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Geodesic and holomorphically-projective mapping of conformally-kählerian spaces

Želko Radulovič

Abstract. This paper is devoted to study of geodesic and holomorphycally - projective mapping of conformally-Kählerian spaces, which are a generalization of the conformal space. A condition admitting geodesic mapping of conformally-Kählerian spaces, has been found. Conformally - Kählerian spaces not admitting nontrivial geodesic-projective mapping has been discussed.

In the theory of almost complex manifolds the geodesic and holomorphically- projective mapping has been studying by many authors. These matters were discussed in D.Beklemishev's survey [1], as well as in K.Jano's book [2].

W.Y.Westlake [3], K.Yano and T.Nagano [4,5] have shown that between Kählerian spaces one cannot establish nontrivial geodesic mapping preserving structure. Those finding were further developed by A.V.Karmasina and I.N.Kurbatova [6], and it has been shown that K-spaces do not admit nontrivial and structure-preserving geodesic mapping onto almost Hermitian spaces.

In works by J.Mikeš [7-10] results of his study of the geodesic mapping of Kählerian spaces not preserving the structure is given.

Since such spaces are a natural generalization of conformally spaces it is natural to call them a conformally-Kählerian. The interest in investigation of these spaces recently has grown due to the possibility of using them as a model of Kaluca-Klein theory [11]. Investigations into conformally-Kählerian spaces are carried out, for example, in [6, 12-14].

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§1. The main properties of conformally-Kählerian spaces geodesic mapping

1. The conformally-Kählerian spaces

Let's preliminary establish basic definitions [1,11].

Definition 1. A Riemannian space H_n is called almost Hermitian if both metric tensor $g_{ij}(x)$ and almost Hermitian structure $F_i^h(x)$ are determined in it, and $F_i^h(x)$ satisfies the conditions

$$F_\alpha^h F_i^\alpha = -\delta_i^h; \qquad F_{(i g_j)\alpha}^\alpha = 0, \tag{1}$$

δ_i^h being the Kronecer symbols, (i, j) denoting symmetrization without division.

Definition 2. An almost Hermtian space with covariantly constant structure is called Kählerian space.

Definition 3. Riemannian space that can be conformally mapped onto Kählerian space is called conformally-Kählerian space K_n .

Evidently, a conformally-Kählerian space K_n is almost Hermitian. Conformalählrian spaces K_n are characterized by the existence of almost Hermitian structure $F_i^h(x)$, satisfying (1) and the following conditions [11,6]:

$$F_{i,j}^h = (n-2)^{-1}(\delta_j^h F_{i,\alpha}^\alpha - g_{ij}F_{\cdot\cdot,\alpha}^{\alpha h} + F_j^h F_{\beta,\alpha}^\alpha F_i^{\beta h} + F_{ji}F_{\beta,\alpha}^\alpha F^{\beta h}) \tag{2}$$

where $F_{ij} = g_{i\alpha}F_j^\alpha; F^{ij} = g^{j\alpha}F_\alpha^i$.

We would point out, that conformally-Kählerian space, determining by (1) and (2), is one of the particular classes established by A.Gray [11] among almost Hermitian spaces.

To facility the investigation and discussion of the almost-Hermitian spaces, in general, and conformally-Kählerian spaces, in particular, the following procedure of indexes conjugation has been introduced:

$$T_{\bar{i}\dots}^{\dots} \equiv T_{\alpha\dots}^{\dots}F_i^\alpha; \qquad T_{\dots}^{\bar{i}\dots} \equiv T_{\dots}^{\alpha\dots}F_\alpha^i. \tag{3}$$

This procedure is possessing the following properties:

$$T_{\bar{\bar{i}}} = -T_i; \qquad T^{\bar{\bar{i}}} = -T^i; \qquad T_{\bar{\alpha}}U^\alpha = T_\alpha U^{\bar{\alpha}}$$

Evidently, that $\delta_{\bar{i}}^h = \delta_i^{\bar{h}} = F_i^h$ and both tensor, metric tensor (g_{ij}) and conjugated (g^{ij}) imply that

$$g_{i\bar{j}} + g_{\bar{i}j} = 0; \quad g_{\bar{i}\bar{j}} = g_{ij}; \quad g^{i\bar{j}} + g^{\bar{i}j} = 0; \quad g^{\bar{i}\bar{j}} = g^{ij}. \tag{4}$$

Then (2) may be rewritten in the more compact form:

$$F^h_{i,j} = \delta^h_j \phi_i - g_{ij}\phi^h + \delta^h_{\bar{j}}\phi_{\bar{i}} - g_{\bar{i}j}\phi^{\bar{h}} \tag{5}$$

where $\phi_i = (n-2)^{-1}F^{\alpha}_{i,\alpha}$; $\phi^h = g^{h\alpha}\phi_\alpha$. Omitting index h, we obtain

$$F_{hi,j} = g_{hj}\phi_i - g_{ij}\phi_h - g_{\bar{h}j}\phi_{\bar{i}} + g_{\bar{i}j}\phi_{\bar{h}}. \tag{6}$$

