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Numerical Reckoning Fixed Points for Suzuki's Generalized Nonexpansive Mappings via New Iteration Process

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Abstract. In this paper we propose a new three-step iteration process, called M iteration process, for approximation of fixed points. Some weak and strong convergence theorems are proved for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces. Numerical example is given to show the efficiency of new iteration process. Our results are the extension, improvement and generalization of many known results in the literature of iterations in fixed point theory.

1. Introduction

Once the existence of a fixed point of some mapping is established, then to find the value of that fixed point is not an easy task, that is why we use iterative processes for computing them. By time, many iterative processes have been developed and it is impossible to cover them all. The well-known Banach contraction theorem use Picard iteration process for approximation of fixed point. Some of the other well-known iterative processes are Mann [11], Ishikawa [7], Agarwal [2], Noor [12], Abbas [1], SP [15], S^* [8], CR [4], Normal-S [18], Picard Mann [10], Picard-S [6], Thakur New [22] and so on.

Speed of convergence play important role for an iteration process to be preferred on another iteration process. In [16], Rhoades mentioned that the Mann iteration process for decreasing function converge faster than the Ishikawa iteration process and for increasing function the Ishikawa iteration process is better than the Mann iteration process. Also the Mann iteration process appears to be independent of the initial guess (see also [17]). In [2], the authors claimed that Agarwal iteration process converge at a rate same as that of the Picard iteration process and faster than the Mann iteration process for contraction mappings. In [1], the authors claimed that Abbas iteration process converge faster than Agarwal iteration process. In [4], the authors claimed that CR iteration process is equivalent to and faster than Picard, Mann, Ishikawa, Agarwal, Noor and SP iterative processes for quasi-contractive operators in Banach spaces. Also in [9] the authors proved that CR iterative process converge faster than the S^* iterative process for the class of contraction mappings. In [6], authors claimed that Picard-S iteration process converge faster than all Picard, Mann, Ishikawa, Noor, SP, CR, Agarwal, S^*, Abbas and Normal-S for contraction mappings. In [22], the authors proved with the help of numerical example that Thakur New iteration process converge faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration processes for the class of Suzuki generalized nonexpansive mappings.

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Rhoades [16] made an universal remark on the rate of convergence of these iteration processes that:
"It is doubtful if any global statement can be made, since there is nothing about these iteration procedures to cause their analysis to be different from that of the other approximation method."

Motivated by above, in this paper, we introduce a new iteration process namely M iteration process which is the first three-step iteration process with a single set of parameters. Using our new iteration process, we prove some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings, which is the generalization of nonexpansive as well as contraction mappings, in the setting of uniformly convex Banach spaces. Finally, an example of Suzuki generalized nonexpansive mapping is given which is not nonexpansive. Numerically we compare the speed of convergence of our new M iteration process with the well-known iteration processes like two-step S iteration process and three-step Picard-S iteration process for given example.

2. Preliminaries

First we recall some definitions, propositions and lemmas to be used in the next two sections.

A Banach space X is called uniformly convex [5] if for each $\varepsilon \in (0, 2]$ there is a $\delta > 0$ such that for $x, y \in X$,

$$\left.\begin{array}{r} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x-y\| > \varepsilon \end{array}\right\} \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq \delta.$$

A Banach space X is said to satisfy the Opial property [13] if for each sequence $\{x_n\}$ in X, converging weakly to $x \in X$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in X$ such that $y \neq x$.

A point p is called fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of mapping T. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called contraction if there exists $\theta \in (0, 1)$ such that $\|Tx - Ty\| \leq \theta \|x - y\|$, for all $x, y \in C$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have $\|Tx - p\| \leq \|x - p\|$. In 2008, Suzuki [21] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|.$$

Suzuki [21] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The mapping satisfy condition (C) is called Suzuki generalized nonexpansive mapping.

Suzuki [21] obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mappings. In 2011, Phuengrattana [14] proved convergence theorems for Suzuki generalized nonexpansive mappings using the Ishikawa iteration in uniformly convex Banach spaces and $CAT(0)$ spaces. Recently, fixed point theorems for Suzuki generalized nonexpansive mappings have been studied by a number of authors see e.g. [22] and references therein.

We now list some properties of Suzuki generalized nonexpansive mappings.

