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LOCALLY FINITE HYPERSPACE TOPOLOGY OF ISOCOMPACT SPACES

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Abstract. The purpose of this paper is to investigate some properties of the hyperspace $(\exp X, \tau_{lf})$, with the locally finite topology, when the space (X, τ) is a normal space and every closed countably compact subset of X is compact (*isc*-space). Some properties of *isc*-spaces and *iscc*-spaces are given. If (X, τ) is a normal *isc*-space, then the space $\mathcal{Z}(X) = \{F \subset X : F \text{ is compact}\}$ is a closed subspace of $(\exp X, \tau_{lf})$. Applications to paracompactness are given.

1. Introduction

A topological space X with a topology τ will be denoted by (X, τ). The "base" space (X, τ) is assumed to be Hausdorff. The closure of a set $A \subset (X, \tau)$ is denoted by $[A]_X$ or $[A]$ and the cardinality of A by $|A|$.

We use the following notation ([3],[4],[8]):

$$\exp X = 2^X = \{F \subset X : F \text{ is closed and not empty}\},$$
$$\mathcal{Z}(X) = \{F \subset X : F \text{ is compact}\} \subset \exp X,$$
$$\exp(X_0, X) = \langle X_0 \rangle = \{F \in \exp X : F \subset X_0\},$$
$$\mathcal{C}ouc(X) = \{F \in \exp X : F \text{ is countably compact}\},$$
$$\mathcal{J}_n(X) = \{F \in \exp X : |F| \leq n\},$$
$$\mathcal{J}(X) = \{F \in \exp X : F \text{ is finite}\}.$$

Let $\mathcal{U} = \{U_s : s \in S\}$ be a collection of subsets of X. Then

$$\langle \mathcal{U} \rangle = \langle U_s : s \in S \rangle = \{F \in \exp X : F \subset \bigcup U_s \wedge F \cap U_s \neq \varnothing, \forall s \in S\}.$$

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The *finite topology* τ_f (or 2^τ) is the one generated by an open collection of the form

$$\langle U_1, \ldots, U_n\rangle$$

with $U_1, \ldots U_n$ open subsets of X.

The *locally finite topology* τ_{lf} is the one generated by the sets of the form

$$\langle \mathcal{U}\rangle = \langle U_s : s \in S\rangle,$$

where $\mathcal{U} = \{U_s : s \in S\}$ is a locally finite collection of open subsets of X.

The *countably locally finite topology* τ_{clf} is the one generated by the sets of the form

$$\langle \mathcal{U}\rangle = \langle U_s : s \in S\rangle, |S| \leq \aleph_0,$$

where $\mathcal{U} = \{U_s : s \in S\}$ is a locally finite collection of open subsets of X.

If (X, d) is a metric space and H_d is the Hausdorff metric on $\exp X$, then τ_{H_d} is the topology on $\exp X$ generated by the metric H_d.

A space X is said to be *feebly compact* provided each locally finite family of open subsets of X is finite. Clearly, every countably compact space is feebly compact, and any feebly compact space is pseudocompact. Conversely, a completely regular, pseudocompac space is feebly compact and for normal spaces, the three concepts coincide ([3], [9]).

The collection $\mathcal{J}(X)$ of a space X is always dense in $(\exp X, \tau_f)$. However, if X contains an infinite, locally finite collection of open sets, $\mathcal{J}(X)$ is not dense in $(\exp X, \tau_{lf})$. This leads us to the following result ([9]).

Theorem 1.1. *A space* (X, τ) *is feebly compact iff* $\tau_f = \tau_{lf}$. *A normal space* (X, τ) *is countably compact iff* $\tau_f = \tau_{lf}$. □

Definition 1.2. A space X is called an *isocompact space* (= *isc-space*) if every closed countably compact subset of X is compact.

A space X is called a *iscc-space* if every countably compact subset of X is compact.

If X is a normal *isc*-space (normal *iscc*-space), then X is called a T_4*isc*-space (T_4*iscc*-space).

Note. The class of *isc*-spaces contains: compact spaces, metrizable spaces, Lindelöf spaces, σ-compact spaces, paracompact spaces, Q-spaces (= realcompact spaces), Dieudonné complete spaces.