Now, let covariantly differentiate (6) at x^k and alternate at indexes j and k. Changing the notations and taking into consideration Ricci identity, we get

$$R_{\bar{h}ijk} + R_{h\bar{i}jk} = g_{ij}\phi^*_{hk} - g_{ik}\phi^*_{hj} - g_{hj}\phi^*_{ik} + g_{hk}\phi^*_{ij} -$$

$$g_{\bar{i}j}\tilde{\phi}_{hk} + g_{\bar{i}k}\tilde{\phi}_{hj} + g_{\bar{h}j}\tilde{\phi}_{ik} - g_{\bar{h}k}\tilde{\phi}_{ij}, \tag{7}$$

where $\phi^*_{ij} \equiv \phi_{i,j} - \phi_{\bar{i}}\phi_{\bar{j}}$; $\tilde{\phi}_{ij} \equiv (\phi_{\bar{i}})_{,j} - \phi_{\bar{i}}\phi_{\bar{j}}$.

After contracting (7) with $F^h_{h'}F^i_{i'}$, we omitted prime at indexes, added by respective components with (7), and received

$$g_{ij}\Phi_{hk} - g_{ik}\Phi_{hj} - g_{hj}\Phi_{ik} + g_{hk}\Phi_{ij} +$$

$$g_{\bar{i}j}\Phi_{\bar{h}k} - g_{\bar{i}k}\Phi_{\bar{h}j} - g_{\bar{h}j}\Phi_{\bar{i}k} + g_{\bar{h}k}\Phi_{\bar{i}\bar{j}} = 0 \tag{8}$$

where $\Phi_{hk} \equiv \phi_{hk} + \phi_{\bar{h}k}$.

Let us contract (8) with g^{ij}:

$$(n-1)\Phi_{hk} - \Phi_{\bar{h}\bar{k}} + \Phi g_{hk} + \bar{\Phi} g_{\bar{h}k},$$

where $\Phi \equiv \Phi_{\alpha\beta}g^{\alpha\beta}$; $\bar{\Phi} \equiv \Phi_{\bar{\alpha}\beta}g^{\alpha\beta}$. Contracting the latter with g^{hk}, we see, that $\Phi = 0$. So,

$$(n-1)\Phi_{hk} - \Phi_{\bar{h}\bar{k}} + \bar{\Phi} g_{\bar{h}k} = 0. \tag{9}$$

From (9) we conclude that

$$(n-1)\Phi_{\bar{h}\bar{k}} - \Phi_{hk} + \bar{\Phi} g_{\bar{h}k} = 0.$$

Thus

$$\Phi_{hk} = \alpha g_{\bar{h}k},$$

when α is a certain invariant. According to the definition of tensor Φ_{hk} we have

$$\phi^*_{hk} + \tilde{\phi}_{\bar{h}k} = \alpha g_{\bar{h}k},$$

from where it follows that

$$\tilde{\phi}_{hk} = \phi_{\bar{h}k} + \alpha g_{hk} \tag{10}$$

Then (7) takes the form

$$R_{\bar{h}ijk} + R_{h\bar{i}jk} = g_{ij}\phi_{hk} - g_{ik}\phi_{hj} - g_{hj}\phi_{ik} + g_{hk}\phi_{ij} -$$

$$g_{\bar{i}j}\phi_{\bar{h}k} + g_{\bar{i}k}\phi_{\bar{h}j} + g_{\bar{h}j}\phi_{\bar{i}k} - g_{\bar{h}k}\phi_{\bar{i}j}. \tag{11}$$

From (11) it follows that

$$R_{\bar{h}\bar{i}jk} + R_{hijk} = g_{\bar{i}j}\phi_{hk} - g_{\bar{i}k}\phi_{hj} - g_{hj}\phi_{\bar{i}k} + g_{hk}\phi_{\bar{i}j} +$$

$$g_{\bar{i}j}\phi_{\bar{h}k} - g_{ik}\phi_{\bar{h}j} - g_{\bar{h}j}\phi_{ik} + g_{\bar{h}k}\phi_{ij}. \tag{12}$$

It is easily checked that ϕ_{ij} can be expressed by means of Riemannian tensor components and structure and metric of Conformally - Kählerian space K_n.

Taking into consideration the Riemannian tensor properties (12) can be rewritten as the following:

$$R_{hi\bar{j}\bar{k}} + R_{hijk} = g_{hk}\phi_{\bar{j}i} - g_{hj}\phi_{\bar{k}j} + g_{ij}\phi_{\bar{k}h} - g_{ik}\phi_{\bar{j}h} +$$

$$g_{h\bar{k}}\phi_{ji} - g_{h\bar{j}}\phi_{ki} + g_{i\bar{j}}\phi_{kh} - g_{i\bar{k}}\phi_{jh}. \tag{13}$$

2. Geodesic mapping of Riemanian spaces.

Definition 4. Diffeomorphism f of Riemannian space V_n into $\bar{V}_n$ is called a geodesic mapping if all geodesic lines in V_n are mapped as geodesic lines in $\bar{V}_n$.

