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Bounds for Generalized Normalized δ-Casorati Curvatures for Bi-slant Submanifolds in $T-$space Forms

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Abstract. In this paper, we prove the inequality between the generalized normalized δ-Casorati curvatures and the normalized scalar curvature for the bi-slant submanifolds in $T-$space forms and consider the equality case of the inequality. We also develop same results for semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, slant submanifolds, invariant and anti-invariant submanifolds in $T-$space forms.

1. Introduction

In 1993, Chen [9] establish the simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds know as the theory of Chen invariants, which is one of the most interesting research area of differential geometry. Chen has given a basic inequality in terms of the intrinsic invariant δ_M and the squared mean curvature $||H||^2$ of the immersion as

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)}||H||^2 + \frac{1}{2}(m+1)(m-2)c,$$

for $m-$dimensional submanifold M of a real space form $\overline{M}(c)$.This inequality also holds good if M in anti-invariant submanifold of complex space form $\overline{M}(c)$[10]. Similar inequality is also obtained for $C-$totally real submanifolds of a Sasakian space form with constant $\varphi-$sectional curvature c [15], given by

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)}||H||^2 - \frac{1}{2}(m+1)(m-2)\frac{c+3}{4}.$$

In the initial paper Chen established inequalities between the scalar curvature, the sectional curvature and the squared norm of the mean curvature of a submanifold in a real space form. He also obtained the inequalities between k-Ricci curvature, the squared mean curvature and the shape operator for the submanifolds in the real space form with arbitrary codimension [8]. Since then different geometers obtained the similar inequalities for different submanifolds and ambient spaces [6, 7, 23, 27, 28].

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In 2002, Carraiazzo [3] introduced bi-slant submanifolds of an almost Hermitian manifold as a natural generalization of semi-slant submanifolds.

Casorati [4] introduced Casorati curvature (extrinsic invariant) of a submanifold of a Riemannian manifold and defined as the normalized square length of the second fundamental form, which extends the concept of the principal direction of a hypersurfaces of a Riemannian manifold [19]. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [13, 14, 22, 32, 33]. Therefore it attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [17, 24, 25, 31].

In this paper, we will study the optimal inequalities for the generalized normalized δ-Casorati curvature for the bi-slant submanifolds of $T-$space forms. We also develop same results for semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, slant submanifolds, invariant and anti-invariant submanifolds in $T-$space form.

2. Preliminaries

Let $(\overline{M}, g)$ be a Riemannian manifold with $dim(\overline{M}) = 2m + s$ and the Lie algebra of vector field in $\overline{M}$ denote by $T\overline{M}$. Then $\overline{M}$ is said to be an $S-$Manifold if there exist on $\overline{M}$ an $f-$structure ϕ [34] of rank $2m$ and s global vector fields $\xi_1, ..., \xi_s$ (structure vector fields) such that [2]
(i) if $\eta^1, ..., \eta^s$ are dual $1-$forms of $\xi_1, ..., \xi_s$, then:

$$\phi\xi_i = 0, \quad \eta^i \circ \phi = 0, \quad \phi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i \tag{1}$$

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{2}$$

$$\overline{g}(X, \xi_i) = \eta^i(X) \tag{3}$$

for any $X, Y \in T\overline{M}, i = 1, ..., s$.
(ii) The $f-$structure ϕ is normal, that is

$$[\phi, \phi] + 2\sum_{i=1}^{s} \xi_i \otimes d\eta^i = 0$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.
(iii) $\eta^1 \wedge ... \wedge \eta^s \wedge (d\eta^i) \neq 0$ and for each i, $d\eta^i = 0$

In a $T-$manifold $\overline{M}$, beside the relation (1) and (2) the following also hold:

$$(\nabla_X \phi)Y = 0 \tag{4}$$

$$\nabla_X \xi_i = 0 \tag{5}$$

for any vector fields $X, Y \in T\overline{M}$

Let $\overline{D}$ denote the distribution determined by $-\phi^2$ and $\overline{D}^{\perp}$ the complementary distribution. $\overline{D}^{\perp}$ is determined by $\phi^2 + I$ and spanned by $\xi_1, ..., \xi_n$. If $X \in \overline{D}$, then $\eta^i(X) = 0$ for any i and if $X \in \overline{D}^{\perp}$, then $\phi X = 0$.

