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Ricci Semisymmetric Almost Kenmotsu Manifolds with Nullity Distributions

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Abstract. The object of the present paper is to characterize Ricci semisymmetric almost Kenmotsu manifolds with its characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution respectively. Finally, an illustrative example is given.

1. Introduction

Among Riemannian manifolds, the most interesting and most important for applications are the symmetric ones. From the local point of view it was introduced independently by Shirokov [15] as a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where ∇ is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was worked out by Cartan in 1927. As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor R satisfies $R(X,Y)\cdot R = 0$, where $R(X,Y)$ is considered as a field of linear operators, acting on R. Semisymmetric manifolds were classified by Szabó, locally in [16]. The classification results of Szabó were presented in the book [4].
A Riemannian manifold is said to be Ricci semisymmetric if $R(X,Y)\cdot S = 0$ where S denotes the Ricci tensor of type $(0,2)$. A general classification of these manifolds has been worked out recently by V.A. Mirzoyan [11]. Recently, De and Velimirović [5] studied spacetimes with semisymmetric energy momentum tensor. On the other hand, an odd dimensional manifold $M^{2n+1} (n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying [1, 2]

$$\phi^2 = -I + \eta \otimes \xi, \; \eta(\xi) = 1, \; \phi\xi = 0, \; \eta \circ \phi = 0. \tag{1}$$

The first and one of the remaining three relations in (1) imply the other two relations in (1). An almost contact structure is said to be normal if the induced almost complex structure J on $M^{2n+1} \times \mathbb{R}$ defined by

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$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X)$$

for all vector fields $X, Y \in \chi(M)$ = the set of all differentiable vector fields on M.
A Kenmotsu manifold [10] can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ where $\Phi = g(X, \phi Y)$. It is well known that Kenmotsu manifolds can be characterize by $(\nabla_X \phi)Y = g(\phi X, Y) - \eta(Y)\phi X$, for any vector fields X, Y, Z.
Recently in ([7],[8],[12],[13]), almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. In [6] G. Dileo and A.M. Pastore studied locally symmetric almost Kenmotsu manifolds. Moreover almost Kenmotsu manifolds satisfying some nullity conditions were also investigated by G. Dileo and A.M. Pastore [7]. Also for more results on $(k, \mu)'$-nullity distribution and (k, μ)-nullity distribution on almost Kenmotsu manifolds, we refer to A.M. Pastore and V. Saltarelli ([12],[13]). In recent papers ([17],[18],[19],[20]) Y. Wang and X.M. Liu study almost Kenmotsu manifolds with nullity distributions. In [18] Y. Wang and X.M. Liu study ξ-Riemannian semisymmetric almost Kenmotsu manifolds satisfying $(k, \mu)'$-nullity distribution and (k, μ)-nullity distribution. Since semisymmetry $(R \cdot R = 0)$ implies Ricci semisymmetry $(R \cdot S = 0)$, but the converse is not true, in general, in the present paper we generalize the results of [18] and [6].
The paper is organized as follows:
In section 2, some basic results of almost Kenmotsu manifolds are given. Section 3 deals with Ricci semisymmetic almost Kenmotsu manifolds with ξ belonging to $(k, \mu)'$-nullity distribution. In the next section we consider Ricci semisymmetric almost Kenmotsu manifolds with ξ belonging to the (k, μ)-nullity distribution. As a consequence of these results we obtain several corollaries. Finally, an illustrative example is given.