V_n and $\bar{V}_n$ admits a geodesic correspondence, if and only if the following condition is satisfied in the "common" mapping coordinate system [15]:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\psi_j + \delta^h_j\psi_i, \tag{14}$$

where $\bar{\Gamma}^h_{ij}$ and Γ^h_{ij} are corresponding Cristoffel symbols of the second type for $\bar{V}_n$ and V_n, ψ_i is a certain vector field, which is necessarily a gradient type, that is $\psi_i = \partial_i\psi; \partial_i \equiv \partial/\partial x^i$.

When $\psi_i \not\equiv 0$ the geodesic mapping is said nontrivial.

The relations (14) are equivalent to the following:

$$\bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi_j\bar{g}_{ik}. \tag{15}$$

In [15], it is proved, that V_n admits geodesic mapping if in V_n there exists a solution of the following equation

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \tag{16}$$

on an unknown symmetrical regular tensor a_{ij} and a vector λ_i.

If $\lambda \neq 0$ then a geodesic mapping is nontrivial.

It is well known, that the condition of integrabilty of these equations has the form

$$a_{\alpha(i} R^{\alpha}{}_{j)kl} = \lambda_{l(i} g_{j)k} - \lambda_{k(i} g_{j)l}, \tag{17}$$

where $\lambda_{ij} = \lambda_{i,j}$.

§2. Geodesic mapping of conformally-Kählerian spaces

1. Geodesic structure preserving mapping.

Let H_n and $\bar{H}_n$ are almost Hermitian spaces with structures F_i^h and $\bar{F}_i^h$ respectively, and let a certain diffeomorphism (mapping) be established between them.

If, with respect to a "common" coordinate system, the condition

$$\bar{F}_i^h(x) = F_i^h(x), \tag{18}$$

is satisfied, then it is said, that the structure is preserved by the mapping.

As we mentioned above, the geodesic mapping preserving the structure of certain almost Hermitian spaces has been investigated in [3-6].

In the paper by A.V.Karmasina and I.N.Kurbatova [6] geodesic mapping of conformally-Kählerian space onto almost Hermitian space structure preserving were studied. Their investigations were completed by the following theorem.

Theorem 1. *A conformally-Kählerian space K_n $(n > 2)$ does not admit nontrivial structure-preserving geodesic mapping onto almost Hermitian spaces.*

Proof. Let us suppose the opposite. Let conformally-Kählerian space $K_n(g_{ij}, F_i^h)$ admit a nontrivial geodesic structire-preserving mapping (18) onto almost Hermitian space $\bar{H}_n(\bar{g}_{ij}, \bar{F}_i^h)$.

The condition

$$\bar{g}_{\alpha i} \bar{F}_j^{\alpha} + \bar{g}_{\alpha j} \bar{F}_i^{\alpha} = 0,$$

existing in $\bar{H}_n$ spaces, takes the following form, taking into account (18)

$$\bar{g}_{\alpha i} F_j^{\alpha} + \bar{g}_{\alpha j} F_i^{\alpha} = 0. \tag{19}$$

This condition can be covariantly differentiated at x^k. Taking into account (6) and (11), after the reduction we get

$$\chi_i\bar{g}_{jk} + \chi_j\bar{g}_{ik} - \chi_{\bar{i}}\bar{g}_{\bar{j}k} - \chi_{\bar{j}}\bar{g}_{\bar{i}k} -$$

$$\Theta_i g_{jk} - \Theta_j g_{ik} + \Theta_{\bar{i}} g_{\bar{j}k} + \Theta_{\bar{i}} g_{\bar{i}k} = 0, \tag{20}$$

where $\chi_i \equiv \phi_i + \psi_{\bar{i}}$; $\Theta_i \equiv \bar{g}_{i\alpha}\phi^\alpha$.

Let's make the relation (20) symmetrical by all indexes:

$$\chi_i\bar{g}_{jk} + \chi_j\bar{g}_{ki} + \chi_k\bar{g}_{ij} -$$

$$\Theta_i g_{jk} - \Theta_j g_{ki} - \Theta_k g_{ij} = 0. \tag{21}$$

And now, let us consider the case, when vectors χ_i and Θ_i are noncolinear. Then there exist a vector ϵ^i such that $\epsilon^\alpha\Theta_\alpha = 0$ and $\epsilon^\alpha\chi_\alpha = 1$.

If we contract (21) with $\epsilon^i\epsilon^j\epsilon^k$, we shall see that

$$\bar{g}_{\alpha\beta}\epsilon^\alpha\epsilon^\beta = 0. \tag{22}$$

After contraction of (21) with $\epsilon^j\epsilon^k$, by (22) we get

$$\bar{g}_{i\beta}\epsilon^\alpha = \alpha\Theta_i, \tag{23}$$

where α is an invariant.