A plane section Π in T_pM of an $T-$manifold $\overline{M}$ is called a $\phi-$section if $\pi \perp \overline{D}^{\perp}$ and $\phi(\pi) = \pi$. $\overline{M}$ is of constant $\phi-$sectional curvature [2] if at each point $p \in \overline{K}(\pi)$, the sectional curvature $\overline{K}(\pi)$ does depend on the choice of the $\phi-$section π of $T_p\overline{M}$. If $\overline{K}(\pi)$ is constant for all non-null vectors in π, we call $\overline{M}$ to be of constant $\phi-$sectional curvature at point p. The function of c defined by $c(p) = \overline{K}(\pi)$ is called the $\phi-$sectional curvature of $\overline{M}$. A $T-$manifold $\overline{M}$ with constant $\phi-$sectional curvature c is said to be a $T-$space form and is denoted by $\overline{M}(c)$.

The curvature tensor $\overline{R}$ of a $T-$space form $\overline{M}(c)$ is given in [20]

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,W) &= \frac{c}{4}\{\overline{g}(X,Z)\overline{g}(Y,W)-\overline{g}(Y,Z)\overline{g}(X,W)-\overline{g}(X,Z)\sum u^i(Y)u^i(W)\\ &\quad -\overline{g}(Y,W)\sum u^i(Z)u^i(X)+\overline{g}(X,W)\sum u^i(Y)u^i(Z)+\overline{g}(Y,Z)\sum u^i(X)u^i(W)\\ &\quad +(\sum u^i(Z)u^i(X))(\sum u^i(Y)u^i(W))-(\sum u^i(W)u^i(X))+(\sum u^i(Y)u^i(Z))\\ &\quad +\overline{g}(W,\phi X)\overline{g}(Y,\phi Z)+\overline{g}(Y,\phi W)\overline{g}(X,\phi Z)-2\overline{g}(X,\phi Y)\overline{g}(W,\phi Z)\}.\end{aligned} \tag{6}$$

When $s = 0$, a $T-$manifold $\overline{M}$ becomes a Kaehler manifold. When $s = 1$, a $T-$manifold $\overline{M}$ becomes a cosymplectic manifold [20].

The Equation of Gauss for submanifold M of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \tag{7}$$

for any vectors X, Y,Z and W tangent to M, where we denote as usual $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.
From (6) and Gauss equation (7), we have

$$\begin{aligned}R(X,Y,Z,W) &= \frac{c}{4}\{\overline{g}(X,Z)\overline{g}(Y,W)-\overline{g}(Y,Z)\overline{g}(X,W)-\overline{g}(X,Z)\sum u^i(Y)u^i(W)\\ &\quad -\overline{g}(Y,W)\sum u^i(Z)u^i(X)+\overline{g}(X,W)\sum u^i(Y)u^i(Z)+\overline{g}(Y,Z)\sum u^i(X)u^i(W)\\ &\quad +(\sum u^i(Z)u^i(X))(\sum u^i(Y)u^i(W))-(\sum u^i(W)u^i(X))+(\sum u^i(Y)u^i(Z))\\ &\quad +\overline{g}(W,\phi X)\overline{g}(Y,\phi Z)+\overline{g}(Y,\phi W)\overline{g}(X,\phi Z)-2\overline{g}(X,\phi Y)\overline{g}(W,\phi Z)\}\\ &\quad -g(h(X,Z),h(Y,W))+g(h(X,W),h(Y,Z)).\end{aligned} \tag{8}$$

From now on, we suppose that the structure vector fields are tangent to M and we denote by $n + s$ the dimension of M. We consider $n \geq 2$. Hence, if we denote by $L = D_1 \oplus D_2$ the orthogonal distribution to $\overline{D}^{\perp}$ in TM. We can write orthogonal direct decomposition $TM = L \oplus \overline{D}^{\perp}$.

