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THE GENERALIZATION OF THE I. N. VECUA EQUATION WITH THE ANALYTIC COEFFICIENTS IN Z, $\overline{Z}$

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Abstract. In this paper it is presented that the methods for the solving of Vecua equation is not only used for the elliptic type of equations. It can be proved that the general methods of areolar series, analytic substitution, and analytic iteration are valid for the all types of partial equations with the analytic coefficients.

1. Introduction

The general system of partial linear equations of I-st order, with two unknown functions $u = u(x, y)$ and $v = v(x, y)$ and with analytic coefficients on the (x, y)

$$\begin{aligned} a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} + b_{11}\frac{\partial v}{\partial x} + b_{12}\frac{\partial v}{\partial y} + a_1 u + b_1 v &= f_1 \\ a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} + b_{21}\frac{\partial v}{\partial x} + b_{22}\frac{\partial v}{\partial y} + a_2 u + b_2 v &= f_2 \end{aligned} \tag{1}$$

with the elliptic condition [1]:

$$a > 0, \quad \Delta = 0 \tag{2}$$

and the condition of symmetry

$$a_{12} = -a_{21}, \quad a_{11} = a_{22}, \tag{3}$$

can be transformed in the canonical form

$$U_x - V_y + a_* U + b_* V = f, \quad U_y - V_x + c_* U + d_* V = g. \tag{4}$$

If we introduce the complex function

$$W = W(z, \overline{z}) = U(x, y) + iV(x, y), \qquad z = x + iy, \overline{z} = x - iy \tag{5}$$

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and its formal differentials

$$\frac{\partial W}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y}\right) = \frac{1}{2}\left[U_x - V_y + i(U_y + V_x)\right], \tag{6}$$

$$\frac{\partial W}{\partial z} = \frac{1}{2}\left(\frac{\partial W}{\partial x} - i\frac{\partial W}{\partial y}\right) = \frac{1}{2}\left[U_x + V_y - i(U_y - V_x)\right], \tag{7}$$

$$\frac{\partial \overline{W}}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial x} + i\frac{\partial \overline{W}}{\partial y}\right) = \frac{1}{2}\left[U_x + V_y + i(U_y + V_x)\right], \tag{8}$$

$$\frac{\partial \overline{W}}{\partial z} = \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial x} - i\frac{\partial \overline{W}}{\partial y}\right) = \frac{1}{2}\left[U_x - V_y - i(U_y + V_x)\right], \tag{9}$$

then it is easy to prove the next basic features:

Theorem 1. *All four differentials from (6) to (9) are linearly independent.*

Theorem 2. *The operation "$\overline{W}$" is distributive as for the common quotient, and for differentials quotient, so for the partial differentials, i.e. for the (6)–(9) there are relations:*

$$\overline{\left(\frac{\partial W}{\partial \overline{z}}\right)} = \frac{\partial \overline{W}}{\partial z}, \tag{10}$$

$$\overline{\left(\frac{\partial W}{\partial z}\right)} = \frac{\partial \overline{W}}{\partial \overline{z}}, \tag{11}$$

which are not linear, but **conjugated**.

By this, the system (1) can be written in a form of one complex equation with four differential:

$$a\frac{\partial W}{\partial \overline{z}} + b\frac{\partial W}{\partial z} + c\frac{\partial \overline{W}}{\partial \overline{z}} + d\frac{\partial \overline{W}}{\partial z} = AW + B\overline{W} + F, \tag{12}$$

where the a, b, c, d, A, B and F are some given analytic coefficients from z, $\overline{z}$. The equation (12) obviously contains the Beltramy equation

$$\frac{\partial W}{\partial \overline{z}} = q(z, \overline{z})\frac{\partial W}{\partial z}$$

and the Vecua equation

$$\frac{\partial W}{\partial \overline{z}} = A(z, \overline{z}) + B(z, \overline{z})\overline{W} + F(z, \overline{z}). \tag{13}$$

2. The generalization of Vecua equation

There is a question - in which form the Vecua equation will be transformed by the pseudo - **linear substitution:**

$$W = \alpha V + \beta \overline{V} + \gamma, \tag{14}$$

where the α, β and γ are the analytic functions in z and $\overline{z}$. By the substitution directly (14) to (13) we get:

$$\begin{aligned}(15)\qquad &\alpha\frac{\partial v}{\partial\overline{z}}+\beta\frac{\partial\overline{V}}{\partial\overline{z}}=\\ &V\left[A\alpha+B\overline{\beta}-\frac{\partial\alpha}{\partial\overline{z}}\right]+\overline{V}\left[A\beta-B\overline{\alpha}-\frac{\partial\beta}{\partial\overline{z}}\right]+\left[F+A\gamma+B\overline{\gamma}-\frac{\partial\gamma}{\partial\overline{z}}\right].\end{aligned}$$

That means, if we resolved (13) we resolved also the classes (15) for every analytic triads (α,β,γ), so we can find V, as a resolution of (15) in a form of

$$V=V(z,\overline{z};A,B,F;\alpha,\beta,\gamma).$$

From (14) re determine, by the conjugation

$$V=\frac{1}{|\beta|^2+|\alpha|^2}-\left[-\overline{\alpha}W+\beta\overline{W}+(\overline{\alpha}\gamma-\overline{\gamma}\beta)\right]. \tag{16}$$