2. Almost Kenmotsu Manifolds

Let M^{2n+1} be an almost Kenmotsu manifold with structure (ϕ, ξ, η, g). Let $h = \frac{1}{2}\pounds_\xi \phi$ on an almost Kenmotsu manifold, where £ is the Lie differentiation. We denote by $l = R(\cdot, \xi)\xi$. The two $(1, 1)$- type tensor l and h are symmetric and satisfy [12]

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0. \tag{3}$$

Also we have the following results

$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{4}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{5}$$

$$R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X, \tag{6}$$

where $h' = h \circ \phi$.
Finally, we recall the definition of the nullity distribution. D.E. Blair, T. Koufogiorgos and B.J. Papantoniou [3] introduced (k, μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ as follows:

$$\begin{aligned} N_p(k, \mu) = \{Z \in T_pM : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &+\mu[g(Y, Z)hX - g(X, Z)hY]\}, \end{aligned} \tag{7}$$

where $h = \frac{1}{2}\pounds_\xi \phi$ and $(h, k) \in \mathbb{R}^2$.
In [7], G. Dileo and A.M. Pastore introduced the notion of $(k,\mu)'$-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_pM : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \end{aligned} \tag{8}$$

where $h' = h \circ \phi$ and $(k,\mu) \in \mathbb{R}^2$.

3. ξ Belongs to the $(k,\mu)'$-Nullity Distribution

In this section we consider an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and we recall some results stated in [7]. From (8) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{9}$$

where $(k,\mu) \in \mathbb{R}^2$. We denote by $\mathfrak{D}$ the contact distribution defined by $\mathfrak{D} = ker(\eta) = Im(\phi)$. Replacing Y by ξ in (9) gives $lX = k(X - \eta(X)\xi) + \mu h'X$. Using (1) and (3) in the above equation we obtain

$$\phi l \phi X = -k(X - \eta(X)\xi + \mu h'X).$$

Substituting the above equation in (5) gives

$$h'^2 = (k+1)\phi^2 \quad (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{10}$$

Now let $X \in \mathfrak{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (10) it follows that $\lambda^2 = -(k+1)$. Hence $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote the eigenspaces associated with h' by $[\lambda]'$ and $[-\lambda]'$ corresponding to the eigen value $\lambda \neq 0$ and $-\lambda$ of h' respectively. Before proving our main theorem, we state the following result due to G. Dileo and A.M. Pastore [6, Prop. 4.3].

Lemma 3.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distribution $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves, respectively. Furthermore, the sectional curvature are given as following:*

(a) $K(X,\xi) = k - 2\lambda$ if $X \in [\lambda]$ and
$K(X,\xi) = k + 2\lambda$ if $X \in [-\lambda]'$,

(b) $K(X,Y) = k - 2\lambda$ if $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ if $X, Y \in [-\lambda]'$ and
$K(X,Y) = -(k+2)$ if $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} has constant negative scalar curvature $r = 2n(k - 2n)$.

Theorem 3.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If M^{2n+1} is Ricci semisymmetric, then either M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, or the manifold is an Einstein manifold.*

Proof. From (9) we get on contraction by using (3),

$$S(X,\xi) = 2nk\eta(X). \tag{11}$$

We suppose that the manifold is Ricci semisymmetric. Then $(R(X,Y) \cdot S)(U,V) = 0$ for all vector fields X, Y, U, V, which implies

$$S(R(X,Y)U, V) + S(U, R(X,Y)V) = 0. \tag{12}$$

Also from (9) it follows that

$$R(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi] + \mu[\eta(Y)h'X - g(h'X,Y)\xi]. \tag{13}$$

Substituting $Y = \xi$ in (12) and using (13) we obtain

$$\begin{aligned}
&k\eta(U)S(X,V) - kg(X,U)S(\xi,V) + \mu\eta(U)S(h'X,V)\\
&-\mu g(h'X,U)S(\xi,V) + k\eta(V)S(U,X)\\
&-kg(X,V)S(U,\xi) + \mu\eta(V)S(U,h'X) - \mu g(h'X,V)S(U,\xi) = 0.
\end{aligned} \tag{14}$$

Again putting $U = \xi$ in (14) and using (11) we get

$$kS(X,V) + \mu S(h'X,V) - 2nk^2 g(X,V) - 2nk\mu g(h'X,V) = 0. \tag{15}$$

Replacing X by $h'X$ in (15) and using the fact ${h'}^2 = (k+1)\phi^2$ yields

$$kS(h'X,V) - \mu(k+1)S(X,V) - 2nk^2 g(h'X,V) + 2nk\mu(k+1)g(X,V) = 0. \tag{16}$$