For any orthonormal basis $\{e_1, ..., e_n, ..., e_{n+s}\}$ of T_pM, the scalar curvature

$$\tau = \sum_{i<j} K(e_i \wedge e_j) \tag{9}$$

where $K(e_i \wedge e_j)$ denoted the sectional curvature of M associated with the plane section spanned by e_i,e_j. In particular, if we put $e_{n+\alpha} = \xi_\alpha$ for $\alpha = 1, 2, ..., s$, then (9) implies

$$2\tau = \sum_{i\neq j} K(e_i \wedge e_j) + 2\sum_{i=1}^{n}\sum_{\alpha=1}^{s} K(e_i \wedge \xi_\alpha) \tag{10}$$

Let M be an $(n + s)$-dimensional submanifold of a $T-$space form $\overline{M}(c)$ of dimension $2m + s$. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on M and $\overline{M}(c)$ respectively. The Gauss and Weingarten equations are respectively defined as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\overline{\nabla}_X \xi = -S_\xi X + \nabla^{\perp}_X Y,$$

for vector fields $X, Y \in TM$ and $\xi \in T^{\perp}M$. Where h, S and $\nabla^{\perp}$ is the second fundamental form, the shape operator and the normal connection respectively. The second fundamental form and the shape operator are related by the following equation

$$g(h(X,Y),\xi) = g(S_{\xi}X,Y),$$

for vector fields $X, Y \in TM$ and $\xi \in T^{\perp}M$.

Let M be an $(n+s)$-dimensional submanifold of a $T-$space form $\overline{M}(c)$ of dimension $2m+s$. For any tangent vector field $X \in TM$, we can write $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX respectively. If $P = 0$, the submanifold is said to be an anti-invariant submanifold and if $F = 0$, the submanifold is said to be an invariant submanifold. The squared norm of P at $p \in M$ is defined as

$$||P||^2 = \sum_{i,j=1}^{n+s} g^2(\phi e_i, e_j), \tag{11}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be a slant submanifold if for any $p \in M$ and a non zero vector $X \in T_pM$, the angle between JX and T_pM is constant, i.e., the angle does not depend on the choice of $p \in M$ and $X \in T_pM$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of M in $\overline{M}$.

A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be a bi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2, such that (i) TM admits the orthogonal direct decomposition i.e $TM = D_1 + D_2$. (ii) For i=1,2, D_i is the slant distribution with slant angle θ_i.

In fact, semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, slant submanifolds can be obtained from bi-slant submanifolds in particular. We can see the case in the following table:

Table 1: Defination

S.N.	$\overline{M}(c)$	M	D_1	D_2	θ_1	θ_2
(1)	$\overline{M}$	semi-slant	invariant	slant	0	slant angle
(2)	$\overline{M}$	hemi-slant	slant	anti-invariant	slant angle	$\frac{\pi}{2}$
(3)	$\overline{M}$	CR	invariant	anti-invariant	0	$\frac{\pi}{2}$
(4)	$\overline{M}$	slant	either $D_1 = 0$ or $D_2 = 0$		either $\theta_1 = \theta_2 = \theta$ or $\theta_1 = \theta_2 \neq \theta$	

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively and when $0 < \theta < \frac{\pi}{2}$, then slant submanifold is called proper slant submanifold.

If M is a bi-slant submanifold in $T-$space form $\overline{M}(c)$, then one can easily see that

$$||P||^2 = \sum_{i,j}^{n+s} g^2(Pe_i, e_j) = 2(d_1 cos^2\theta_1 + d_2 cos^2\theta_2). \tag{12}$$

Let M be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \ldots, e_{n+s}\}$ and $\{e_{n+s+1}, \ldots, e_{2m+s}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ at any $p \in M$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1\le i<j\le n+s} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{(n+s)(n+s-1)}.$$

The mean curvature vector denoted by H is defined as

$$H = \frac{1}{n+s}\sum_{i,j=1}^{n+s} h(e_i, e_i).$$

We also put

$$h_{ij}^{\gamma} = g(h(e_i, e_j), e_\gamma),\ i, j \in 1, 2, .., n+s,\ \gamma \in \{n+s+1, n+s+2, ..., 2m+s\}.$$