Finally, we contract (8) with ϵ^k, and conclude from (23) that

$$\bar{g}_{ij} = \theta_{(i}\xi_{j)}, \tag{24}$$

where ξ_i is a certain vector. So we obtained a contradiction to $Rg||\bar{g}_{ij}|| = n > 2$. Consequently, vectors χ_i and Θ_i are collinear: $\chi_i = \alpha\Theta_i$. Then (21) takes the form

$$\Theta_i(g_{ik} - \alpha\bar{g}_{jk}) + \Theta_j(g_{ki} - \alpha\bar{g}_{ki}) + \Theta_k(g_{ij} - \alpha\bar{g}_{ij}) = 0.$$

One can see, that the latter relation implies either $\Theta_i = 0$ or $g_{ij} - \alpha\bar{g}_{ij} = 0$.

The first case $\Theta_i = 0 \iff \phi^h = 0$ means that conformally - Kählerian space is Kählerian space, but for the Kählerian spaces the theorem 1 is proved.

The second case, where $g_{ij} - \alpha\bar{g}_{ij} = 0$ and K_n and $\bar{H}_n$ are in a conformal correspondence, is in contradiction with nontriviality of the geodesic correspondence.

Thus the theorem 1 has been proved.

2. General properties of geodesic mapping of conformally-Kählerian spaces.

Let us study the general properties of geodesic mapping of conformally-Kählerian spaces.

Let conformally-Kählerian space K_n admit a nontrivially geodesic mapping onto certain Riemannian spaces $\bar{V}_n$. Then in K_n there exist a solution of the equation (16) and satisfying the condition of integrability (17).

Let contract (17) with $F^{j}_{j'}F^{k}_{k'}$, and after omitting primes, we get:

$$a_{\alpha(h}R^{\alpha}_{i)\bar{j}\bar{k}} = \lambda_{\bar{k}(h}g_{i)\bar{j}} - \lambda_{\bar{j}(h}g_{i)\bar{k}}. \tag{25}$$

Subtracting (17) from (25), taking into consideration (13), and after the grouping, we obtain

$$\begin{aligned}&a_{kh}\phi_{\bar{j}i} - a_{jh}\phi_{\bar{k}i} + g_{kh}\Phi_{\bar{j}i} - g_{jh}\Phi_{\bar{k}i} +\\ &a_{\bar{k}h}\phi_{ji} - a_{\bar{j}h}\phi_{ki} + g_{\bar{k}h}\Phi_{ji} - g_{\bar{j}h}\Phi_{ki} +\\ &a_{ki}\phi_{\bar{j}h} - a_{jh}\phi_{\bar{k}h} + g_{ki}\Phi_{\bar{j}h} - g_{ji}\Phi_{\bar{k}h} +\\ &a_{\bar{k}i}\phi_{jh} - a_{\bar{j}i}\phi_{kh} + g_{\bar{k}i}\Phi_{jh} - g_{\bar{j}i}\Phi_{kh},\end{aligned} \tag{26}$$

where

$$\Phi_{ji} \equiv \lambda_{i\bar{j}} - \phi_{j\alpha}a^{\alpha}_{i}. \tag{27}$$

If an arbitrary vector ϵ^h satisfies the condition

$$a_{i\alpha}\epsilon^{\alpha} = \alpha g_{i\alpha}\epsilon^{\alpha} + \beta g_{\bar{i}\alpha}\epsilon^{\alpha}, \tag{28}$$

then $a_{ij} = \alpha g_{ij}$, which is contrary to the nontriviality of the geodesic mapping. So, there exist a vector ϵ^i such that (28) is not true for it. Moreover, for the ϵ^i

$$g_i, \quad g_{\bar{i}} \quad a_i, \quad a_{\bar{i}} \tag{29}$$

is a linearly independent system of vectors, where

$$g_i \equiv g_{i\alpha}\epsilon^{\alpha}; \qquad a_i \equiv a_{i\alpha}\epsilon^{\alpha}.$$

Let us contract (26) with ϵ^h:

$$\begin{aligned}&a_k\phi_{\bar{j}i} - a_j\phi_{\bar{k}i} + g_k\Phi_{\bar{j}i} - g_j\Phi_{\bar{k}i} +\\ &a_{\bar{k}}\phi_{ji} - a_{\bar{j}}\phi_{ki} + g_{\bar{k}}\Phi_{\bar{j}i} - g_{\bar{j}}\Phi_{ki} +\end{aligned}$$

$$a_{kj}\phi_{\bar{j}*} - a_{ij}\phi_{\bar{k}*} + g_{ki}\Phi_{\bar{j}*} - g_{ij}\Phi_{\bar{k}*} +$$

$$a_{\bar{k}j}\phi_{j*} - a_{\bar{j}i}\phi_{k*} + g_{\bar{k}i}\Phi_{j*} - g_{\bar{j}i}\Phi_{k*} = 0 \tag{30}$$

where

$$\phi_{j*} \equiv \phi_{j\alpha}\epsilon^{\alpha}; \qquad \Phi_{j*} \equiv \Phi_{j\alpha}\epsilon^{\alpha}.$$

Let's also contract (30) with ϵ^i:

$$a_k\phi_{\bar{j}*} - a_j\phi_{\bar{k}*} + g_k\Phi_{\bar{j}*} - g_j\Phi_{\bar{k}*} +$$

$$a_{\bar{k}}\phi_{j*} - a_{\bar{j}}\phi_{k*} + g_{\bar{k}}\Phi_{j*} - g_{\bar{j}}\Phi_{k*} = 0 \tag{31}$$

Now, from (29) and (31) it follows that

$$\phi_{j*} = \alpha a_j + \beta a_{\bar{j}} + \gamma g_j + \delta g_{\bar{j}}; \tag{32}$$

$$\Phi_{j*} = \bar{\alpha}_j + \bar{\beta} a_{\bar{j}} + \bar{\gamma} g_j + \bar{\delta} g_{\bar{j}}, \tag{33}$$

where $\alpha, \ldots, \bar{\delta}$ are certain invariants.