Subtracting k multiple of (15) and μ multiple of (16) we have

$$(k^2 + \mu^2(k+1))[S(X,V) - 2nkg(X,V)] = 0. \tag{17}$$

Since $\mu = -2$, the above equation reduces to

$$(k+2)^2[S(X,V) - 2nkg(X,V)] = 0. \tag{18}$$

Now we consider the following two cases:

case 1: $k \neq -2$. It follows from (18) that

$$S(X,V) = 2nkg(X,V), \tag{19}$$

which implies that the manifold is an Einstein manifold.

case 2: $k = -2$. It follows from ${h'}^2X = (k+1)\phi^2 X$ for any $X \in \chi(M)$ that the nonzero eigenvalue of h' is either 1 or −1, that is, $\lambda = \pm 1$. Without loss of any generality we now choose $\lambda = 1$, noticing $\mu = -2$ and then it follows from Lemma 3.1 that $K(X,\xi) = -4$ and for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Also from Lemma 3.1 we see that $K(X,Y) = -4$ for any $X,Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X,Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [7] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Noticing that $\lambda = 1$, then we known that two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we conclude that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.
This completes the proof. □

In [18] the authors studied semisymmetric almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution and in this case they obtain $k = -2$. Also $R \cdot R = 0$ implies $R \cdot S = 0$. Therefore from Theorem 3.2 we obtain the result of [18] by Y. Wang and X. Liu.

Corollary 3.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If M^{2n+1} is semisymmetric, then M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Again Ricci symmetry ($\nabla S = 0$) implies $R \cdot S = 0$, therefore we can state the following:

Corollary 3.4. *A Ricci symmetric almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution with $h' \neq 0$ is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, or an Einstein manifold.*

The above corollary generalizes the results of [6].
A Riemannain manifold is said to be Ricci-recurrent [14] if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z),$$

where A is a non-zero 1-form.
In [9] J.B. Jun, U.C. De and Gautam Pathak prove that a Ricci-recurrent Riemannian manifold is Ricci semisymmetric. Hence we conclude the following:

Corollary 3.5. *A Ricci-recurrent almost Kenmotsu manifold M^{2n+1} with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ is either locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, or an Einstein manifold.*

4. ξ Belongs to the (k,μ)-Nullity Distribution

This section is devoted to study Ricci semisymmetric almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution.
From (7) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{20}$$

where $(k,\mu) \in \mathbb{R}^2$. Now we state the following:

Lemma 4.1. *([7], Theorem 4.1) Let M^{2n+1} be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a warped product of an open interval and an almost Kähler manifold.*

From (20) we get by using Lemma 4.1,

$$S(X,\xi) = -2n\eta(X). \tag{21}$$

By hypothesis the manifold under consideration is Ricci semisymmetric, therefore

$$(R(X,Y)\cdot S)(U,V) = 0.$$

Replacing Y by ξ in the above equation gives

$$S(R(X,\xi)U,V) + S(U,R(X,\xi)V) = 0. \tag{22}$$

From (20) it follows that

$$R(X,\xi)U = k[\eta(U,X) - g(X,U)\xi] + \mu[\eta(U)hX - g(hX,U)\xi].$$

By Lemma 4.1 we get

$$R(X,\xi)U = g(X,U)\xi - \eta(U)X. \tag{23}$$

Now using (23) and (21) in (22) yields

$$S(X,V) = -2ng(X,V),$$

which implies that the manifold is an Einstein manifold.
Conversely, if the manifold is an Einstein manifold, then obviously the manifold is Ricci semisymmetric. Hence we can state the following:

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the (k, μ)-nullity distribution. Then M^{2n+1} is Ricci semisymmetric if and only if the manifold is an Einstein manifold.*