Proposition 2.1. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be any mapping. Then*
(i) [21, Proposition 1] If T is nonexpansive then T is Suzuki generalized nonexpansive mapping.

(ii) [21, Proposition 2] If T is Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.

(iii) [21, Lemma 7] If T is Suzuki generalized nonexpansive mapping, then $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$ for all $x, y \in C$.

Lemma 2.2. *[21, Proposition 3] Let T be a mapping on a subset C of a Banach space X with the Opial property. Assume that T is Suzuki generalized nonexpansive mapping. If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$.*

Lemma 2.3. *[21, Theorem 5] Let C be a weakly compact convex subset of a uniformly convex Banach space X. Let T be a mapping on C. Assume that T is Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.4. *[19, Lemma 1.3] Suppose that X is a uniformly convex Banach space and $\{t_n\}$ be any real sequence such that $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(C, \{x_n\})$ consists of exactly one point.

3. *M* Iteration Process and Convergence Results

Throughout this section we have $n \geq 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ and C ba a nonempty subset of Banach Space X. In this section, we prove some weak and strong convergence theorems for the fixed point of Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces using our new M iteration process.

Following is the one step Mann iteration process:

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \end{cases} \tag{1}$$

Agarwal iteration process introduced in [2], also called S iteration process, is defined as:

$$\begin{cases} x_0 \in C \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1 - \alpha_n)T x_n + \alpha_n T y_n \end{cases} \tag{2}$$

Recently in 2014, Gursoy and Karakaya in [6] introduced new iteration process called Picard-S iteration process, as follow

$$\begin{cases} x_0 \in C \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \\ y_n = (1-\alpha_n)T x_n + \alpha_n T z_n \\ x_{n+1} = T y_n \end{cases} \tag{3}$$

They proved that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings. Also, by providing an example, it is shown that the Picard-S iteration process converge faster than all Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas, Normal-S and Two-step Mann iteration processes.

After this in 2015, Thakur et. al. [22] used a new iteration process, defined as:

$$\begin{cases} x_0 \in C \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n) \\ x_{n+1} = T y_n. \end{cases} \tag{4}$$

With the help of numerical example they proved that (4) is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration process for suzuki generalized nonexpansive mappings.

We note that the speed of convergence of iteration process (3) and (4) are almost same.

Problem 3.1. *Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes* (2) *and* (3)*?*

To answer this, we introduce a new three-step iteration process known as "M Iteration Process", defined as:

$$\begin{cases} x_0 \in C \\ z_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = T z_n \\ x_{n+1} = T y_n \end{cases} \tag{5}$$

We now establish the following useful result:

Lemma 3.2. *Let C be a nonempty closed convex subset of a Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (5)*, then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T Suzuki generalized nonexpansive mappings, so

$$\frac{1}{2}\|p - Tp\| = 0 \leq \|p - z\| \text{ implies that } \|Tp - Tz\| \leq \|p - z\|.$$

So by Proposition 2.1(*ii*), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n T x_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|T x_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{6}$$

Using (6) together with Proposition 2.1(*ii*), we get

$$\begin{aligned} \|y_n - p\| &= \|T z_n - p\| \\ &\leq \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{7}$$

Similarly, by using (7) together with Proposition 2.1(ii), we have

$$\begin{aligned} \left\|x_{n+1} - p\right\| &= \left\|Ty_n - p\right\| \\ &\leq \left\|y_n - p\right\| \\ &\leq \left\|x_n - p\right\|. \end{aligned} \tag{8}$$

This implies that $\{\left\|x_n - p\right\|\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists, as required. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (5) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 3.2, $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} \left\|x_n - p\right\| = r. \tag{9}$$

From (6) and (9), we have

$$\limsup_{n\to\infty} \left\|z_n - p\right\| \leq \limsup_{n\to\infty} \left\|x_n - p\right\| = r. \tag{10}$$

By Proposition 2.1(ii), we have

$$\limsup_{n\to\infty} \left\|Tx_n - p\right\| \leq \limsup_{n\to\infty} \left\|x_n - p\right\| = r. \tag{11}$$

On the other hand by suing Proposition 2.1(ii), we have

$$\begin{aligned} \left\|x_{n+1} - p\right\| &= \left\|Ty_n - p\right\| \\ &\leq \left\|y_n - p\right\| \\ &= \left\|Tz_n - p\right\| \\ &\leq \left\|z_n - p\right\|. \end{aligned}$$