Substituting (32) and (31) into (33) we see that $\bar{\alpha} = \gamma$ and $\bar{\beta} = \delta$. Therefore, we can rewrite (33) as

$$\Phi_{j*} = \gamma a_j + \delta a_{\bar{j}} + \bar{\gamma} g_j + \bar{\delta} g_{\bar{j}}; \tag{34}$$

Transforming (30) with components from (32) and (34) we get:

$$a_k M_{\bar{j}i} - a_j M_{\bar{k}i} + g_k \bar{M}_{\bar{j}i} - g_j \bar{M}_{\bar{k}i} +$$

$$a_{\bar{k}} M_{ji} - a_j M_{ki} + g_{\bar{k}} \bar{M}_{ji} - g_{\bar{j}} \bar{M}_{\bar{k}i} = 0,$$

where

$$M_{ji} \equiv \phi_{ji} - \alpha a_{ji} - \beta a_{\bar{j}i} - \gamma g_{ji} - \delta g_{\bar{j}i};$$

$$\bar{M}_{ji} \equiv \Phi_{ji} - \gamma g_{ji} - \delta g_{\bar{j}i} - \gamma a_{ji} - \delta a_{\bar{j}i}; \tag{36}$$

it follows from (35)

$$M_{ji} = a_j A_i + a_{\bar{j}} B_i + g_j C_i + g_{\bar{j}} D_i;$$

$$\bar{M}_{ji} = a_j C_i + a_{\bar{j}} D_i + g_j E_i + g_{\bar{j}} F_i, \tag{37}$$

where $A_i, \ldots, F_i$ are certain vectors.

Based on (36) and (37) we have:

$$\phi_{ji} = \alpha a_{ij} + \beta a_{\bar{j}i} + \gamma g_{ji} + \delta g_{\bar{j}i} + a_j A_i + a_{\bar{j}} B_i + g_j C_i + g_{\bar{j}} D_i;$$

$$\Phi_{ji} = \bar{\gamma} g_{ij} + \bar{\delta} g_{\bar{j}i} + \gamma a_{ji} + \delta a_{\bar{j}i} + a_j C_i + a_{\bar{j}} D_i + g_j E_i + g_{\bar{j}} F_i. \tag{38}$$

Set

$$S_{ji} \equiv a_j A_i + a_{\bar{j}} B_i + g_j C_i + g_{\bar{j}} D_i;$$

$$\bar{S}_{ji} \equiv a_j C_i + a_{\bar{j}} D_i + g_j E_i + g_{\bar{j}} F_i. \tag{39}$$

Substituting (38) for (26) we can rewrite (26) as the following:

$$a_{kh} S_{\bar{j}i} - a_{jh} S_{\bar{k}i} + g_{kh} \bar{S}_{\bar{j}i} - g_{jh} \bar{S}_{\bar{k}i} +$$

$$a_{\bar{k}h} S_{ji} - a_{\bar{j}h} S_{ki} + g_{\bar{k}h} \bar{S}_{ji} - g_{\bar{j}h} \bar{S}_{\bar{k}i} +$$

$$a_{ki} S_{\bar{j}h} - a_{ji} S_{\bar{k}h} + g_{ki} \bar{S}_{\bar{j}h} - g_{ji} \bar{S}_{\bar{k}h} +$$

$$a_{\bar{k}i} S_{jh} - a_{\bar{j}h} S_{ki} + g_{\bar{k}i} \bar{S}_{jh} - g_{\bar{j}i} \bar{S}_{kh} = 0. \tag{40}$$

A detailed analysis of the (40) for $n \geq 8$ leads us to a conclusion that the vectors A_i, B_i, E_i, F_i are complanarly with vectors C_i and D_i. Therefore (38) can be transformed into

$$\phi_{ji} = \alpha g_{ij} + \beta a_{\bar{j}i} + \gamma g_{ji} + \delta g_{\bar{j}i} + c_j C_i + d_j D_i;$$

$$\Phi_{ji} = \bar{\gamma} g_{ij} + \bar{\delta} g_{\bar{j}i} + \gamma a_{ji} + \delta a_{\bar{j}i} + \bar{c}_j C_i + \bar{d}_j D_i. \tag{41}$$

where $c_i, d_i, \bar{c}_i, \bar{d}_i$ are certain vectors.