The norm of the squared mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{(n+s)^2}\sum_{\gamma=n+s+1}^{2m+s}\left(\sum_{i=1}^{n+s} h_{ii}^{\gamma}\right)^2$$

and the squared norm of second fundamental form h is denoted by C defined as

$$C = \frac{1}{n+s}\sum_{\gamma=n+s+1}^{2m+s}\sum_{i,j=1}^{n+s}\left(h_{ij}^{\gamma}\right)^2$$

known as Casorati curvature of the submanifold.

If we suppose that L is an r-dimensional subspace of TM, $r \ge 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of L. Then the scalar curvature of the r-plane section L is given as

$$\tau(L) = \sum_{1\le\gamma<\beta\le r} K(e_\gamma \wedge e_\beta)$$

and the Casorati curvature C of the subspace L is as follows

$$C(L) = \frac{1}{r}\sum_{\gamma=n+s+1}^{2m+s}\sum_{i,j=1}^{n+s}\left(h_{ij}^{\gamma}\right)^2.$$

A point $p \in M$ is said to be an invariantly quasi-umbilical point if there exist $2m-n$ mutually orthogonal unit normal vectors $\xi_{n+s+1}, \ldots, \xi_{2m+s}$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $n+s-1$ and that for each ξ_γ the distinguished eigne direction is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_c(n+s-1)$ and $\widehat{\delta_c}(n+s-1)$ are defined as

$$[\delta_c(n+s-1)]_p = \frac{1}{2}C_p + \frac{n+s+1}{2(n+s)} inf\{C(L)|L : \text{a hyperplane of } T_pM\} \tag{13}$$

and

$$[\widehat{\delta_c}(n+s-1)]_p = 2C_p + \frac{2(n+s)-1}{2(n+s)} sup\{C(L)|L : \text{a hyperplane of } T_pM\}. \tag{14}$$

For a positive real number $t \neq (n+s)(n+s-1)$, put

$$a(t) = \frac{1}{(n+s)t}(n+s-1)(n+s+t)((n+s)^2 - (n+s) - t), \tag{15}$$

then the generalized normalized δ-Casorati curvatures $\delta_c(t; n+s-1)$ and $\widehat{\delta_c}(t; n+s-1)$ are given as

$$[\delta_c(t; n+s-1)]_p = tC_p + a(t) inf\{C(L)|L : \text{a hyperplane of } T_pM\},$$

if $0 < t < (n+s)^2 - (n+s)$, and

$$[\widehat{\delta_c}(t; n+s-1)]_p = tC_p + a(t) sup\{C(L)|L : \text{a hyperplane of } T_pM\},$$

if $t > (n+s)^2 - (n+s)$.

3. Main Theorem

Theorem 3.1. *Let M be a $(n+s)$-dimensional bi-slant submanifold in $T-$space forms $\overline{M}(c)$ of dimension $2m+s$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n+s-1)$ satisfies

$$\rho \leq \frac{\delta_c(t; n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\} \tag{16}$$

for any real number t such that $0 < t < (n+s)(n+s-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n+s-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta_c}(t; n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\} \tag{17}$$

for any real number $t > (n+s)(n+s-1)$. Moreover , the equality holds in (16) and (17) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_{n+s}\}$ and normal orthonormal frame $\{e_{n+s+1}, \ldots, e_{2m+s}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+s+1, \ldots, 2m+s\}$, take the following form

$$S_{n+s+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{(n+s)(n+s-1)}{t}a \end{pmatrix}, \qquad S_{n+s+2} = \cdots = S_{2m+s} = 0. \tag{18}$$

Proof. Let $\{e_1, \ldots, e_n, e_{n+1} = \xi_1, \ldots, e_{n+s} = \xi_s\}$ and $\{e_{n+s+1}, \ldots, e_{2m+s}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ respectively at any point $p \in M$. Then from (8), (10) and (12), we have

$$2\tau = (n+s)^2\|H\|^2 - (n+s)C + \frac{c}{4}\{n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\}. \tag{19}$$