Therefore

$$r \leq \liminf_{n\to\infty} \left\|z_n - p\right\|. \tag{12}$$

From (10) and (12), we get

$$\begin{aligned} r &= \lim_{n\to\infty} \left\|z_n - p\right\| \\ &= \lim_{n\to\infty} \left\|(1 - \beta_n)x_n + \beta_n Tx_n - p\right\| \\ &= \lim_{n\to\infty} \left\|\beta_n(Tx_n - p) + (1 - \beta_n)(x_n - p)\right\|. \end{aligned} \tag{13}$$

Using (9), (11) and (13) togather with Lemma 2.4, we get

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Let $p \in A(C, \{x_n\})$. By Proposition 2.1(*iii*), we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty} \left\|x_n - Tp\right\| \\ &\leq \limsup_{n\to\infty} (3\|Tx_n - x_n\| + \left\|x_n - p\right\|) \\ &\leq \limsup_{n\to\infty} \left\|x_n - p\right\| \\ &= r(p, \{x_n\}). \end{aligned}$$

This implies that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex, $A(C, \{x_n\})$ is singleton, hence we have $Tp = p$. Hence $F(T) \neq \emptyset$. □

Now we are in the position to prove weak convergence theorem.

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (5) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, so by Theorem 3.3, we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex hence reflexive, so by Eberlin's theorem there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $q_1 \in X$. Since C is closed and convex, by Mazur's theorem $q_1 \in C$. By Lemma 2.2, $q_1 \in F(T)$. Now, we show that $\{x_n\}$ converges weakly to q_1. In fact, if this is not true, so there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_2 \in C$ and $q_2 \neq q_1$. By Lemma 2.2, $q_2 \in F(T)$. Since $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists for all $p \in F(T)$. By Theorem 3.3 and Opial's property, we have

$$\begin{aligned} \lim_{n\to\infty} \left\|x_n - q_1\right\| &= \lim_{j\to\infty} \left\|x_{n_j} - q_1\right\| \\ &< \lim_{j\to\infty} \left\|x_{n_j} - q_2\right\| \\ &= \lim_{n\to\infty} \left\|x_n - q_2\right\| \\ &= \lim_{k\to\infty} \left\|x_{n_k} - q_2\right\| \\ &< \lim_{k\to\infty} \left\|x_{n_k} - q_1\right\| \\ &= \lim_{n\to\infty} \left\|x_n - q_1\right\|, \end{aligned}$$

which is contradiction. So $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to a fixed point of T. □

Next we prove the strong convergence theorem.

Theorem 3.5. *Let C be a nonempty compact convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (5) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.3, we have that $F(T) \neq \emptyset$ so by Theorem 3.3, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since C is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.1(*iii*), we have

$$\left\|x_{n_k} - Tp\right\| \leq 3\left\|Tx_{n_k} - x_{n_k}\right\| + \left\|x_{n_k} - p\right\|, \text{ for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, *i.e.*, $p \in F(T)$. Since, by Lemma 3.2, $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists for every $p \in F(T)$, so x_n converge strongly to p. □

Senter and Dotson [20] introduced the notion of a mappings satisfying condition (I) as.

A mapping $T : C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p\in F(T)} \left\|x - p\right\|$.

Now we prove the strong convergence theorem using condition (I).

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (5) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.2, we have $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} \left\|x_n - p\right\| = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Suppose $r > 0$, from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\|. \tag{14}$$

Since $F(T) \neq \emptyset$, so by Theorem 3.4, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. So (14) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{15}$$

Since f is nondecreasing function, so from (15) we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$\left\|x_{n_k} - y_k\right\| < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

So using (8),we get

$$\left\|x_{n_{k+1}} - y_k\right\| \leq \left\|x_{n_k} - y_k\right\| < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} \left\|y_{k+1} - y_k\right\| &\leq \left\|y_{k+1} - x_{k+1}\right\| + \left\|x_{k+1} - y_k\right\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists, we have that $x_n \to p \in F(T)$. Hence proved. □

4. Numerical Example

For numerical interpretations first we construct an example of Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 4.1. *Define a mapping $T : [0,1] \to [0,1]$ by*

$$Tx = \begin{cases} 1 - x \text{ if } x \in \left[0, \frac{1}{8}\right) \\ \frac{x+7}{8} \text{ if } x \in \left[\frac{1}{8}, 1\right]. \end{cases}$$

Need to prove that T is Suzuki generalized nonexpansive mapping but not nonexpansive.