Thus, (26) takes the form of (40), where

$$S_{ji} = c_j C_i + d_j D_i; \qquad \bar{S}_{ji} = \bar{c}_j C_i + \bar{d}_j D_i.$$

This formula may be rewritten as

$$C_{(i} A_{h)jk} + D_{(i} B_{h)jk} = 0, \tag{42}$$

where

$$A_{hjk} \equiv a_{kh} c_{\bar{j}} - a_{jh} c_{\bar{k}} + a_{\bar{k}h} c_j - a_{\bar{j}h} c_k +$$

$$g_{kh} \bar{c}_{\bar{j}} - g_{jh} \bar{c}_{\bar{k}} + g_{\bar{k}h} \bar{c}_{\bar{j}} - g_{\bar{j}h} \bar{c}_k;$$

$$B_{hjk} \equiv a_{kh}d_{\bar{j}} - a_{jh}d_{\bar{k}} + a_{\bar{k}h}d_j - a_{\bar{j}h}d_k +$$

$$g_{kh}\bar{d}_{\bar{j}} - g_{jh}\bar{d}_{\bar{k}} + g_{\bar{k}h}\bar{d}_{\bar{j}} - g_{\bar{j}h}\bar{d}_k.$$

Now, we assume that C_i and D_i are noncolinear vectors.

Then there exist η^i such that $\eta^\alpha C_\alpha = 1$ and $\eta^\alpha D_\alpha = 0$. Contracting (42) with $\eta^h\eta^i$ we see, that $\eta^\alpha A_{\alpha jk} = 0$. After contracting (42) with η^i, on the base of the previously considerations, we get $A_{hik} = D_h D_{jk}$ where D_{jk} is a certain tensor.

The latter can be rewritten as

$$a_{kh}c_{\bar{j}} - a_{jh}c_{\bar{k}} + a_{\bar{k}h}c_j - a_{\bar{j}h}c_k +$$

$$g_{kh}\bar{c}_{\bar{j}} - g_{jh}\bar{c}_{\bar{k}} + g_{\bar{k}h}\bar{c}_{\bar{j}} - g_{\bar{j}h}\bar{c}_k = D_h D_{jk}. \tag{43}$$

If c_i and $\bar{c}_i$ are noncolinear, then there exists a vector Θ^i such that $\Theta^\alpha c_\alpha = 0$ and $\Theta^\alpha c_{\bar{\alpha}} = 1$. Contracting (43) with Θ^j, we get

$$g_{\bar{k}h} + a g_{kh} + b a_{kh} = \sum_{\sigma=1}^{5} \xi_{\sigma^k}\eta_{\sigma^h}, \tag{44}$$

$\xi_{\sigma^k}\eta_{\sigma^h}$ – here and further are certain vectors.

Alternating (44) we obtained

$$g_{\bar{k}h} = \sum_{\sigma=1}^{10} \xi_{\sigma^k}\eta_{\sigma^h},$$

which leads us to a contradiction when $n > 10$.

Now, let us consider the case of colinearity of C_i and $\bar{C}_i$. For example, let $\bar{c}_i = -\rho c_i$. Then (43) can be rewritten as

$$A_{kh}c_{\bar{j}} - A_{jh}c_{\bar{k}} + A_{\bar{k}h}c_j - A_{\bar{j}h}c_k = D_h D_{jk}, \tag{45}$$

where $A_{ij} = a_{ij} - \rho g_{ij}$.

From (45) and noncolinearity of c_i and $\bar{c}_i$ it follows that

$$a_{ij} = \rho g_{ij} + \xi_{1^i}c_j + \xi_{2^i}c_{\bar{j}} + D_i\xi_{3^j}, \tag{46}$$

which implies

$$Rg||a_{ij} - \rho g_{ij}|| \leq 3. \tag{47}$$

It remained to consider the case of collinearity of C_i and D_i. Here, if $c_i \neq 0$ (47) will be obtained.

As a result, we get for $n > 10$ the following situation: either (47) is true or

$$\phi_{ji} = \alpha a_{ji} + \beta a_{\bar{j}i} + \gamma g_{ij} + \delta g_{\bar{j}i}; \tag{48a}$$

$$\lambda_{i\bar{j}} - \phi_{j\alpha} a_i^\alpha \equiv \Phi_{ji} = \bar{\gamma} g_{ji} + \bar{\delta} g_{\bar{j}i} + \delta a_{\bar{j}i} + a_{ji}. \tag{48b}$$

For $\alpha^2 + \beta^2 \neq 0$ (48 a) implies that

$$a_{ji} = \alpha\phi_{ji} + \beta\phi_{\bar{j}i} + \gamma g_{ij} + \delta_{\bar{i}j},$$

where $\alpha, \dots, \delta$ are certain invariant.

After symetrisation of the last expression, we have:

$$a_{ij} = \alpha\phi_{(ij)} + \beta\phi_{(\bar{i}j)} + \gamma g_{ij}. \tag{49}$$

It follows then, that the mobility rate of K_n relative to geodesical mapping introduced in [11], is not greater than 3.