Define the following function, denoted by Q, a quadratic polynomial in the components of the second fundamental form

$$Q = tC + a(t)C(L) - 2\tau + \frac{c}{4}\{n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\}, \tag{20}$$

where L is the hyperplane of T_pM. Without loss of generality, we suppose that L is spanned by $e_1, \ldots, e_{n+s-1}$, it follows from (20) that

$$\mathcal{Q} = \frac{n+s+t}{n+s}\sum_{\gamma=n+s+1}^{2m+s}\sum_{i,j=1}^{n+s}(h_{ij}^{\gamma})^2 + \frac{a(t)}{n+s-1}\sum_{\gamma=n+s+1}^{2m+s}\sum_{i,j=1}^{n+s-1}(h_{ij}^{\gamma})^2 - \sum_{\gamma=n+s+1}^{2m+s}\Big(\sum_{i=1}^{n+s}h_{ii}^{\gamma}\Big)^2,$$

which can be easily written as

$$\begin{aligned}\mathcal{Q} \;=\; & \sum_{\gamma=n+s+1}^{2m+s}\sum_{i=1}^{n+s-1}\Big[\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)(h_{ii}^{\gamma})^2 + \frac{2(n+s+t)}{n+s}(h_{in+s}^{\gamma})^2\Big] \\ & + \sum_{n+s+1}^{2m+s}\Big[2\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)\sum_{(i<j)=1}^{n+s}(h_{ij}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n+s}h_{ii}^{\gamma}h_{jj}^{\gamma} + \frac{t}{n+s}(h_{n+sn+s}^{\gamma})^2\Big].\end{aligned} \tag{21}$$

From (21), we can see that the critical points

$$h^c = (h_{11}^{n+s+1}, h_{12}^{n+s+1}, \ldots, h_{n+sn+s}^{n+s+1}, \ldots, h_{11}^{2m+s}, \ldots, h_{n+sn+s}^{2m+s})$$

of $\mathcal{Q}$ are the solutions of the following system of homogenous equations:

$$\begin{cases}\frac{\partial\mathcal{Q}}{\partial h_{ii}^{\gamma}} = 2\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)(h_{ii}^{\gamma}) - 2\sum_{k=1}^{n+s}h_{kk}^{\gamma} = 0 \\ \frac{\partial\mathcal{Q}}{\partial h_{n+sn+s}^{\gamma}} = \frac{2t}{n+s}h_{n+sn+s}^{\gamma} - 2\sum_{k=1}^{n+s-1}h_{kk}^{\gamma} = 0 \\ \frac{\partial\mathcal{Q}}{\partial h_{ij}^{\gamma}} = 4\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)(h_{ij}^{\gamma}) = 0 \\ \frac{\partial\mathcal{Q}}{\partial h_{in+s}^{\gamma}} = 4(\frac{n+s+t}{n+s}(h_{in+s}^{\gamma}) = 0,\end{cases} \tag{22}$$

where $i,j = \{1,2,\ldots,n+s-1\}$, $i \neq j$, and $\gamma \in \{n+s+1,\ldots,2m+s\}$.

Hence, every solution h^c has $h_{ij}^{\gamma} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{Q}$ is of the following form

$$\mathcal{H}(\mathcal{Q}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)-2 & -2 & \ldots & -2 & -2 \\ -2 & 2\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)-2 & \ldots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \ldots & 2\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)-2 & -2 \\ -2 & -2 & \ldots & -2 & \frac{2t}{n+s} \end{pmatrix},$$

H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\Big(4\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big), 4\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big), \ldots, 4\Big(\frac{n+s+t}{n+s}+\frac{a(t)}{n+s-1}\Big)\Big),$$

and

$$H_3 = diag\Big(\frac{4(n+s+t)}{n+s}, \frac{4(n+s+t)}{n+s}, \dots, \frac{4(n+s+t)}{n+s}\Big).$$

