\le 1} r^{-1}(x,t,\rho)\int_{-1}^{1} r(x,s,\rho)r(s,t,\rho)|q(s)|\,ds.\end{aligned} \tag{33}$$

It follows from (25) that $r(x,s,\rho)r(s,t,\rho) \le 3r(x,t,\rho)$ and, therefore, the estimate (33) yields

$$\gamma_{p+1} \le 3C_\varepsilon \gamma_p |\rho|^{-1}\int_{-1}^{1} |q(s)|\,ds.$$

The latter inequality delivers the estimate (32) if

$$6C_\varepsilon|\rho|^{-1}\int\limits_{-1}^{1}|q(s)|\,ds \le 1.$$

Estimate (32) provides that the series $\sum_{p=1}^{\infty}\Big(G^{(p+1)}(x,t;\lambda)-G^{(p)}(x,t;\lambda)\Big)$ converges uniformly for $x\in[-1,1]$. As its m-th partial sum equals $G^{(m+1)}(x,t;\lambda)-G_0(x,t;\lambda)$, the sequence $G^{(m)}(x,t;\lambda)$ also converges uniformly and its limit $G(x,t;\lambda)$ satisfies (31). $\square$

Remark 1. It follows from Lemma 5 that the Green's function $G(x,t;\lambda)$ of $\mathcal{L}$ could have poles (and, therefore, the operator $\mathcal{L}$ could have eigenvalues) that are asymptotically close to the eigenvalues $\lambda_k=\rho_k^2$ of $\mathcal{L}_0$.

Corollary 1. *Under the assumptions of Lemma 5, the Green's function $G(x,t;\lambda)$ satisfies the estimate*

$$|G(x,t;\lambda)|\le 2C_\varepsilon|\rho|^{-1}r(x,t,\rho). \tag{34}$$

Let us consider the biorthogonal eigenfunction expansion related to $\mathcal{L}$

$$f(x)\sim\sum_{k\in\mathbb{Z}}(f,\widetilde{v}_k)\,\widetilde{u}_k(x) \tag{35}$$

where $\widetilde{u}_k(x)$ are the root functions of $\mathcal{L}$, $\widetilde{v}_k(x)$ form the biorthogonal system (they are the root functions of $\mathcal{L}^*$). Denote by $\sigma_m(x,f)$ the partial sum of (35) that includes all the root functions $\widetilde{u}_k(x)$ corresponding to the eigenvalues $\widetilde{\lambda}_k=\widetilde{\rho}_k^2$ that satisfy the condition $|\widetilde{\rho}_k|<R_m$.

Then, for the same sequence of contours L_m and sufficiently large m, the partial sums $\sigma_m(x,f)$ have the integral representation

$$\sigma_m(x,f)=-\frac{1}{2\pi i}\int\limits_{L_m}\left(\int\limits_{-1}^{1}G(x,t;\lambda)f(t)\,dt\right)d\lambda$$

and therefore, due to (18),

$$\sigma_m(x,f)-S_m(x,f)=-\frac{1}{2\pi i}\int\limits_{L_m}\left(\int\limits_{-1}^{1}\Big[G(x,t;\lambda)-G_0(x,t;\lambda)\Big]f(t)\,dt\right)d\lambda. \tag{36}$$

Theorem 2. *For an arbitrary function $f(x)\in L_1(-1,1)$, the eigenfunction expansions of $f(x)$ related to $\mathcal{L}$ and $\mathcal{L}_0$ equiconverge and the estimate (3) holds uniformly with respect to $x\in[-1,1]$.*

Proof. It follows from the estimates (24) and (34) that, for sufficiently large m and $\rho^2\in L_m$, we have

$$\Big|G(x,t;\rho^2)-G_0(x,t;\rho^2)\Big|\le 4C_\varepsilon^2|\rho|^{-2}r(x,t,\rho)\int\limits_{-1}^{1}|q(s)|\,ds.$$

Hence relation (36) yields

$$|\sigma_m(x,f)-S_m(x,f)|\le\frac{1}{\pi}\int\limits_{\rho^2\in L_m}\left\{\int\limits_{-1}^{1}\Big|G(x,t;\rho^2)-G_0(x,t;\rho^2)\Big|\,|f(t)|\,dt\right\}|\rho\,d\rho|\le C_1\int\limits_{\rho^2\in L_m}\left\{\int\limits_{-1}^{1}r(x,t,\rho)|f(t)|\,dt\right\}\left|\frac{d\rho}{\rho}\right| \tag{37}$$

where $C_1 = 4C_\varepsilon^2\pi^{-1}\int_{-1}^{1}|q(s)|\,ds$.

Taking sufficiently small $\delta > 0$, we split the interval $(-1,1)$ into two parts:

$$\Delta_1 = (-1,1)\setminus\Delta_2, \quad \Delta_2 = (-1,-1+\delta)\cup(-x-\delta,-x+\delta)\cup(x-\delta,x+\delta)\cup(1-\delta,1).$$

Therefore, the estimate (37) turns into the relation

$$|\sigma_m(x,f)-S_m(x,f)| \le C_1 \int\limits_{\rho^2\in L_m}\int\limits_{\Delta_1}\Big(\exp(-\rho_*||x|-|t||)+\exp(-\rho_*(2-|x|-|t|))\Big)|f(t)|\,dt\left|\frac{d\rho}{\rho}\right| + 2C_1\pi\int\limits_{\Delta_2}|f(t)|\,dt. \quad (38)$$

Due to Lebesgue Theorem, for any $\varepsilon_0 > 0$, there exists $\delta > 0$ such that the second term in the right-hand side of (38) is less than $\varepsilon_0/2$. The first term satisfies the estimate

$$\int\limits_{\rho^2\in L_m}\int\limits_{\Delta_1}\Big(\exp(-\rho_*||x|-|t||)+\exp(-\rho_*(2-|x|-|t|))\Big)|f(t)|\,dt\left|\frac{d\rho}{\rho}\right| \le 3\int\limits_{-1}^{1}|f(t)|\,dt\int\limits_{\rho^2\in L_m}\exp(-\rho_*\delta)\left|\frac{d\rho}{\rho}\right|.$$

Since

$$\int\limits_{\rho^2\in L_m}\exp(-\rho_*\delta)\left|\frac{d\rho}{\rho}\right| = 4\int\limits_{0}^{\pi/4}\exp(-\delta R_m|\sin\tau|)\,d\tau + 2\int\limits_{\pi/4}^{3\pi/4}\exp(-\delta R_m|\cos\tau|)\,d\tau \le C_2R_m^{-1}.$$

Therefore, the second term in the right-hand side of (38) could be also made less than $\varepsilon_0/2$ provided m is sufficiently large. □

As the equiconvergence property (3) is obtained uniformly on $[-1,1]$ for any integrable function $f(x)$ and the orthogonal series (17) converges for any function $f(x)\in L_2(-1,1)$ in the metric of $L_2(-1,1)$, Theorem 2 directly provides that, for any function $f(x)\in L_2(-1,1)$,

$$\left\|\sigma_m(x,f)-f(x)\right\|_2 = o(1), \quad m\to\infty.$$

Theorem 3. *The system of root functions of $\mathcal{L}$ with an arbitrary complex-valued coefficient $q(x)\in L_2(-1,1)$ forms a basis in $L_2(-1,1)$.*

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