Theorem 1.3. *If $f : X \longrightarrow Y$ is a continuous one-to-one mapping and Y is an iscc-space, then X is also an iscc-space.*

Proof. Let F be a countably compact set of X. Then $f(F)$ is a countably compact set of Y and thus $f(F)$ is compact. The mapping $f|_F : F \longrightarrow f(F)$ is one-to-one and continuous. To see that $f|_F$ is a homeomorphism, we will show that $f|_F$ is a closed mapping. Let $F_1 \subset F$ and $[F_1]_F = F_1$. Then F_1 is countably compact and $f(F_1)$ is also countably compact. It follows that $f(F_1)$ is compact. Thus, $f|_F$ is a homeomorphism and F is a compact subset of X, which completes the proof. □

Example. Let $Y = [0, \omega_1]$ and $X = [0, \omega_1) \cup \{p\}$ (p isolated point) and $f : X \longrightarrow Y$ defined by $f(\alpha) = \alpha$ for all $\alpha \in [0, \omega_1)$ and $f(p) = \omega_1$. The mapping f is a continuous one-to-one mapping, Y is an *isc*-space and X is not an *isc*-space.

Corollary 1.4. *If (X, τ) is an iscc-space and $\tau \subset \tau'$, then (X, τ') is also an iscc-space.*

Proof. Since $\tau \subset \tau'$, then the identity mapping $Id : (X, \tau') \longrightarrow (X, \tau)$ is a continuous one-to-one mapping. □

Theorem 1.5. *A normal isc-space (X, τ) is an iscc-space if and only if every compact subspace of X is an iscc-space.*

Proof. Let X_0 be a countably compact subspace of X. Since X is a normal space then $[X_0]_X$ is countably compact and since X is an *isc*-space it follows that $[X_0]_X$ is compact. The space $[X_0]_X$ is an *iscc*-space and we have that X_0 is a compact subspace of X. Therefore X is an *iscc*-space. □

2. Results

Theorem 2.1. *Let (X, τ) be a normal space. The following are equivalent:*

(1) X is an isc-space ($\mathcal{Z}(X) = \mathcal{C}ouc(X)$).

(2) $\mathcal{Z}(X)$ is a closed subspace of $(\exp X, \tau_{lf})$.

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{C}ouc(X) = \mathcal{Z}(X)$ and $F_0 \in \exp X \setminus \mathcal{Z}(X)$. Then there exists a locally finite collection $\mathcal{U} = \{U_s : s \in S\}, |S| \geq \aleph_0$, $U_s \in \tau$, such that $F_0 \in \langle \mathcal{U} \rangle$.

Let $F \in \langle \mathcal{U} \rangle$. For all $s \in S$, let $x_s \in F \cap U_s$. The set $A = \{x_s : s \in S\}$ is a discrete, closed and infinite subset of X. $A \notin \mathcal{Z}(X) \Longrightarrow F \notin \mathcal{Z}(X)$, and we have

$$[\mathcal{Z}(X)]_{\tau_{lf}} = \mathcal{Z}(X).$$

$\neg(1) \Longrightarrow \neg(2)$. Let $F \in \mathcal{C}ouc(X) \wedge F \notin \mathcal{Z}(X)$. Then each locally finite open covering of F is finite. Let

$$\mathcal{U} = \{U_1, U_2, \ldots, U_n\} \quad \text{and} \quad F \in \langle \mathcal{U} \rangle.$$

Let $x_i \in U_i, \quad i = 1, 2, \ldots, n$. Then $\{x_1, x_2, \ldots, x_n\} \in \langle \mathcal{U} \rangle$ and

$$\{x_1, x_2, \ldots, x_n\} \in \mathcal{Z}(X) \Longrightarrow F \in [\mathcal{Z}(X)]_{\tau_{lf}} \Longrightarrow [\mathcal{Z}(X)]_{\tau_{lf}} \neq \mathcal{Z}(X). \quad \square$$

By Theorem 2.1 we have

Corollary 2.2. a) *If* (X, τ) *is a locally compact* $T_4 isc-$ *space, then* $\mathcal{Z}(X)$ *is an open and closed subspace of* $(\exp X, \tau_{lf})$.

b) *If* (X, τ) *is a metrizable locally compact space, then* $(\exp X, \tau_{lf})$ *is a connected space iff* X *is a compact connected space (continuum).* $\square$

Theorem 2.3. *Let* (X, τ) *be a* $T_4 isc-$ *space. Then* $\mathcal{Z}(X)$ *is an isc-space.*

Proof. Let $\mathcal{B}$ be a closed countably compact subspace of $\mathcal{Z}(X)$ ($\mathcal{B} \in \mathcal{C}ouc(\mathcal{Z}(X))$ and let

$$B = \cup\{K : K \in \mathcal{B}\}.$$

We will show that B is a countably compact subset of X.