If $\alpha^2 + \beta^2 = 0$, then from (48 a) we have

$$\phi_{ij} = \gamma g_{ij} + \delta g_{\bar{j}i}. \tag{50}$$

Excepting the tenzor ϕ_{ij} from (48 b) by means of (50) we obtained

$$\lambda_{i,\bar{j}} = \mu g_{i\bar{j}} + \nu g_{ij} + B a_{i\bar{j}} + C a_{ij},$$

where μ, ν, B, C are certain invariants.

The following relation has been obtained by conjugation at index j

$$\lambda_{i,j} = \mu g_{ij} - \nu g_{i\bar{j}} + B a_{ij} - C a_{i\bar{j}}, \tag{51}$$

Alternating (51) we have obtained

$$\nu g_{i\bar{j}} + C(a_{i\bar{j}} - a_{j\bar{i}}) = 0.$$

Differentiating covariantly the latter, and considering (2) and (16), it is easily seen that either the conditions (47) are satisfied, or $\nu = C = 0$. Thus, formula (51) takes the form

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij}. \tag{52}$$

In [16] the Riemannian space V_n is called $V_n(B)$-space, if it admits a geodesic mapping satisfying (16) and (52). The basic properties of these spaces were considered in [16] too. In particular, V_n-spaces, admitting non-circular vector fields are the $V_n(B)$-spaces.

The results of the present investigation are summed up in the following theorem.

Theorem 2. *If a Conformally-Kählerian space K_n $(n > 10)$ admits a geodesic mapping onto a Riemannian space, then either space K_n is a $V_n(B)$-space, or the a_{ij} solution of the equations (16) satisfies one of the following conditions:*

$$Rg||a_{ij} - \rho g_{ij}|| \leq 3,$$

or

$$a_{ij} = \alpha\phi_{(ij)} + \beta\phi_{(\bar{i}j)} + \gamma g_{ij}.$$

3. Conformally-Kählerian spaces, admitting concircular fields.

A vector field ξ_i is called [17] concircular, if the following relations are true:

$$\xi_{i,j} = \rho g_{ij}, \tag{52}$$

where ρ is a certain invariant. According to A.P.Shirokov, such a field is called a convergence field when the case $\rho = const$ is fulfilled.

Riemanian spaces V_n, where exist concircular vector fields with $\rho \neq 0$, admit nontrivial geodesic mapping [15].

It is known, that V_n where exists nonisotropic vector ξ_i, may be referred to the coordinate system, with

$$ds^2 = e(dx^1)^2 + f(x_1)d\tilde{s}^2, \tag{53}$$

where $e = \pm 1$, $f(\neq 0)$ is a function of the corresponding argument, $d\tilde{s}^2$, is a metrics of a certain Riemanian space $\tilde{V}_{n-1}$.

Geodesically, $\bar{V}_n$ -space corresponds to this one. The metric form of $\bar{V}_n$ can be written as quoted in [18]:

$$d\bar{s}^2 = e\alpha(1+\beta f)^{-2}(dx^1)^2 + \alpha f(1+\beta f)^{-1}d\tilde{s}^2, \tag{54}$$

where α and β are some constants such that

$$\alpha \neq 0; \qquad 1 + \beta f \neq 0. \tag{55}$$

In [9] it is shown, that Kählerian spaces, with a concircular nonconstant vector field, may be put to coordinate system (53), where $f = const$ and $d\tilde{s}^2$ is a metrics of a certain Sasaki space.

V_n space is called Sasaki space, if there exists a nontrivial structure of F_i^h, satisfying the following relations:

$$F_\alpha^h F_i^\alpha = -\delta_i^h + \chi^h \chi_i; \quad \chi_{,i}^h = F_i^h; \quad \chi^\alpha + \chi_\alpha = \pm 1,$$

$$F^h_{i,j} = -\chi^h g_{ij} + \delta^h_j \chi_i; \quad \chi_{i,j} \quad \chi_{j,i} = 0,$$

where χ^i is a certain Kiling vector, $\xi_i \equiv g_{i\alpha}\xi^\alpha$.

Theorem 3. *Metrics*

$$ds^2 = e(dx^1)^2 + f(x^1)d\tilde{s}^2, \tag{56}$$

determines conformally-Kählerian spaces, where $d\tilde{s}^2$, is a metrics of a certain Sasaki space. Proof. Taking a coordinate system

$$y^1 = y^1(x^1), y^2 = x^2, \dots, y^n = x^n,$$

such that

$$ds'^2 = g(y^1)e(dy^1)^2 + g(y^1) \cdot (y^1)^2 d\tilde{s}^2(y^2, \dots, y^n).$$

Then

$$ds'^2 = g(y^1)(e(dy^1)^2 + (y^1)^2 d\tilde{s}^2).$$

This means, that initial space is conformed to Kählerian space, which completes the proof.

Taking into account the form of the metrics (54), which, as a matter of fact, geodesically corresponds to the metrics (53), we have constructed a family of geodesically corresponding conformally-Kählerian spaces.

Theorem 4. *Let $d\tilde{s}^2$ be a metrics of Sasaki spaces. Then conformally - Kählerian spaces with the metrics* (56) *admit geodesical mapping onto conformally-Kählerian spaces with the metrics* (54).