If $x = \frac{3}{25}$, $y = \frac{1}{8}$ we see that

$$\begin{aligned} \left\|Tx - Ty\right\| &= \left|Tx - Ty\right| \\ &= \left|1 - \frac{3}{25} - \frac{57}{64}\right| \\ &= \left|\frac{1600 - 192 - 1425}{1600}\right| \\ &= \frac{17}{1600} \\ &> \frac{1}{200} \\ &= \left\|x - y\right\|. \end{aligned}$$

Hence T is not nonexpansive mapping.

To verify that T is Suzuki generalized nonexpansive mapping, consider the following cases:

***Case I**: Let $x \in \left[0, \frac{1}{8}\right)$, then $\frac{1}{2}\|x - Tx\| = \frac{1-2x}{2} \in \left(\frac{3}{8}, \frac{1}{2}\right]$. For $\frac{1}{2}\|x - Tx\| \leq \left\|x - y\right\|$ we must have $\frac{1-2x}{2} \leq y - x$, i.e., $\frac{1}{2} \leq y$, hence $y \in \left[\frac{1}{2}, 1\right]$. We have*

$$\left\|Tx - Ty\right\| = \left|\frac{y+7}{8} - (1 - x)\right| = \left|\frac{y + 8x - 1}{8}\right| < \frac{1}{8},$$

and

$$\left\|x - y\right\| = \left|x - y\right| > \left|\frac{1}{8} - \frac{1}{2}\right| = \frac{3}{8}.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \left\|x - y\right\| \Longrightarrow \left\|Tx - Ty\right\| \leq \left\|x - y\right\|$.

***Case II**: Let $x \in \left[\frac{1}{8}, 1\right]$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\left|\frac{x+7}{8} - x\right| = \frac{7-7x}{16} \in \left[0, \frac{49}{128}\right]$. For $\frac{1}{2}\|x - Tx\| \leq \left\|x - y\right\|$ we must have $\frac{7-7x}{16} \leq \left|y - x\right|$, which gives two possibilities:*

***(a).** Let $x < y$, then $\frac{7-7x}{16} \leq y - x \Longrightarrow y \geq \frac{7+9x}{16} \Longrightarrow y \in \left[\frac{65}{128}, 1\right] \subset \left[\frac{1}{8}, 1\right]$. So*

$$\left\|Tx - Ty\right\| = \left|\frac{x+7}{8} - \frac{y+7}{8}\right| = \frac{1}{8}\left\|x - y\right\| \leq \left\|x - y\right\|.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \left\|x - y\right\| \Longrightarrow \left\|Tx - Ty\right\| \leq \left\|x - y\right\|$.

***(b).** Let $x > y$, then $\frac{7-7x}{16} \leq x - y \Longrightarrow y \leq x - \frac{7-7x}{16} = \frac{23x-7}{16} \Longrightarrow y \in \left[-\frac{33}{128}, 1\right]$. Since $y \in [0,1]$, so $y \leq \frac{23x-7}{16} \Longrightarrow x \geq \frac{16y+7}{23} \Longrightarrow x \in \left[\frac{7}{23}, 1\right]$. So the case is $x \in \left[\frac{7}{23}, 1\right]$ and $y \in [0,1]$.*

Now $x \in \left[\frac{7}{23}, 1\right]$ and $y \in \left[\frac{1}{8}, 1\right]$ is already included in case (a). So let $x \in \left[\frac{7}{23}, 1\right]$ and $y \in \left[0, \frac{1}{8}\right)$, then

$$\begin{aligned} \left\|Tx - Ty\right\| &= \left|\frac{x+7}{8} - (1 - y)\right| \\ &= \left|\frac{x + 8y - 1}{8}\right|. \end{aligned}$$

Table 1: Sequences generated by M, Picard-S and S iteration processes.