Hence, we find that $\mathcal{H}(Q)$ has the following eigenvalues

$$\lambda_{11} = 0,\ \lambda_{22} = 2\Big(\frac{2t}{n+s} + \frac{a(t)}{n+s-1}\Big),\ \lambda_{33} = \cdots = \lambda_{n+sn+s} = 2\Big(\frac{n+s+t}{n+s} + \frac{a(t)}{n+s-1}\Big),$$

$$\lambda_{ij} = 4\Big(\frac{n+s+t}{n+s} + \frac{a(t)}{n+s-1}\Big), \lambda_{in} = \frac{4(n+s+t)}{n+s},\ \forall\ i,j \in \{1,2,\dots,n+s-1\},\ i \neq j.$$

Thus, Q is parabolic and reaches at minimum $Q(h^c) = 0$ for the solution h^c of the system (22). Hence $Q \geq 0$ and hence

$$2\tau \leq tC + a(t)C(L) + \frac{c}{4}\{n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\},$$

whereby, we obtain

$$\begin{aligned}\rho \ &\leq\ \frac{t}{(n+s)(n+s-1)}C + \frac{a(t)}{(n+s)(n+s-1)}C(L)\\ &\quad + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\}\end{aligned}$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$h^{\gamma}_{ij} = 0,\ \forall\ i,j \in \{1,\dots,n+s\},\ i \neq j \text{ and } \gamma \in \{n+s+1,\dots,2m+s\} \tag{23}$$

and

$$h^{\gamma}_{n+sn+s} = \frac{(n+s)(n+s-1)}{t}h^{\gamma}_{11} = \cdots = \frac{(n+s)(n+s-1)}{t}h^{\gamma}_{n+s-1n+s-1}, \quad \forall \gamma \in \{n+s+1,\dots,2m+s\}. \tag{24}$$

From (23) and (24), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\overline{M}$, such that the shape operator takes the form (18) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

□

Corollary 3.2. *Let M be a $(n+s)$-dimensional bi-slant submanifold in $T-$space form $\overline{M}$. Then*
(i) The normalized δ-Casorati curvature $\delta_c(n+s-1)$ satisfies

$$\rho \leq \delta_c(n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2 + s(1-s))\}.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to suitable tangent orthonormal frame $\{e_1,\dots,e_{n+s}\}$ and normal orthonormal frame $\{e_{n+s+1},\dots,e_{2m+s}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1,\dots,2m+s\}$, take the following form

$$S_{n+s+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0\\ 0 & a & 0 & \dots & 0 & 0\\ 0 & 0 & a & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & a & 0\\ 0 & 0 & 0 & \dots & 0 & 2a \end{pmatrix},\ S_{n+s+2} = \cdots = S_{2m+s} = 0.$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n+s-1)$ *satisfies*

$$\rho \leq \widehat{\delta}_c(n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1cos^2\theta_1 + d_2cos^2\theta_2 + s(1-s))\}.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$*, such that with respect to suitable tangent orthonormal frame* $\{e_1, \ldots, e_{n+s}\}$ *a and normal orthonormal frame* $\{e_{n+s+1}, \ldots, e_{2m+s}\}$*, the shape operator* $S_r \equiv S_{e_r}$*,* $r \in \{n+s+1, \ldots, 2m+s\}$*, take the following form*

$$S_{n+s+1} = \begin{pmatrix} 2a & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2a & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, S_{n+s+2} = \cdots = S_{2m+s} = 0.$$

Theorem 3.3. *Let M be a* $(n+s)$*-dimensional submanifold in* $T-$*space form* $\overline{M}(c)$ *of dimension* $2m+s$*. Then we have the following table for generalized normalized* $\delta-$*Casorati curvatures:*