Since semisymmetric manifold ($R \cdot R = 0$) implies Ricci semisymmetric ($R \cdot S = 0$) and locally symmetric manifold ($\nabla R = 0$) implies Ricci symmetric ($\nabla S = 0$) and also $\nabla S = 0$ implies $R \cdot S = 0$, therefore Theorem 4.2 generalizes the Theorem of [18].
From Theorem 4.2 and the above discussions we can conclude the following:

Corollary 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the (k, μ)-nullity distribution. Then the following conditions are equivalent:*

(a) M^{2n+1} *is an Einstein manifold;*

(b) $\nabla S = 0$;

(c) $R \cdot S = 0$.

5. Example of a 5-Dimensional Almost Kenmotsu Manifold

In this section, we construct an example of an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$, which is an Einstein manifold. We consider 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 are five vector fields in $\mathbb{R}^5$ which satisfies [7]

$$[\xi, e_2] = -2e_2, \ [\xi, e_3] = -2e_3, \ [\xi, e_4] = 0, \ [\xi, e_5] = 0,$$
$$[e_i, e_j] = 0, \text{ where } i, j = 2, 3, 4, 5.$$

Let g be the Riemannian metric defined by

$$g(\xi, \xi) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1$$
$$\text{and } g(\xi, e_i) = g(e_i, e_j) = 0 \text{ for } i \neq j; i, j = 2, 3, 4, 5.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \xi),$$

for any $Z \in \chi(M)$. Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi(\xi) = 0, \ \phi(e_2) = e_4, \ \phi(e_3) = e_5, \ \phi(e_4) = -e_2, \ \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g we have

$$\eta(\xi) = 1, \ \phi^2 Z = -Z + \eta(Z)\xi$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $Z, U \in \chi(M)$. Moreover,

$$h'\xi = 0, \ h'e_2 = e_2, \ h'e_3 = e_3, \ h'e_4 = -e_4, \ h'e_5 = -e_5.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\ \nabla_\xi e_4 = 0,\ \nabla_\xi e_5 = \xi,$$
$$\nabla_{e_2} \xi = 2e_2,\ \nabla_{e_2} e_2 = -2\xi,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,$$
$$\nabla_{e_3} \xi = 2e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -2\xi,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,$$
$$\nabla_{e_4} \xi = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = 0,\ \nabla_{e_4} e_5 = 0,$$
$$\nabla_{e_5} \xi = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h' X,$$

for any $X \in \chi(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$R(\xi, e_2)\xi = 4e_2,\ R(\xi, e_2)e_2 = -4\xi,\ R(\xi, e_3)\xi = 4e_3,\ R(\xi, e_3)e_3 = -4\xi,$$
$$R(\xi, e_4)\xi = R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0,$$
$$R(e_2, e_3)e_2 = 4e_3,\ R(e_2, e_3)e_3 = -4e_2,\ R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0,$$
$$R(e_2, e_5)e_2 = R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0,$$
$$R(e_3, e_5)e_3 = R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0.$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distribution, with $k = -2$ and $\mu = -2$.

Using the expressions of the curvature tensor we find the values of the Ricci tensor S as follows:

$$S(\xi, \xi) = S(e_2, e_2) = S(e_3, e_3) = -8,\ S(e_4, e_4) = S(e_5, e_5) = 0.$$

This shows that the manifold is Ricci semisymmetric.
Since $\{\xi, e_2, e_3, e_4, e_5\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1\xi + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$$

and

$$Y = b_1\xi + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5,$$

where $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 \in \mathbb{R} \setminus \{0\}$ such that $a_4b_4 + a_5b_5 = 0$. Hence,

$$g(X, Y) = a_1b_1 + a_2b_2 + a_3b_3$$

and

$$S(X, Y) = -8(a_1b_1 + a_2b_2 + a_3b_3).$$

Therefore, we see that $S(X, Y) = -8g(X, Y)$, that is, the manifold M is an Einstein manifold.
Thus Theorem 3.2 is verified.

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