	M	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.9984375	0.9984375	0.9875
x_2	0.999986267089844	0.999978963614444	0.998653671324426
x_3	0.999999942021466	0.999999737837406	0.999865772752021
x_4	0.999999999962418	0.999999996885298	0.999987242180259
x_5	1	0.999999999964181	0.999998826296532
x_6	1	0.999999999999598	0.999999894547974
x_7	1	0.999999999999996	0.999999990694634
x_8	1	1.	0.999999999190409
x_9	1	1.	0.999999999930363
x_{10}	1	1.	0.999999999994067

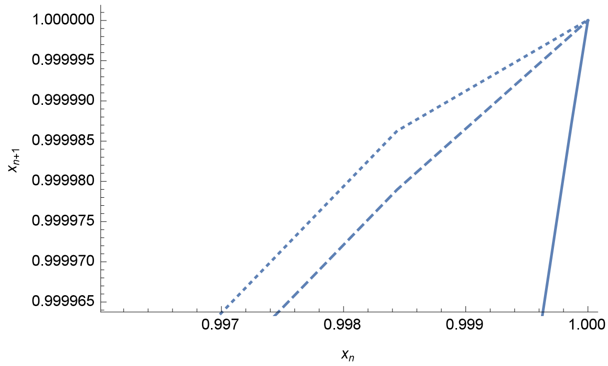

Figure 1: Convergence of iterative sequences generated by M (dots), Picard-S (dashes) and S (line) iteration processes to the fixed point 1 of mapping T defined in Example 4.1.

For convenience, first we consider $x \in \left[\frac{7}{23}, \frac{1}{2}\right]$ *and* $y \in \left[0, \frac{1}{8}\right)$, *then* $\left\|Tx - Ty\right\| \leq \frac{1}{16}$ *and* $\left\|x - y\right\| > \frac{33}{184}$. *Hence* $\left\|Tx - Ty\right\| \leq \left\|x - y\right\|$.

Next consider $x \in \left[\frac{1}{2}, 1\right]$ *and* $y \in \left[0, \frac{1}{8}\right)$, *then* $\left\|Tx - Ty\right\| \leq \frac{1}{8}$ *and* $\left\|x - y\right\| > \frac{3}{8}$. *Hence* $\left\|Tx - Ty\right\| \leq \left\|x - y\right\|$. *So* $\frac{1}{2}\left\|x - Tx\right\| \leq \left\|x - y\right\| \Longrightarrow \left\|Tx - Ty\right\| \leq \left\|x - y\right\|$.

Hence T is Suzuki generalized nonexpansive mapping.

Numerically we compare our new iteration process with two existing iteration processes. First is S iteration process (2) and the second is Picard-S iteration process (3).

In "Table 1" we can see some of the first terms of a sequence generated by M, Picard-S and S iteration processes for $\alpha_n = \frac{2n}{\sqrt{7n+9}}$, $\beta_n = \frac{1}{\sqrt{3n+7}}$, where initial value $x_0 = 0.9$ and operator T is that of Example 4.1. Set the stop parameter to $\|x_n - 1\| \leq 10^{-15}$, where "1" is the fixed point of T. Graphic representation in given in Figure 1. We can easily see that the new M iterations are the first converging one than the S iterations and the Picard-S iterations.

In order to see how far from exactly "1" the value of x_n is for a certainly value of n, we resort to arbitrary precision calculations and get the Figure 2.

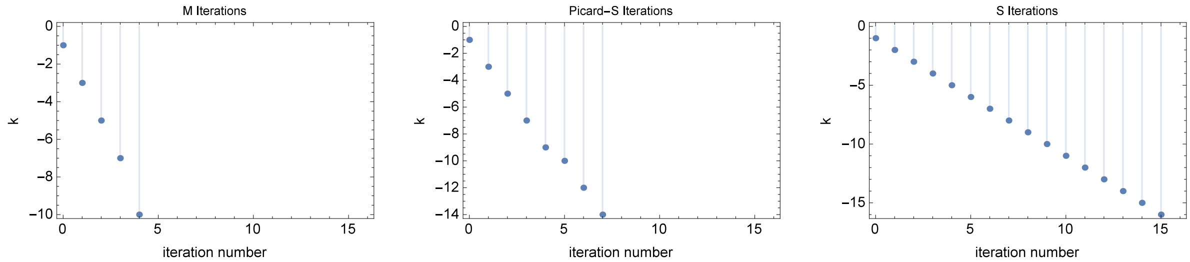

Figure 2: Graphs for M, Picard-S and S iteration processes where the value of k indicates that the value of the recursion after a certain number of steps is only 10^k units away from fixed point 1 of mapping T defined in Example 4.1.

Author's contributions

The authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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