Generalized Normalized δ−Casorati curvatures			
S.N.	$\overline{M}(c)$	*M*	*Inequality*
(1)	$\overline{M}(c)$	*semi-slant*	1. $\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+d_2cos^2\theta_2+s(1-s))\}$ 2. $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+d_2cos^2\theta_2+s(1-s))\}$
(2)	$\overline{M}(c)$	*hemi-slant*	1. $\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1cos^2\theta_1+s(1-s))\}$ 2. $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1cos^2\theta_1+s(1-s))\}$
(3)	$\overline{M}(c)$	*CR*	1. $\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+s(1-s))\}$ 2. $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+s(1-s))\}$
(4)	$\overline{M}(c)$	*slant*	1. $\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3((n+s)cos^2\theta+s(1-s))\}$ 2. $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3((n+s)cos^2\theta+s(1-s))\}$
(5)	$\overline{M}(c)$	*invariant*	1. $\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-2)+3s(2-s))\}$ 2. $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-2)+3s(2-s))\}$

S.N.	$\overline{M}(c)$	M	Inequality
(6)	$\overline{M}(c)$	anti-invariant	1. $\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3s(1-s))\}$ 2. $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3s(1-s))\}$

Moreover, the equality holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_{n+s}\}$ and normal orthonormal frame $\{e_{n+s+1}, \ldots, e_{2m+s}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+s+1, \ldots, 2m+s\}$, take the following form

$$S_{n+s+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{(n+s)(n+s-1)}{t}a \end{pmatrix}, \qquad S_{n+s+2} = \cdots = S_{2m+s} = 0. \tag{25}$$

Proof. First four results of the Theorem 3.3 can be simply obtained with the help of Table 1 and the results in Theorem 3.1. And the next two results of the Theorem 3.3 can be seen by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in case of invariant and anti-invariant submanifold respectively in result of slant submanifold given in Theorem 3.3. □

Corollary 3.4. *Let M be a (n+s)-dimensional submanifold in T−space form $\overline{M}(c)$. Then for the normalized $\delta-$Casorati we have the following table*

Normalized $\delta-$Casorati curvatures			
S.N.	$\overline{M}(c)$	M	Inequality
(1)	$\overline{M}(c)$	semi-slant	1. $\rho \leq \delta_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+d_2cos^2\theta_2+s(1-s))\}$ 2. $\rho \leq \widehat{\delta}_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+d_2cos^2\theta_2+s(1-s))\}$
(2)	$\overline{M}(c)$	hemi-slant	1. $\rho \leq \delta_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1cos^2\theta_1+s(1-s))\}$ 2. $\rho \leq \widehat{\delta}_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1cos^2\theta_1+s(1-s))\}$
(3)	$\overline{M}(c)$	CR	1. $\rho \leq \delta_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+s(1-s))\}$ 2. $\rho \leq \widehat{\delta}_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3(d_1+s(1-s))\}$
(4)	$\overline{M}(c)$	slant	1. $\rho \leq \delta_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3((n+s)cos^2\theta+s(1-s))\}$ 2. $\rho \leq \widehat{\delta}_c(t;n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1)+3((n+s)cos^2\theta+s(1-s))\}$

S.N.	$\overline{M}(c)$	*M*	*Inequality*
(5)	$\overline{M}(c)$	*invariant*	1. $\rho \leq \delta_c(t; n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-2) + 3s(2-s))\}$ 2. $\rho \leq \widehat{\delta_c}(t; n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-2) + 3s(2-s))\}$
(6)	$\overline{M}(c)$	*anti-invariant*	1. $\rho \leq \delta_c(t; n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3s(1-s))\}$ 2. $\rho \leq \widehat{\delta_c}(t; n+s-1) + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3s(1-s))\}$

Moreover, the equality sign for the inequalities δ_c in the above holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_{n+s}\}$ and normal orthonormal frame $\{e_{n+s+1}, \ldots, e_{2m+s}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+s+1, \ldots, 2m+s\}$, take the following form

$$S_{n+s+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2a \end{pmatrix}, \ S_{n+s+2} = \cdots = S_{2m+s} = 0.$$

and the equality sign for the inequalities $\widehat{\delta_c}$ in the above table holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_{n+s}\}$ and normal orthonormal frame $\{e_{n+s+1}, \ldots, e_{2m+s}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+s+1, \ldots, 2m+s\}$, take the following form

$$S_{n+s+1} = \begin{pmatrix} 2a & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2a & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, S_{n+s+2} = \cdots = S_{2m+s} = 0.$$

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