Let $\mathcal{U} = \{U_n : n \in N\}$ be a countable cover of B and let

$$\mathcal{U}^* = \{\gamma : \gamma \subset \mathcal{U}, \gamma \text{ is finite}\}$$

For every $K \in \mathcal{B}$ let $\mathcal{U}'(K) = \{U \in \mathcal{U} : U \cap K \neq \varnothing\}$ be a covering of K. Then there exists a finite subcovering $\mathcal{U}(K)$ of $\mathcal{U}'(K)$,

$$\mathcal{U}(K) \in \mathcal{U}^* \wedge \mathcal{U}(K) = \{U \in \mathcal{U} : U \cap K \neq \varnothing\}.$$

The collection $\{\langle \mathcal{U}(K) \rangle : K \in \mathcal{B}\}$ is a countable covering of $\mathcal{B}$. Since $\mathcal{B}$ is a countably compact set of $\mathcal{Z}(X)$ then there exists a finite subcovering of $\mathcal{B}$

$$\{\langle \mathcal{U}(K_1) \rangle, \langle \mathcal{U}(K_2) \rangle, \ldots, \langle \mathcal{U}(K_n) \rangle\}.$$

The collection

$$\{\mathcal{U}(K_1), \mathcal{U}(K_2), \ldots, \mathcal{U}(K_n)\}$$

is a finite subcovering of B and B is a countably compact subset of X. The set $[B]_X = [B]$ is also countably compact ((X, τ) is normal). Therefore $[B]$ is a compact subset of X. The set $\langle [B] \rangle = \exp([B], X)$ is a compact subspace of $\mathcal{Z}(X)$ and $\mathcal{B} \subset \langle [B] \rangle$, it follows that $\mathcal{B}$ is a compact subspace of $\mathcal{Z}(X)$. $\square$

Theorem 2.4. a) *If* (X,τ) *is a metrizable space, then* $(\exp(X,\tau_{lf})$ *is an iscc-space.*

b) *If* (X,τ) *is a Lindelöf space, then* $(\exp(X,\tau_{lf})$ *is an isc-space.*

Proof. a) If (X,τ) is a metrizable space, then $\tau_{lf}=\tau_{sup}$, where $\tau_{sup}=sup$ $\{\tau_{H_d} : d$ metrizes $X\}$ (see [2],Theorem 2).

Every space of the class

$$\{(\exp X,\tau_{H_d}) : d \text{ metrizes } X\}$$

is an $iscc-$ space and $\tau_{H_d}\subset\tau_{lf}$. By Corollary 1.4. it follows that $(\exp(X,\tau_{lf})$ is a cc-space.

b) If (X,τ) is a Lindelöf space, then $(\exp X,\tau_{lf})$ is a real - compact space (see [11], Theorem 3.5), and it follows that $(\exp X,\tau_{lf})$ is an isc-space. If (X,τ) is a Lindelöf space, then $\tau_{lf}=\tau_{clf}$. □

Question. If X is a $T_4 isc-$ space, it is true that $(\exp(X,\tau_{lf}))$ is an isc-space?

Theorem 2.5. *If* $(\exp X,\tau_{lf})$ *is a normal space, then* X *is a* $T_4 isc-$ *space.*

Proof. Let $F\notin\mathcal{Z}(X)\ \wedge\ F\in\mathcal{C}ouc(X)$. Then $(\exp(F,X),\tau_{lf})=(\exp(F,X),\tau_f)$. The space $(\exp(F,X),\tau_f)$ is a closed noncompact subspace of $(\exp X,\tau_{lf})$ and by results of Keesling ([5], [6]) and Velichko ([12]) the subspace $(\exp(F,X),\tau_f)$ is not normal. □

Example. Let K be the Sorgenfrey line. Since the space K is a Lindelöf space, then $\mathcal{Z}(X)=\mathcal{C}ouc(X)$. The product $K\times K$ is a nonnormal space ([3]) and $\mathcal{J}_2(K)$ is also a nonnormal subspace of $(\exp K,\tau_{lf})$. Therefore $(\exp K,\tau_{lf})$ is a nonnormal isc-space.

The set of limit points of a space X is denoted by X'.

In [9], it has been proved the following theorem.

Theorem 2.6. ([9]) *Let* (X,τ) *be normal. If* $(\exp X,\tau_{lf})$ *is first countable, then* X' *is countably compact.* □

By this theorem we have the following.

Corollary 2.7. *Let* (X,τ) *be a* $T_4 isc-$ *space. If* $(\exp X,\tau_{lf})$ *is first countable,* X' *must be compact.* □

Since a paracompact T_2-space is a $T_4 isc-$ space, we have the following:

Corollary 2.8. ([9]) *Let* (X,τ) *be a paracompact* T_2*- space. If* $(\exp X,\tau_{lf})$ *is first countable,* X' *must be compact.* □

Corollary 2.9. *Let (X, τ) be a dense-in-itself $T_4 isc-$ space. Then the following conditions are equivalent:*

(a) $(\exp X, \tau_{lf})$ *is first countable,*

(b) $(\exp X, \tau_{lf})$ *is compact and first countable,*

(c) (X, τ) *is compact and first countable.* □

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