Besides, if we terminate the Vecua equation (13) by the methods from [2] or [3], for example by the linear series method or analytic substitution method or iteration

$$\begin{aligned}W(z,\overline{z})&=W(z,\overline{z},A,B,F)=\phi(z)+\int A(z,\overline{z})\phi(z)d\overline{z}+\int B(z,\overline{z})\overline{\phi}(z)d\overline{z}\\ &+\int Ad\overline{z}\int A\phi d\overline{z}+\int Ad\overline{z}\int B\overline{\phi}d\overline{z}+\int Bd\overline{z}\int\overline{A}\,\overline{\phi}dz\\ &+\int Bd\overline{z}\int\overline{B}\phi dz+\int Ad\overline{z}\int Ad\overline{z}\int A\phi d\overline{z}\\ &+\int Ad\overline{z}\int Adz\int B\overline{\phi}dz+\int Ad\overline{z}\int Bd\overline{z}\int\overline{A}\,\overline{\phi}dz\\ &+\int Ad\overline{z}\int Bd\overline{z}\int\overline{B}\phi dz+\int Bd\overline{z}\int\overline{A}dz\int\overline{A}\,\overline{\phi}dz\\ &+\int Bd\overline{z}\int\overline{A}dz\int\overline{B}\phi dz+\int Bd\overline{z}\int\overline{B}dz\int\overline{A\phi}dz\\ &+\int Bd\overline{z}\int\overline{B}dz\int\overline{B}\phi dz+\int F(z,\overline{z})d\overline{z}+\int Ad\overline{z}\int Fd\overline{z}\\ &+\int Bd\overline{z}\int\overline{F}d\overline{z}+\int Ad\overline{z}\int Ad\overline{z}\int Fd\overline{z}+\int Ad\overline{z}\int Bd\overline{z}\int\overline{F}dz\\ &+\int Bd\overline{z}\int\overline{A}dz\int\overline{F}dz+\int Bd\overline{z}\int\overline{B}dz\int\overline{F}dz\\ &+\int Ad\overline{z}\int Ad\overline{z}\int Ad\overline{z}\int Fd\overline{z}+\cdots.\end{aligned}\tag{17}$$

Then, taking W from convergent series (17) to some approximation, and $\overline{W}$ also from (17) to the same approximation and neglecting in (16), we have the resolution the class of the equation (15).

Let us see the other way around: Does the every equation in form

$$a\frac{\partial V}{\partial\overline{z}}+b\frac{\partial\overline{V}}{\partial\overline{z}}=AV+B\overline{V}+F \tag{15'}$$

by the pseudo linear substitution

$$V=\alpha W+\beta\overline{W}+\gamma \tag{14'}$$

is resolved over the some intermediate Vecua equation with the analytic coefficients, i.e. is there such an analog choice for (α,β,γ)?

If we conjugate and substitute equation, we get:

$$
(18)\quad \begin{aligned}
&(a\alpha+b\overline{\beta})\frac{\partial\overline{W}}{\partial\overline{z}}+(a\beta+b\overline{\alpha})\frac{\partial\overline{W}}{\partial\overline{z}}=W(A\alpha+B\overline{\beta}-b\frac{\partial\overline{\beta}}{\partial\overline{z}}-a\frac{\partial\alpha}{\partial\overline{z}}+\\
&\overline{W}\left(A\beta+B\overline{\alpha}-a\frac{\partial\beta}{\partial\overline{z}}-b\frac{\partial\overline{\alpha}}{\partial\overline{z}}\right)+\left(A\gamma+B\overline{\gamma}+F-a\frac{\partial\gamma}{\partial\overline{z}}-b\frac{\partial\overline{\gamma}}{\partial\overline{z}}\right).
\end{aligned}
$$

That would be the Vecua equations in case

$$
(19)\qquad a\beta+b\overline{\alpha}=0.
$$

Then

Theorem 3. *The general Vecua equation (18) with condition (19) is equivalent to Vecua equation (13) by the pseudo linear substitution (14').*

3. Linear equation with four differentials

Let us do the some substitution (14) in the more general equation (12). So, we have:

$$
(20)\quad \begin{aligned}
&(a\alpha+b\beta)\frac{\partial W}{\partial\overline{z}}+(a\beta+b\overline{\alpha})\frac{\partial\overline{W}}{\partial\overline{z}}+(c\beta+d\overline{\beta})\frac{\partial W}{\partial z}+(c\alpha+d\overline{\alpha})\frac{\partial\overline{W}}{\partial z}=\\
&=W\left(A\alpha+B\overline{\beta}-a\frac{\partial\alpha}{\partial\overline{z}}-b\frac{\partial\overline{\beta}}{\partial\overline{z}}-c\frac{\partial\alpha}{\partial z}-d\frac{\partial\overline{\beta}}{\partial z}\right)+\\
&+\overline{W}\left(A\beta+B\overline{\alpha}-a\frac{\partial\beta}{\partial\overline{z}}-b\frac{\partial\overline{\alpha}}{\partial\overline{z}}-c\frac{\partial\beta}{\partial z}-d\frac{\partial\overline{\alpha}}{\partial z}\right)+\\
&+\left(A\gamma+B\overline{\gamma}+F-a\frac{\partial\gamma}{\partial\overline{z}}-b\frac{\partial\overline{\gamma}}{\partial\overline{z}}-c\frac{\partial\gamma}{\partial z}-d\frac{\partial\overline{\gamma}}{\partial z}\right)
\end{aligned}
$$

i.e. the equation is the same type (20) contains also the Vecua equations, Beltramy, as well as many other solvable equations by the analog methods, which we are recommending. So we can terminate, in a sense of **general resolution**, with not considering the type, whatever it is elliptic, hiperbolic, parabolic, or wide classes of elliptic type Vecua.

Considering of this, it can to be designed the enormous member of resolute classes of equations, and illustrations.

References

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