An additional analysis leads us to a conclusion, that whether $f = \cos\alpha x^1$ or $f = \mathrm{ch}\alpha x^1$, $\alpha \neq 0$, then conformally-Kählerian space with the metrics (56) admit nontrivial projective transformations.

§3. Holomophocally-projective mappings of conformally-Kählerian spaces onto an almost Hermitian spaces

Different aspects of holomorphically-projective mapping are reflected in [1,2]. Assuming that we have a conformally-Kählerian spaces $K_n(g_{ij}, F^h_i)$ and an almost Hermitian space $\bar{H}_n(\bar{g}_{ij}, \bar{F}^h_i)$.

In these spaces we shell consider an analogue of analytically planar curves of Kählerian spaces in the following way:

A curve $L : x^h = x^h(t)$ in K_n will be called an analytically planar one, if its tangential vector $\lambda^h \equiv dx^h/dt \neq 0$ under a parallel motion is always complanar to the tangential λ^h and conjugation $\lambda^\alpha F^h_\alpha$ vectors. Similarly we introduce this notation in $\bar{H}_n$.

A diffeomorphism $f : K_n \to \bar{H}_n$ will be called a HP-mapping, if with respect to f, all analytically planar curves in the conformally-Kählerian space is transformed into analytically planar curves in $\bar{H}_n$.

Taking into account the modified results reported in [19], one can show easily that in a HP-mapping "common" coordinate system x

$$\bar{F}^h_i(x) = F^h_i(x)$$

and

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h{}_{(i}\psi_{j)} + F^h{}_{(i}\Theta_{j)}, \tag{57}$$

where ψ_i and Θ_i are certain vectors, Γ^h_{ij}; $\bar{\Gamma}^h_{ij}$ are the connectivities of $\bar{H}_n$ and K_n.

The equations (57) are equivalent to the condition

$$g_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_{(i}\bar{g}_{j)k}\Theta_{(i}\tilde{F}_{j)k}, \tag{58}$$

where $\tilde{F}_{ij} \equiv \bar{g}_{i\alpha}F^\alpha_j$.

Let's differentiate covariantly the relation

$$\bar{g}_{i\alpha}F^\alpha_j + \bar{g}_{j\alpha}F^\alpha_i = 0,$$

which holds in $\bar{H}_n$:

$$\bar{g}_{i\alpha,k}F^\alpha_j + \bar{g}_{j\alpha,k}F^\alpha_i + \bar{g}_{i\alpha}F^\alpha_{j,k} + \bar{g}_{j\alpha}F^\alpha_{i,k} = 0.$$

By (2) and (58) we received

$$\psi_i\bar{g}_{\bar{j}k} + \psi_{\bar{j}}\bar{g}_{ik} + \Theta_i\bar{g}_{jk} + \Theta_{\bar{j}}\bar{g}_{i\bar{k}} +$$

$$\psi_j\bar{g}_{\bar{i}k} + \psi_{\bar{i}}\bar{g}_{jk} + \Theta_j\bar{g}_{ik} + \Theta_{\bar{i}}\bar{g}_{j\bar{k}} +$$

$$\bar{g}_{ik}\phi_j - \bar{g}_{i\alpha}\phi^\alpha g_{jk} + \bar{g}_{i\bar{k}}\phi_{\bar{j}} + \bar{g}_{i\alpha}\phi^{\bar{\alpha}} g_{j\bar{k}} +$$

$$\bar{g}_{jk}\phi_i - \bar{g}_{j\alpha}\phi^\alpha g_{ik} + \bar{g}_{j\bar{k}}\phi_{\bar{i}} + \bar{g}_{j\alpha}\phi^{\bar{\alpha}} g_{i\bar{k}} = 0. \tag{59}$$

Let us symmetrize the obtained expression at all indexes. As the result we get

$$g_{(ij}\xi_{k)} + \bar{g}_{(ij}\bar{\xi}_{k)} = 0, \tag{60}$$

where

$$\xi_k \equiv -\bar{g}_{k\alpha}\phi^\alpha; \qquad \bar{\xi}_k \equiv \psi_{\bar{k}} + \Theta_k + \phi_k.$$

Analyzing (60) similarly to (8), we came to the conclusion that either $\xi_k = \bar{\xi}_k = 0$ or $\bar{g}_{ij} = \alpha g_{ij}$. The first case leads to the condition $\phi_i = 0$, that is a conformally-Kählerian space K_n is a Kählerian space.

The second case, where K_n and $\bar{H}_n$ are consisting in a conformally correspondence with $\bar{g}_{ij} = \alpha g_{ij}$, leads us to the conclusion that the mapping is homotetical.

So, we have proved the following theorem.

Theorem 5. *Conformally-Kählerian spaces $K_n\,(n > 2)$, different from Kählerian spaces, do not admit nontrivial HP- mapping onto almost Hermitian spaces $\bar{H}_n$.*

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Odessa State University, Institute of Mathematics Economics and Mechanics, Department of geometry and topology, Petra Velikogo 2, 270000, Odessa, Ukraine